Real Analysis
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Lecture - 45
More Theorems on Integrals

Well, there are a few things about the integrals that are still to be discussed. If you remember in the last class, we have discussed what is meant by function of a bounded variation. And the whole idea there was that, till then, we had discussed the integrals of this type.

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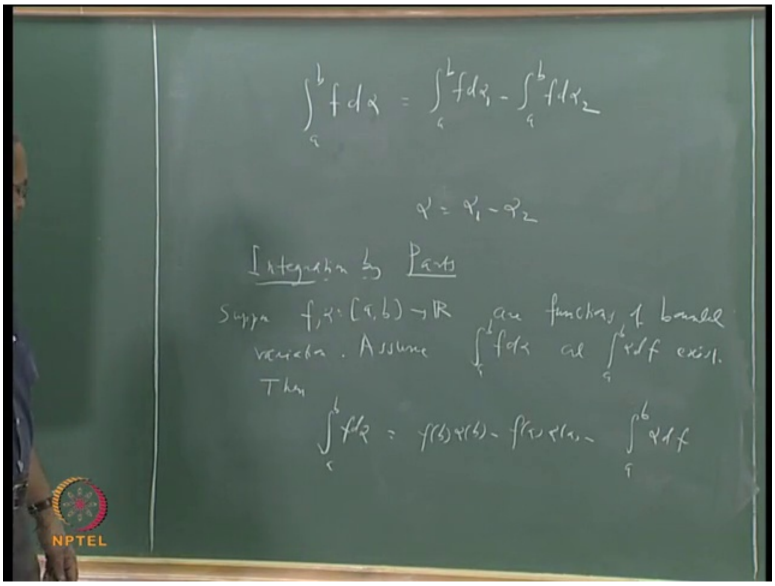

We had discussed integrals of the type integral a to b f d alpha where we had always taken alpha as a monotonically increasing functions and the crux of the discussion in the last class was that we can replace the class of monotonically increasing functions by a class of functions of bounded variation. Every real valued function of bounded variation can be expressed as a difference of two monotonically increasing functions. Hence, if this alpha is a function of bounded variation and if you write this alpha as alpha 1 minus alpha 2, where both of these are monotonically increasing, then you can define this integral as integral a to b f d alpha 1 minus integral a to b f d alpha 2. That was the idea.

Then, what we said was that the main purpose of doing this thing is that we want to prove something what is called integration by part where we have to convert the integral f d alpha in into integral alpha d f. For that, both the things have to be defined integral f d alpha and integral alpha d f and that is what we want to do now. So, that is integration by parts. So, here is let us say that suppose we take two function, suppose f from, let us say f and alpha. Both are functions from a b to r and suppose both are functions of bounded variation f alpha from a b to r are functions for bounded variations, functions of bounded variation.

Let us say that suppose both the integrals exist that is integral f d alpha also exist and integral alpha d f also exist. Further, assume that, assume that integral a to b f d alpha and integral a to b alpha d f exist that means f is Riemann Stieltjes integrable with respect to alpha and alpha is Riemann Stieltjes integrable with respect to f. Suppose both this things happen. Then, integral a to b f d alpha that is equal to f b alpha b minus f a alpha a and minus integral a to b alpha d f. Of course, this integration by parts is something that you would have also seen in the undergraduate calculus courses. If you follow that approach, then the proof is very simple because there you assume lots of things. Let us, let us just quickly recall before we go to the proof of this what we what we do.

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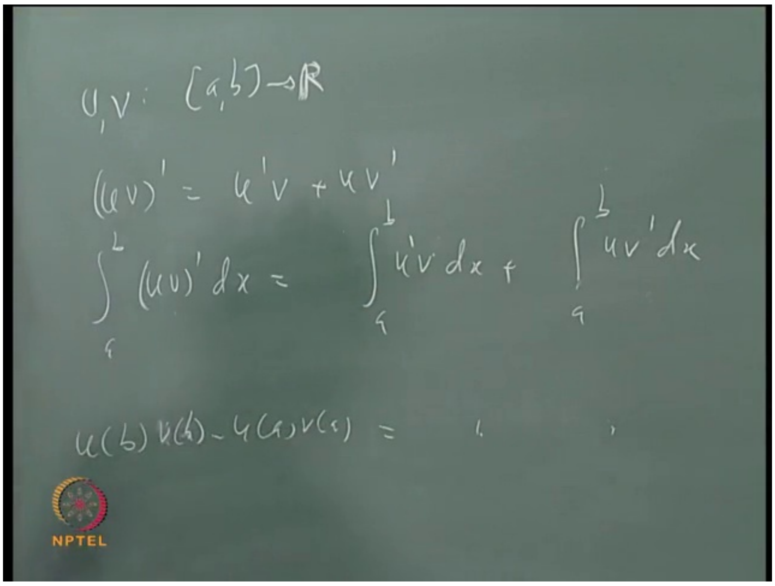

See there what we do is that you look at two functions u and v from a b to r and assume that both of them are differentiable. Assume that u and v both of them are differentiable and then you look at this product formula u v prime is equal to u prime v plus u v prime.

Then, since all these functions are differentiable, they are in particular continuous also. They are all Riemann integrable also. Since anti derivative of this exist, one can use the fundamental theorem of integral calculus here.

So, suppose we use fundamental theorem of integral calculus , what happens is this. That is one can get this integral a to b. Let us say right in the full form that is u v prime d x that is equal to integral a to b u v prime d x, so u prime v d x plus integral a to b u v prime d x. Then, by using fundamental theorem of calculus, since its anti derivative is u v u into v. So, this becomes nothing but u v evaluated between b and a that means nothing but u b into u, u b into v b minus u a into v a and then that is equal to this. Now, we can say that u prime v d x is same as u v prime d x. Remember we had also proved that also integral f d alpha if alpha is differentiable function and such that alpha prime is also integrable, Riemann integrable, then integral f d alpha in that case, we had shown that that is same as integral a to b f d alpha.

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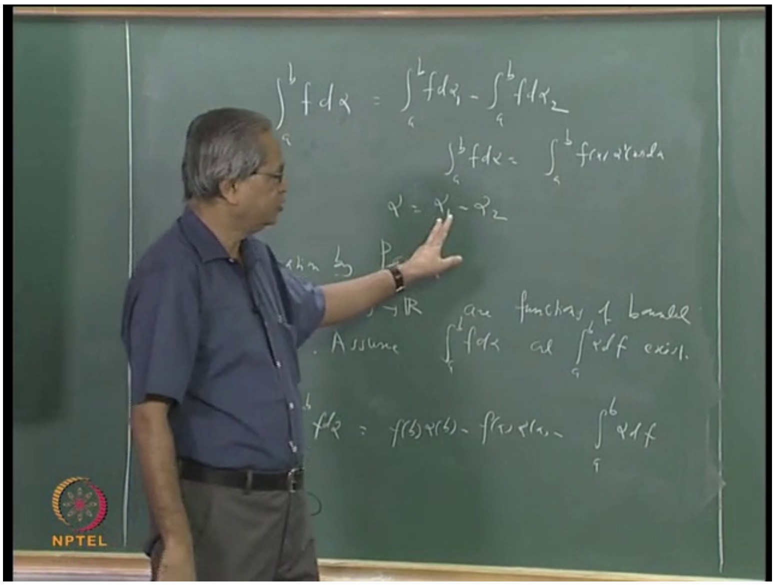

We had shown that is same as integral a to b f x alpha prime x d x. That is how we could convert the Riemann Stieltjes integrals into Riemann integral.

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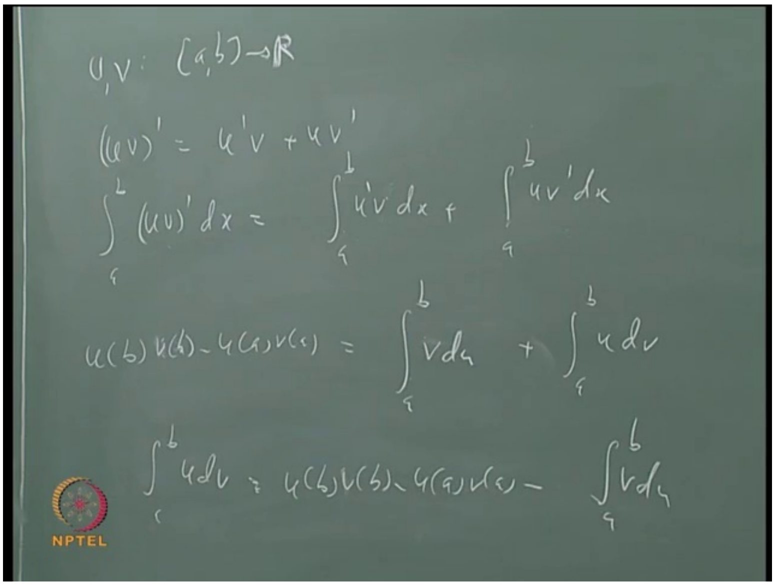

So, suppose you use this notation. Then, this simply becomes, you can write this as integral a to b v d u. This u prime d x is d u and then this becomes integral a to b u d v. So, one can simply write this formula same as this that is let me, let us write this as integral a to b u d v is equal to u b of v b minus u a b a minus integral a to b v d a. That is essentially same as this formula. That is essentially same as if you take, for example f as u and alpha s v this, this, but the difference here is that you have assumed that u and v are differentiable. You assume that u and v are differentiable and that is why the proof and then use fundamental theorem of calculus and then the proof become very easy.

But, here our assumptions are much less. We are only assuming that these are functions of bounded variation. Remember we had seen earlier that if a function has a continuous derivative, then it is a function of bounded variation. If the derivative is bounded, then the function is of bounded variation. So, that is the special case of this. That is the special case of this. Let us now see how this can be proved in a general situation like this.

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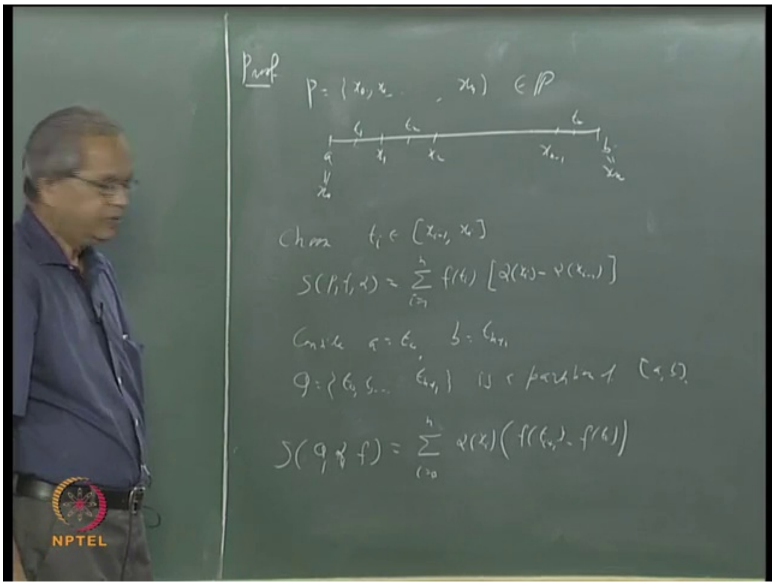

Let us go to the proof of this. We shall be using again the integral as a limit of sum that is we look at the given any integral, either this integral or that integral, we look at the corresponding Riemann Stieltjes sum. Once the function is integrable, the limit of the Riemann Stieltjes sum goes to the value of the integral as mesh of the partition goes to 0. That is the idea we shall use.

So, start with some partition, let us say P is equal to x naught, x 1, x n, and suppose that is a partition of P. I think it may be better to draw this, suppose this is a and that is b. Then, this a is equal to x naught. Let us say x 1 is somewhere here, x 2 is somewhere here etcetera x n minus 1 and this last b is equal to x n. Then, to form the Riemann Stieltjes sum, we choose t i in each of this sub interval, so choose sum t i, choose t i belonging to x i minus 1 into x i. That means what; there will be some t 1 here, some t 2 here etcetera. There will be some t n here. So, what is s P f alpha that is sigma i going from 1 to n f at t i into delta alpha, let this alpha in the full form delta alpha as alpha x i minus alpha x i minus 1.

Now, what we do is that we also look at one more partition and that is a partition given by this t 1, t 2, t n, but only problem is this till now for all partitions, we had assumed that the, for example a partition like this x naught has to be a and x n has to be b. But, about this t 1, t 2, t n, we do not know that, we do not know whether t 1 is equal to a or whether t n is equal to b, but that is not a problem. If it is not the case, we add those points that is add additional points, take t naught is equal to a and take t n plus 1 is equal to b.

So, consider a is equal to t naught and b is equal to t n plus 1. Consider this partition q as t naught, t 1 etcetera t n plus 1. Remember here also you have t naught less than t 1, t 1 less than t 2 etcetera, and finally t n plus 1 is equal to b. So, this is also a partition of a, b. Now, what I do is that with respect to this partition, I write a Riemann Stieltjes sum corresponding to this integral, Riemann stieltjes sum corresponding to this integral.

That means s Q alpha f, consider s Q alpha f, so that is again sigma i going from 1 to n, yes n plus 1. There are n plus 1 intervals now, sub intervals now, i going from, sorry, i equals to 0, 0 to n. But, the problem is that we also have to choose the points in between. In t i minus 1, t i, we have to choose and those points I take as x i, those points. For example, here let us say here you have t 1 here, you have t 2. In between the points, I take as x y.

This is what I do for all. So, I will take this as f at x i into, this will not to be t i plus 1 minus t i, not t i minus t i, so it should be other way because we have alpha f, we want alpha f, so alpha x i into f at t i plus 1 minus f at t i. Now, it will help to write this at least the first few terms of this in the full form. Let me do it here. Yes, yes, that was the whole point, we wanted to increase the, extend the class of monotonically increasing functions. That was the whole point of discussing in the functions of bounded variations. So, let us let us just try to write first two terms so that you will understand what is happening here.

Let us look at this s P f alpha. What is the first step? It is f t i, so f t 1 into alpha x 1 minus alpha x 0, for i equal to 1, f t 1 into alpha x 1 minus alpha x 0 etcetera, then plus f t 2 into alpha x 2 minus alpha x 1 etcetera etcetera. So, what is the term corresponding to i equal to n? It is f at t n multiplied by alpha x n minus alpha at x n minus 1. Similarly, you look at s Q alpha f, s Q alpha f is starts from i equal to 0, so alpha x alpha at x naught into f x 1 minus f, sorry f t 1 minus f t naught f t 1 minus f t naught. What is that?

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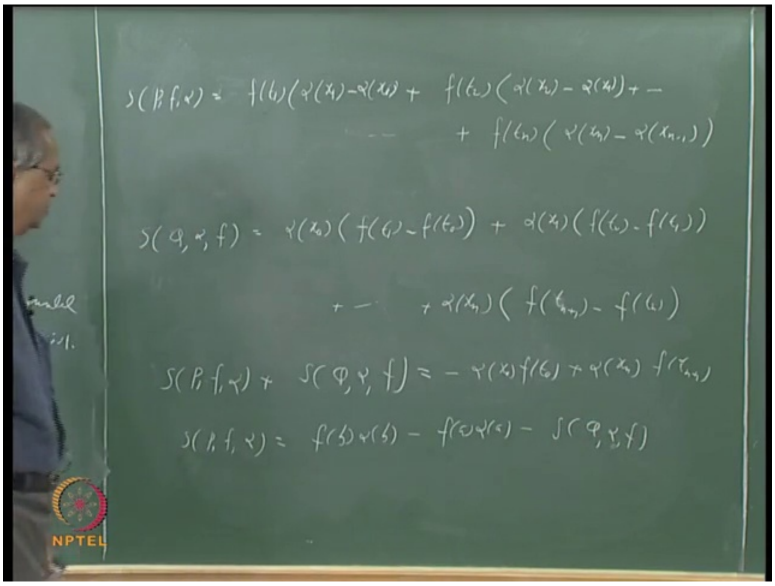

Let me write one more term, for i equal to f 1, it is alpha at x 1 into f t 2 minus f t 1 etcetera. Let me just write the last term plus alpha at t n into f at, sorry alpha at x n f at t n plus 1 minus f at t n. Now, suppose I take the sum of this two, s P f alpha plus s Q alpha f. Look at what is happening. Here, you have the term f t 1 alpha x 1 and here you have minus f t 1 alpha x 1. It will be cancelled. Then, similarly, f t 2 alpha x 2, there will be the next term f t 2; it will be minus f t 2 alpha x 2 here. So, several terms will cancel. What will remain?

For example, here you have alpha x naught f t 1, 0 minus alpha x naught f t 1. So, what will remain is the alpha minus alpha x naught f t naught. Corresponding to that, there is nothing here. Finally, here also, f t n plus 1 alpha x n; corresponding with that also is nothing. So, if you take the sum of what remains is this. That is minus alpha x naught f t naught and then plus alpha x n f t n plus 1. But, again now given by the fact that x naught and t naught both are a, x n is b and t n plus 1 is also b, that is how we have chosen.

So, I can rewrite this whole thing as s P f alpha that is equal to f b alpha b minus f a alpha a and then minus s q alpha f. Then, we know that this integral f d alpha exist that is our assumption integral f d alpha exist. So, now this is Riemann Stieltjes sum corresponding to this integral and so this as mesh of the partition P goes to 0. Then, this will converge to integral a to b f d alpha. Similarly, as the mesh of the partition Q goes to

0, mesh of the partition Q goes to 0 that is q alpha f will converge to integral a to b alpha d f.

Now, the only question is this, can we say that whenever mesh of the partition P goes to 0, mesh of the partition Q also goes to 0? Can we say that how are the mesh of P and mesh of Q related? You can say, for example, if suppose you take t 2 minus t 1, obviously it is smaller than x 2 minus a. So, similarly, if you take any, any t i minus, t i minus 1, if you take any subinterval of the partition Q, you can say that it will be less, not equal to, it is lying, it will be less, not equal to sum of the two consecutive intervals that you can always say.

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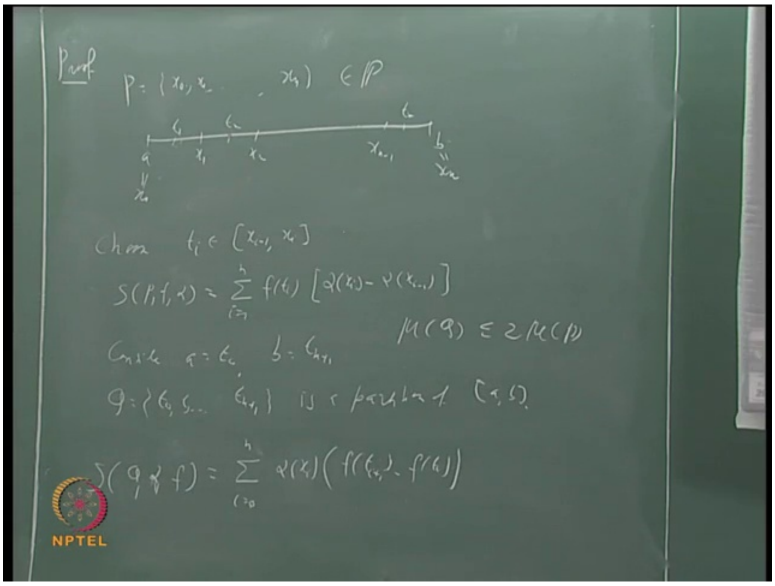

So, I can say this that the mesh of Q is less not equal to two times mesh of P. I mean this may not be a very good estimate, but is certainly true. So, all that we need is this whenever mesh of P goes to 0, mesh of Q also goes to 0.

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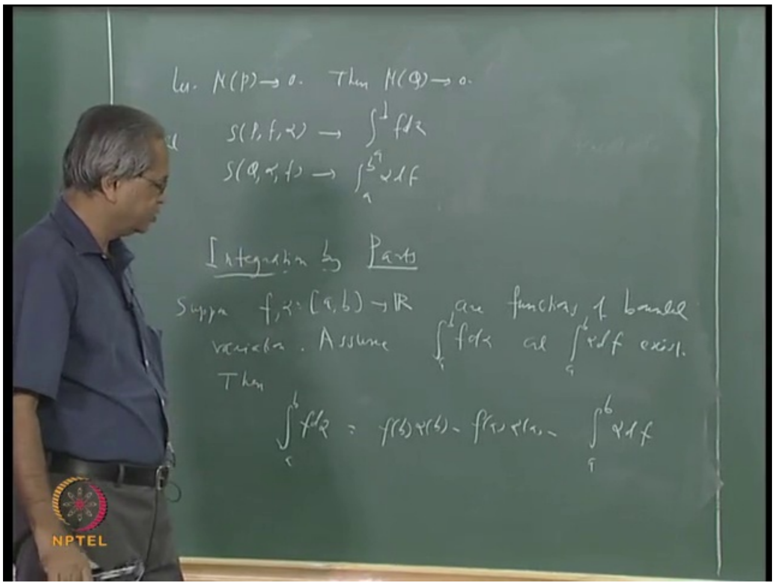

Hence, so let mesh of P tends to 0, and then mesh of Q also tends to 0. What happens is s P f alpha tends to integral a to b f d alpha and s Q alpha f tends to integral a to b alpha d f. So, as mesh of P tends to 0, this left hand side goes to integral a to b f d alpha and this term will go to integral a to b alpha d f and this will be remain as it is f b because that has nothing to do with the partition. That is true for all the partitions. So, in the limit, we get this formula, formula for the integration by part.

Now, in addition to what we have said here, let us assume a few more things. Suppose I have, further assume that this alpha is continuous and f is monotonically increasing. Alpha is continuous and f is monotonically increasing. Then, I can apply, and then this will be as usual Riemann Stieltjes integral. I can apply mean value theorem to this term. I can apply mean value theorem to this term. Then, we will see what happens to this whole thing.

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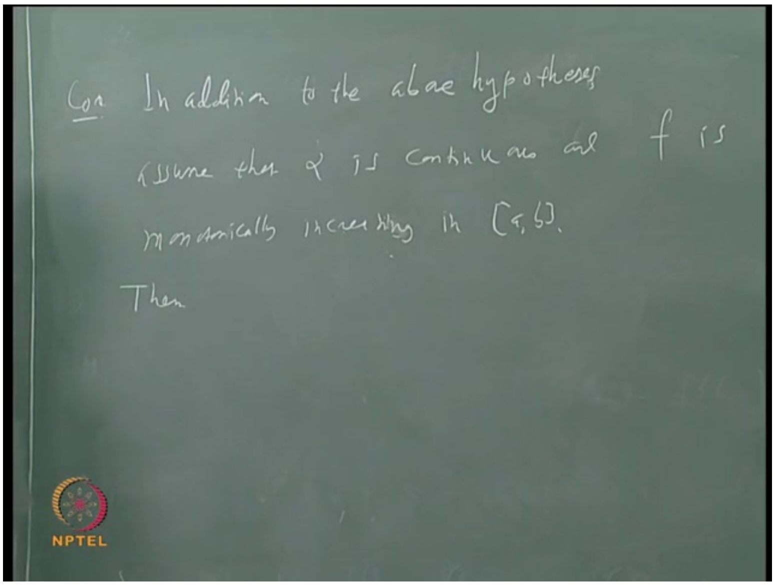

So, let us write that as a conversing. What I mean to say is that in addition to the, in addition to the above hypothesis, assume that alpha is continuous, alpha is continuous and f is monotonically increasing, monotonically increasing in a b. Then, what happens? I will write little later. First we shall see what we are aiming at.

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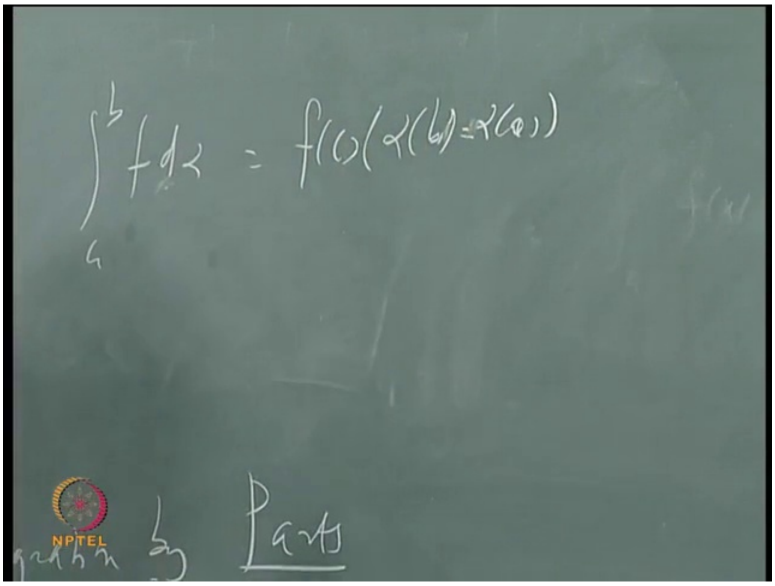

See if you recall the first mean value theorem of integral calculus, what we said there if you look at integral a to b f d alpha, if f is continuous and alpha is protonically increasing, we had shown that there exists a point c, there exists a point c in the interval such that this integral is same as f x c into alpha b minus alpha a. That is what we had called first mean value theorem of the integral calculus. Suppose I apply that to this term here. Now, I am assuming alpha is continuous and f is monotonically increasing. So, we can apply that to this term.

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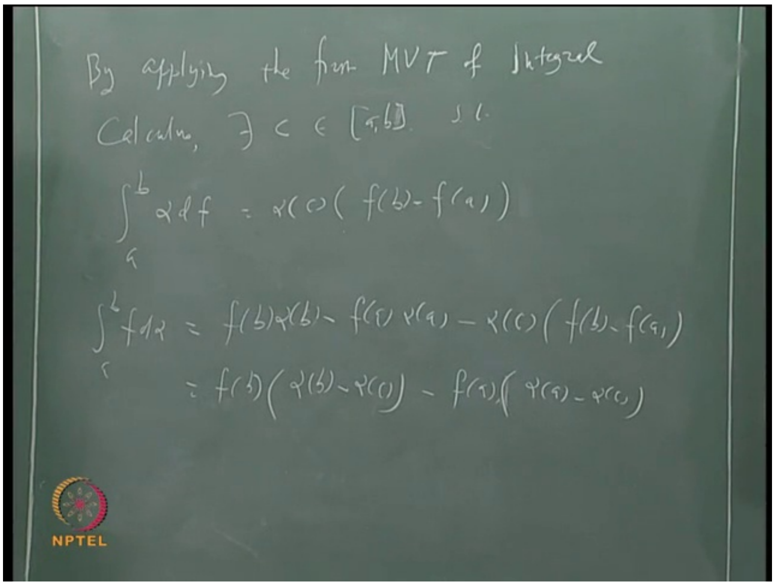

So, let us say by applying the first mean value theorem of integral calculus, we can say there exists some point in c in a, b. There exists some point c in a b such that what should happen? Integral a to b alpha d f, what is this, just imitate this. Now, rules of alpha and f are interchanged. So, this should be alpha at c multiplied by f b minus f a, so alpha of d f that is equal to alpha at c multiplied by f b minus f a. Up to this, we have done nothing new. We just applied first mean value theorem.

Now, you substitute this value here. Then, you substitute this value here. So, what happens? We say integral a to b f d alpha integral f d alpha a to b that is equal to those first two terms will leave as they are f b alpha b minus f a alpha a and then minus for this minus integral alpha d f, I will write this value minus integral alpha d f, so minus alpha at c into f b minus f a. Yes, f of b into alpha b that is right.

Now, we will combine in slightly different manner. You have f b here and here also, so we will combine this two terms, so what will it is f b into alpha b minus alpha c, f b into alpha b minus alpha c. Then, what remains is minus f a alpha a here, so minus f a into again alpha a minus alpha c. So, combining all this, what we can say is that; then I will come back to this.

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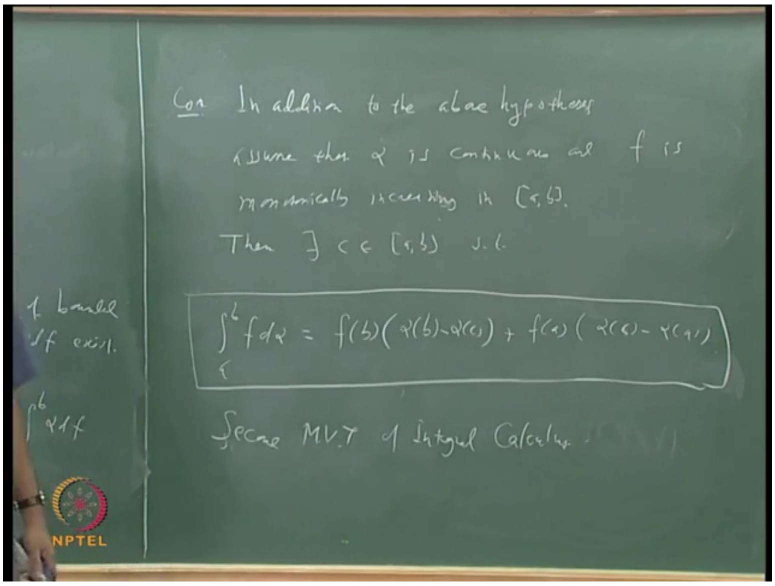

Then, there exists c in a, b such that integral a to b f d alpha, this is equal to f b into alpha b minus alpha c minus f a or if you want you can change it slightly. If I make this plus sign here, then that will be alpha c minus alpha a. So, that is how it is customary written plus f a into alpha c minus alpha a, plus f a into alpha c minus alpha a. So, this conclusion that there exists that is whenever f and alpha satisfy all these conditions that I mentioned here, there exists a c such that integral a to b f b alpha turns out to be f b into alpha b minus alpha c plus f a into alpha c minus alpha a, this is what is called second mean value theorem.

This is called this whole thing that is x is satisfying this; this is called second mean value theorem of integral calculus. So, while discussing the first mean value theorem, I had commented why it is called first because there is something to follow. Now, this is the second mean value theorem. So, that is the idea. Then, there is one more thing that is about the integrals that we need to discuss and may be with that, we can we can close this discussion on integration.

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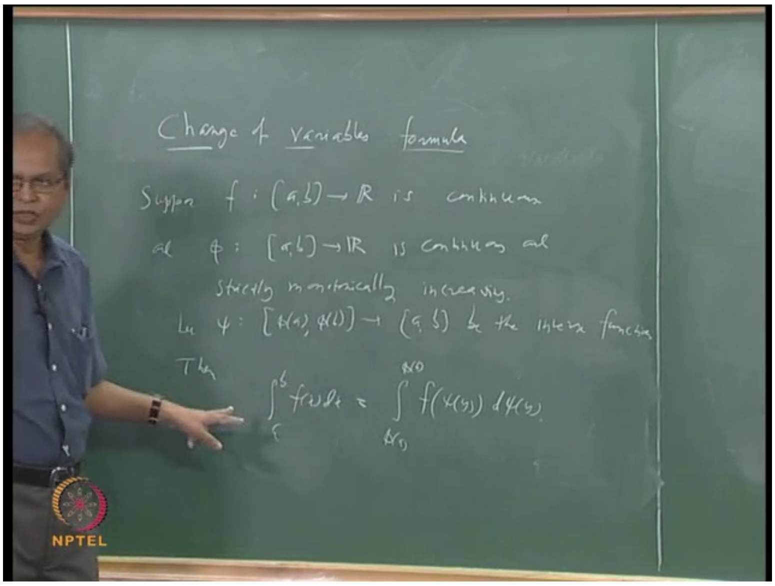

That is what is called change of variable formula; again something that many users quite extensively practice that is change of variables formula. Here also let us to make the matter simple, let us take the function to be continuous suppose and let us talk only about the Riemann integrals, instead of Riemann Stieltjes integrals. Suppose f from a b to r is continuous and let us say and phi from a b to r phi from a b to r is this is also let us take continuous and continuous and strictly monotonically increasing, strictly monotonically increasing.

Now, what is the implication of saying that it is strictly monotonically increasing? What is meant by strictly monotonically increasing? That is whenever x is less than y, phi x is strictly less than phi y. So, in particular it means that the function is one, one. Once you say it is strictly monotonically increasing, it means the function is one, one and hence we can talk about the inverse function, inverse.

Suppose you take this phi will be to interval phi a to phi b, it will map the interval phi a to phi b. So, I can talk about d inverse function. Suppose the inverse function is what I call psi. So, let psi from let us take what is the interval, it will be the interval from phi a to phi b. I can say phi a to phi b to R, but actually it is phi a to phi b to a, b. Let us say psi from a, b be the inverse function, inverse function. Then, remember in a general situation, we may also have to make the statement f is Riemann integrable and whatever occurs it and all that is not necessary because f is continuous and we already prove that continuous functions are Riemann integrable.

So, what I want to say is in that case integral a to b f x d x, this will be transferred to integral from phi a to phi b phi a to phi b f at f at psi y f at psi y d psi y. Again, this is also something that is fairly easy to prove. If you assume that this functions involved are differentiable, then one can use fundamental theorem of integral calculus. Then, the whole proof will become very easy because there it follows merely by chain rule. Then, for example, then this d psi is nothing but psi prime y into d y and that will be and then all that remains is to say.

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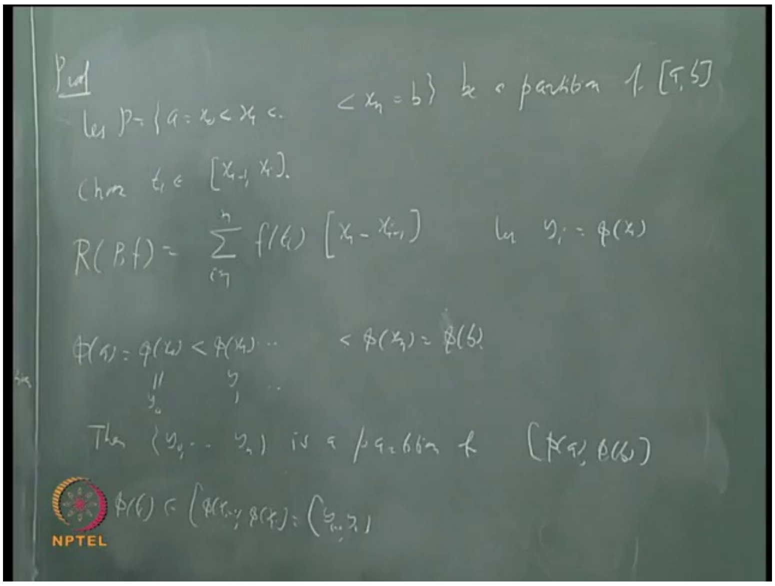

Suppose, if you look at this function f at psi y and then its derivative is nothing but f prime at psi y and multiplied by psi prime y. So, that is all is required. So, for example, suppose you look at, suppose you assume that psi is differentiable. Then, by using standard application of fundamental theorem of integral calculus, all this becomes very easy to prove.

Since, I have done it for several other theorems, I will not discuss that thing. Now, here we shall see this case how we have to prove, but since we are now discussing the general case, we have to represent each of the integral as the limit of a sum. That is what we shall do. So, we shall discuss we shall start with some partition, which represents this integral and look at the corresponding partition here. So, let us say that P be as usual this a is equal to x naught less than x1 etcetera etcetera less than xn that is equal to b be a partition of a, b. When we proceed with the proof, you will also understand what is exactly implication of f is continuous and phi is continuous.

Now, to consider the Riemann sum, we have to look at some points ti in sub interval t xi minus 1, 2. So, choose t i belonging to xi minus 1 to x i. Then, form Riemann sum or P f that is same as sigma i going from 1 to n f ti multiplied by instead delta xi will write it as xi minus xi minus 1, xi minus xi minus 1. Now, let us see the first implication what we should assume about phi. We assume that phi is strictly monotonically increasing, phi is strictly monotonically increasing. So, if I look at phi at x naught that must be strictly less than phi at x1. That must be strictly less than phi at x2 etcetera.

So, what we can say is that phi of a, which is same as phi at x naught, this is strictly less than phi at x1 etcetera etcetera strictly less than phi of xn and that is equal to not d phi of b. What is the meaning of this? It means that this phi x naught x1 phi that is a partition of this interval phi a to phi a, b, that is the partition of this interval phi a to phi b. If we want to use some notation for this, we can call this phi, this we can call this as y naught, this as y1 etcetera. That is in general, we call let yi equal to phi xi.

Then, what we want to say is that, then this y naught, y1, yn is a partition of is a partition of this interval phi a to phi b. As we said about this xis and yis, what about tis if ti is in xi minus 1 to xi, then phi of ti must be in the phi of ti must be in the phi i minus 1 into yi. So, this means phi of ti belongs to phi of xi minus 1 to phi of xi which is same as yi minus 1 to yi. So, if we, if I want to think of Riemann integrals of something, we can take this point phi at ti and multiply that by the length of this intervals. That is possible. That is possible.

Now, which is the function and that is the clue that you get from here. We are not looking at function phi, but now the function psi is from phi a to because remember we want to know, look at the integral on this interval phi a into phi b. So, there should be some function, which is defined on that interval. So, that interval we take the function this f composed with psi. See psi is defined on, psi remember psi is defined from phi a to a b and f is defined from a b to r. So, f is composed with psi defined from phi a phi b to r. So, look at this.

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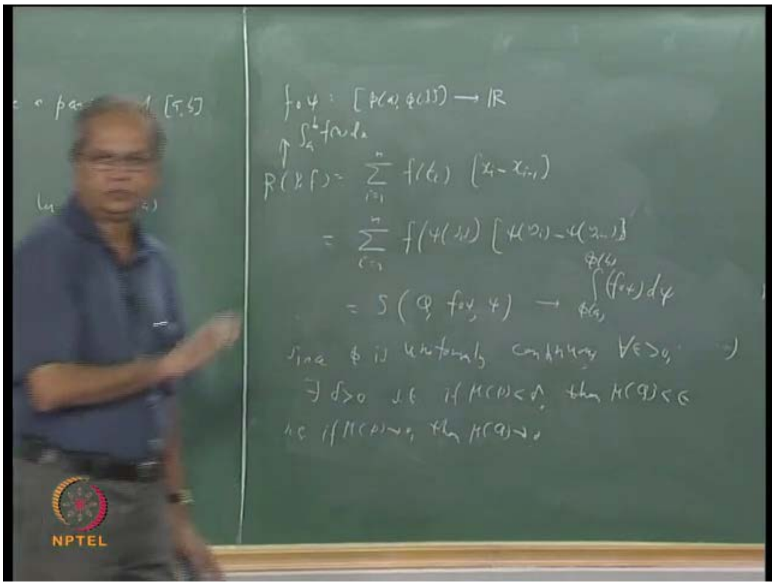

So, this f composed with psi, this will be defined from phi a phi b to r. What we want to know is that we want to look at the Riemann Stieltjes integral of this function f composed with psi with respect to this function this psi. Now, remember here we have assumed that phi is strictly monotonically increasing and psi is its inverse. Does it also follow strictly monotonically increasing? So, there is no problem.

So, if you look at this, now we can express everything now in terms of psi. For example, if you look at this sum here, this let us R P f, what is this? This is nothing but sigma i going from 1 to n; for the time being, let f ti into xi minus xi minus 1. First thing can I write this xi as psi i and xi minus 1 as psi i minus 1? So, this is same as sigma i going from 1 to n, this term I will write later. This is nothing but psi yi minus psi yi minus 1. The only question is how to arrive at f at ti. Now, remember here phi of ti belongs to this sub interval yi minus 1 to yi. I have not given any name for this phi of ti. Let us give it.

Now, let us say, let suppose I call this as si. Let si be equal to phi ti. Let si be equal to phi ti. That is same as saying that ti is equal to psi of si. That is same as ti equals to psi of si. Suppose, I use this notation, this becomes f at psi si multiplied by psi yi minus psi yi minus 1. Now, what can you say about this sum here? It is nothing but the Riemann Stieltjes sum of this. See I have f compose with psi at si and multiplied that by psi yi by minus psi yi minus 1 and si is a point, which lies between yi minus 1 n y. This nothing but Riemann Stieltjes sum of the function f compose with psi with respect to the functions i. So, in other words, with respect to this partition here with respect to this partition here.

Again, we have not given any name to this partition. Let us give it. Now, suppose I call this partition Q. Then, this is nothing but s function is f compose with psi function is f compose with psi. Sorry, first I should write partition s Q f compose with psi. Now, we have we know that f is continuous, so this R P f will go to integral a to b f x d x, wherever mesh of the partition goes to 0. Similarly, if f is continuous, psi is also continuous, f compose with psi is continuous and psi is monotonically increasing. So, this integral also exists that is integral a to b f psi by that also exists and this Riemann Stieltjes sum will also go to this integral provided mesh on the partition Q goes to 0.

Now, that is the only question whether we can show at whenever mesh at the partition P goes to 0, mesh of Q also goes to 0. That is exactly where we make use of this hypothesis. That phi is continuous, phi is continuous. It is a close than border interval. Hence, it is uniformly continuous.

So, what I can say is that given any epsilon, you can find delta such that whenever let us say xi minus xi minus 1 is less than delta phi of xi minus phi of xi minus 1 is less than epsilon, so in other words, whenever mesh of the partition P goes to 0, mesh of the partition Q also goes to 0. That follows from the uniform continuity of phi. That follows from the, let us simply write this that is since we can say since phi is uniformly continuous, we can say that every epsilon bigger than 0, there exists delta bigger than 0 such that if mesh of P is less than delta, then mesh of Q is less than epsilon.

This is because mesh of P is less than delta means what for each i, xi minus xi minus 1 is less than delta. Then, phi xi minus phi xi minus 1 must be less than epsilon, maximum value is mesh of Q. So, mesh of Q must be less than epsilon. So, in other words, if mu of Q goes to 0 that is that in fact, this means that is if mu of P goes to 0, then mu of Q goes to 0 and that is the end of the proof. So, we let mu of P goes to 0.

This will tend to this, will tend to integral a to b f x d n x. This will tend to integral a to b f x d x. This will tend to integral f x d x and this will tend to integral phi a to phi b f converge with psi d psi. That is what is said here that is in the limit integral a to b f x d x should be same as integral phi a to phi bf of psi by d psi y. So, with that, we sort of finish the main points of the usual integrals that we wanted to discuss. There is only one small point, which we need sometimes. So, I will just mention it in passing, but shall not go into too many details of that.

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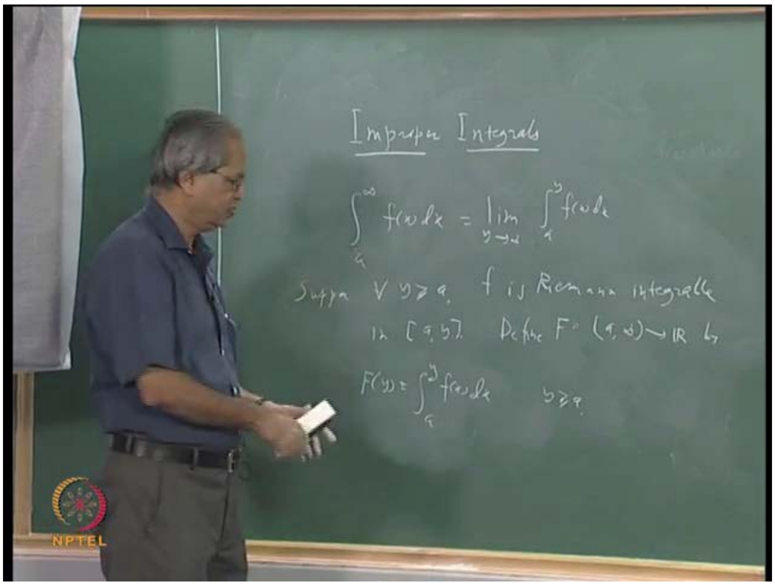

That is what is called improper integrals. I mean the name itself says that there is something less than here what we have discussed so far. In other words, what we discussed so far were proper integrals and some property is not satisfied here, so these are improper integrals and what is that property? Let us stick to the Riemann integrals. For this discussion, we have always assumed that interval a to b is a finite length that is first thing. We have always assumed that function is bounded function a to function a b to r is bounded.

If one of this assumptions is not there that is either if you take the infinite length, for example, a to infinity or minus infinity to b or minus infinity to plus infinity that kind of intervals that is one kind of possibility. That is interval is of infinite length. So, we cannot several things which we did, for example if you would have seen several proofs that you have assumed that whenever a function is continuous, it is uniformly continuous that we could say because the interval was closed end bounded. If it is a interval a to infinity, we cannot do that sort of thing.

So, that is one possibility. The other thing is that function may not be bounded somewhere, for example, it may be of the type interval may be 0 to 1 and function may be type one by x. Then, here x equal to 0, it is not bounded. So, if any such thing happens, then it is called improper integral. How this is done or how those improper integrals are dealt with? Basically, we convert them into series of proper integrals. That is the whole idea; whatever be the case, we convert them into a series of proper integrals.

So, for example, let us just say just take this case. Symbolically, write it like this integral a to infinity f x d x. From the theory, what we have done so far, this is very less because we are not defining what is meant by this, but suppose the following happens. Suppose for every y bigger not equal to a, f is Riemann integrable, f is riemann integrable in the interval a to y. Suppose this happens for every y, then I can talk about this integral a to y f x d x because we assume that it is integrable in this interval a to y.

So, it means this will define a new function in the interval a to infinity. That is for each y, suppose I call this as a, you can say as define f from this interval a to infinity to R by this. Suppose I call this big F of y. So, this is defined for y bigger than not equal to a. What do we do? We consider the limit of this as y goes to infinity. This is the function of y. We consider the limit of this as y goes to infinity. If this limit exists, we say that this improper integral converges. If the limit exists, we say that this improper integral converges and whatever is the value of this limit that is taken as the value of this integral. That means this is nothing but the value of this limit of f y as y goes to infinity.

So, in terms, we can write this as same as limit as y tends to infinity integral a to y f x d x. Of course, this may or may not exist; this limit may or may not exist. Only when limit exists, we say that this improper integral converges. Its value is given by this limit. This limit does not exist. Then, we say that improper integral diverges and once it diverges, there is no question of value. We will stop with that for today.