Integration of Vector Valued Fractions

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Lecture - 44
Integration of Vector Valued Functions

So, far we have discussed main properties of the Riemann integrals as well as its relative Riemann stieltjes integrals. There are two issues that we plan to discuss today. See, so far we have discussed the integrals of real valued functions and can we extend this idea to more general functions. That is what we recall vector value functions that is one thing, other issue is that while discussing the Riemann stieltjes integral, integral of fd alpha we have always taken alpha to be monotonically increasing function, okay?

So, can we remove that restriction on that function alpha? And well what other kinds of functions are possible? That is another question we should discuss. So, the first question is relatively easy, something similar to what we have done in case of derivatives, just we had talked about differentiation of vector valued functions.

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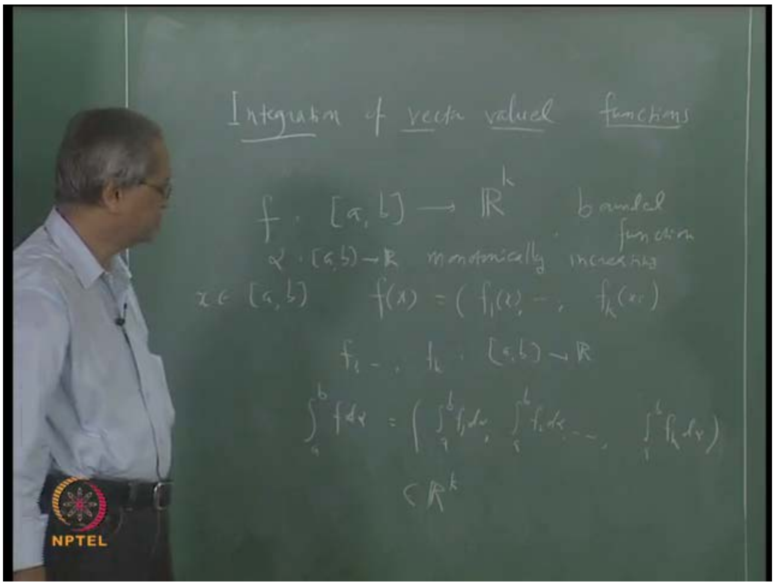

Similarly, we can talk about integration of vector valued functions and what we shall do in all this is that, we shall just discuss the main idea and the main concepts and we will not go into too many details of the proofs, integration of vector valued functions, okay?

Now, what is happening here is that we take a function f from a b to. So, till now we have taken a function from a b to R, instead of that we take a function form a b to any of this, let us say space R k, any of this space R k. So, any element in R k is a vector and we have discussed various norms on this. So, which is analog of absolute value on R and of course, we had to take this as a bounded function, again bounded means it is range is bounded. It is range is a subset of R k, it is range is a subset of R k and R k itself is a matrix space with respect to matrix induce by diagnose and you know what is meant by a bounded set in a matrix space.

So, in that sense it is a bounded function, what we have also observed is that any such vector valued function will give rise to this k scalar valued functions. That is if you take any x in a b then this f x, this is the member of this R k, f x is the member of this R k. So, it will have k components. So, if I call first component as f 1 x, second component as f 2 x, so we can do that we can call this first component as 1 x etcetera, last component as k x then we get that this f 1, f 2, f k each of these is a function from a b to R, all right?

Now, let us say once and once if f is bounded each this f 1, f 2, f, let us say each of this f 1, f 2, f k is also bounded and so let us take alpha as a alpha from a b to R as monotonically increasing. Then we will define this integral a to b, f d alpha as follows. The question is when do we say that this f is integrable and what is value of its integrable? Now, the answer is obvious, if we follow the analogy derivatives what should happen is that each of this f 1, f 2, f k they are functions from a b to. So, each of them should be Riemann stieltjes integrable and then if you take all those k numbers that will form vectors in R k. So, that will be integral of f that is the integral of f.

So, integral a to f, a to b, f 1 d alpha is nothing but this that is this k integral a to b, f 1 d alpha integral a to b f 2 d alpha, and lastly integral a to b f k d alpha. So, to be able to write this what must happen is it if each of this f 1, f 2, f k Riemann stieltjes integrable then, we can define this vector and this will be a member, there is integral also will be a vector, all right?

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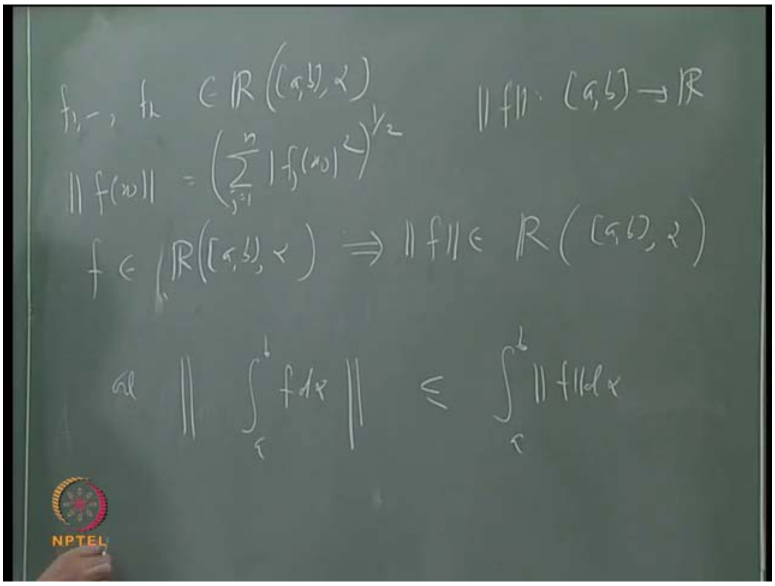

So, in order to be able to say this we must have the following, that is f 1, f 2 f k belong to our td alpha this is the requirement, that is about the definition. The question is what further things we can say about this new types of integrals? We can just imitate whatever we have done in case of real valued functions and most of those theorems we can prove in a similar way, and that is why we shall not go into too many details of those things.

For example, suppose you have two functions f and g, then f plus g also can be defined and then you have integral of f plus g, integral of f plus integral of g only thing is now the whole thing is happening. Now, in R k and the proof will be again to component wise, look at each component and apply the corresponding integral of the real valued functions.

There is only one difference when f just goes a b to R, there is in R, there is a order and with respect to that order R is a totally order space in R k. There is no total order where one can define parcel orders, but there is no total order. So, whatever are the theorems about those orders those will not go, those will not go through immediately. For example, a theorem like whenever f is less, not equal to g integral of f is less not or equal to integral of g, that kind of theorem will not be true here because f less not or equal to g itself does not mean anything here, right?

Then let us recall one more thing, that is in case of real valued functions we have to f is integrable then mod f is also integrable and the absolute value of the integral, that is let me recall is what we have to do is the following. That is f let that be user notification f, that belongs to R a b alpha, this implies mod f is also integrable, that is mod f or and what we know is this integral a to b f d alpha, it is absolute value is less not or equal to integral a to b mod f t, okay?

Now, there is only one change that we want to make in the case of real valued functions, this mod will be replaced by norm. This mod will be replaced by norm and what is remember norm will be a real valued function, norm will be that is f is a function from a b to R b. What is norm of f, norm f of x? There are various norms that you have taken, any one of the norms you can take, but let us say you can take Euclidean norm here. Then that will be same as sigma mod, let us say f j x square, j going from 1 to n and then the square root of this whole, okay?

So, norm f will be a function from a b to R, norm f will be a function from a b to R. So, what we want to say if this function f from a b to R k is integrable. Integrable means what each of this f 1, f 2, f k is integrable then this function norm f is also integrable. This function norm f is also integrable and this also I will replace by this that is norm of this integral because this integral norm is also vector. So, norm of this integral is less not or equal to integral a to b, norm f d f alpha norm f d f alpha. Remember this is a real-world function.

So, this is unusual integral this is this integral is as given here and we take the norm of that, I shall not again go to the detail over this, you can but it is not very difficult. This is something you can try to prove on your own as an exercise, but let me just see, let me just say why this happens. Why this function norm from a b to y in integrable, because f is integrable is what means each of this f 1, f 2, f k is integrable.

Then if f 1, f 2, f k each of this is integrable, than mod f 1, mod f 2, mod f k all of them are integrable and hence this sigma mod f z square is also integrable, right? Because sum of the integrable function and then we take the square root and we have proved that if you compose with any continuous function integrable function with a continuous function, then the composite function is also integrable.

So, norm f is integrable is clear proving, this will require some work, but again not very difficult. So, I will not move into that, you can try to this on your own, so that is about how to go about the deferring indifference of the vector valued functions. This will be quite useful when you deal with vector valued functions and also complex valued functions, etcetera. The second thing is we want to see whether we can replace this condition on alpha. We have seen that so far, we been talking taking alpha as a monotonically decreasing function.

So, can we have replaced this by somewhat bigger class? Why this is required because we can always, we always come across the integrals with respect to some other functions also. For example, low the first thing for example, you must have heard of what is called integration by parts. Now, in integration by part what we want do?

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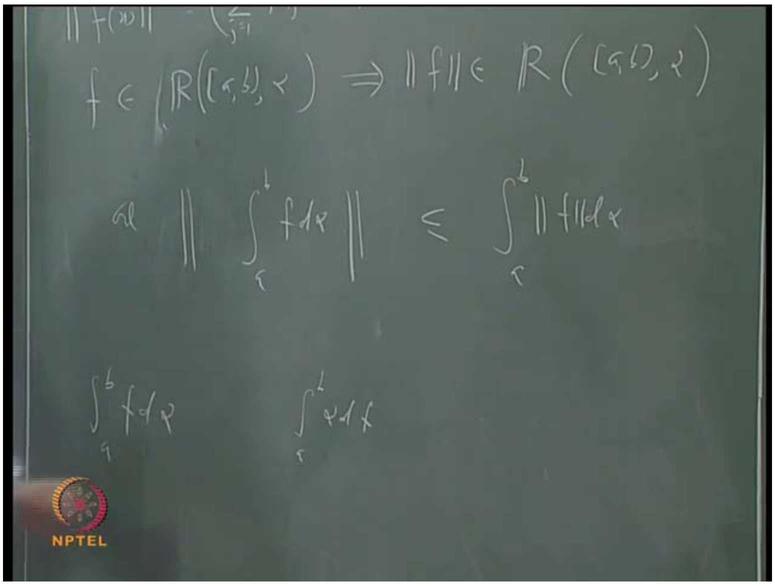

We want to suppose your integral is of this type, integral a to b f d alpha, okay? We want convert it to the integral of this thing, integral a to b alpha d f, right? Plus some more terms here, we want to convert it into this thing. Now, if this integral has to be defined then we will need that f also should be monotonically increasing, but then that puts too many severe restrictions on functions f and alpha. So, that is why we will like to discuss a little bigger class and that bigger class is called functions of bounded variation.

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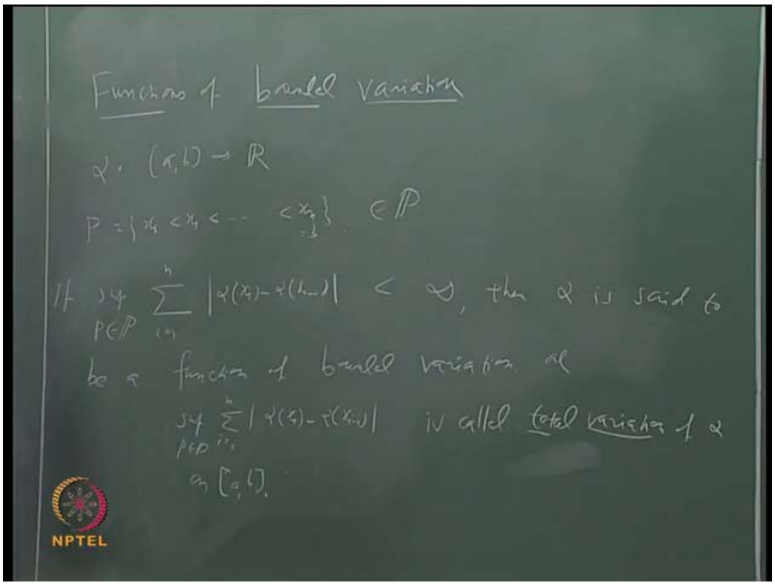

Functions of bounded variation and this class includes the class of all monotonically increasing functions, not only monotonically increasing, but all monotonic functions and also it has better properties. For example, we can prove things like if you take two functions of bonded variation then there are sum is also of bonded variation, that kind of thing we cannot say about monotonic functions. For example, if you take one function which is monotonically increasing and suppose other one is monotonically decreasing, then about there sum we cannot say anything, it need not be at all, okay?

So, those are the kinds of problems are if we just restrict our attention to only monotonic functions. Let us first see what is meant by a function of bonded variation. Let us to do this. Suppose, we take a function alpha from a b to R of course, we can take a b to R, R k also with definition also, but I will discuss that later on. We consider partition P of a b, let us say P is equal to x naught less than x 1, etcetera less than x 1 and suppose this x n is equal to this is the partition of a b. Then what we do is we consider a sigma i going from 1 to n, from alpha xi minus alpha i minus 1 and see till now we have been taking alpha to be monotonically increasing. So, this quantity was always positive, but now we have not made any assumption. So, instead of so will take absolute value mod alpha x i minus alpha x i minus 1.

So, this will be some non negative number for each partition, this will be some non negative number. Then what we do is we take supremum of all such numbers, supremum taken over all the partitions, it is then take the supremum of this for P belonging to script, P supremum of this for P belonging to script P. Now, if this supremum is finite, if this supremum is finite we say that this function alpha is bonded variation and whatever is the number that is called variation of alpha in the interval a to b. That is called variation of alpha in the interval a to b

So, will say that if this number is finite, if this number is finite then alpha is said to be function of bonded variation and whatever is this number, that is called total variation of alpha in the interval a to b, right? This number supremum of P in script, P sigma y going from 1 to n, mod alpha x i minus alpha x i minus 1 is called total variation of alpha on a b of alpha, on a b is this definition, clear? That is you to take all possible partition and for each partition write this number and take the supremum of that, if that supremum is finite then it is called a function of bonded variation, okay?

So, this is the term that we were defining, first of all do we already know any functions of bonded variation. In fact, we started with the idea that we wanted to extend a class of monotonic functions, right? So, our first observation should be that every monotonic function is of bonded variation, does that follow? What will happen? Suppose, this is monotonic function, suppose it is monotonically increasing, let us see if it is monotonically increasing. This mod alpha x i minus alpha x i minus 1, that would be thing, but let us take this, fine?

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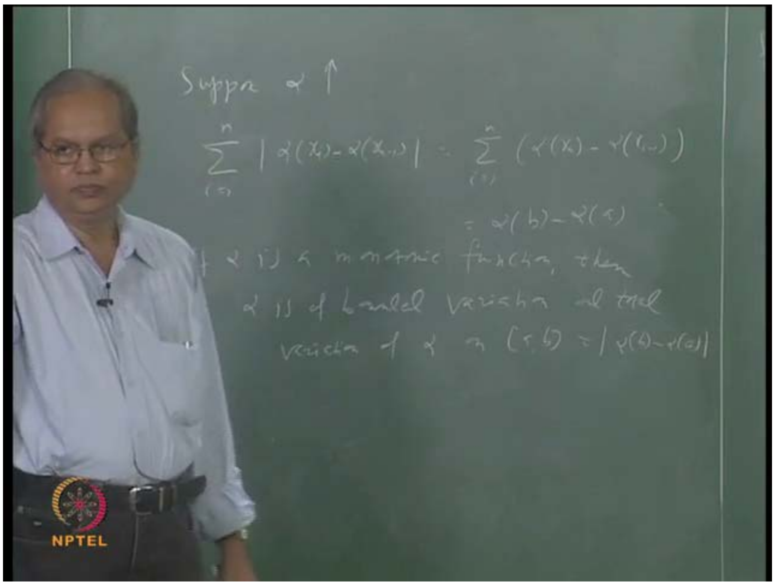

Suppose, alpha is monotonically increasing, let us use this. Suppose, alpha is monotonically increasing then what will happen with this number, sigma i going from point 1 to n mod xi minus alpha xi minus 1. Since, alpha is monotonically increasing, alpha xi is bigger than alpha xi minus 1. So, I can remove this absolute value sigh here or I should say this is same as sigma i going from 1 to n alpha xi minus alpha xi minus 1. And what is this?

It is nothing but alpha b minus alpha a, if it is monotonically decreasing because this will be a h, this will be alpha mod alpha xi, alpha xi minus 1. That will be nothing but alpha xi minus 1 minus alpha xi and so that will become alpha a minus alpha b and the alpha is bigger, okay? So, if alpha is monotonically monotonic function, where increasing or decreasing it is a function of bonded variation and its variation will be either alpha b minus alpha a or alpha a minus alpha b. So, we can say it will be nothing but mod of alpha b minus alpha a. So, we can say that if alpha is a monotonic function then alpha is of bonded variation, okay? And total variation of alpha on a b is same as mod of alpha b minus alpha a.

So, our first aim is satisfied that we have plus which is bigger than the plus of monotonically increasing function. Everybody monotonically increasing function in bonded variation, but not only that every monotonically decreasing function is also bonded variation. So, all monotonic functions are of bonded variations. So, this is at this stage itself it is a bigger plus, but in fact it is much bigger than that. Let us see how that follows. Let us give some protection for this total variation. Suppose, I call the total variation as V.

Let us say V alpha of b, okay? Let us call it, I will explain you why I have taken this V alpha of b because this is something we shall use subsequently, also V this is nothing but supremum of p belong to script p sigma i going from mod alpha xi minus alpha xi minus 1. This is total variation alpha on in, okay? Since, we are going to consider this functions of bonded variation for some time, instead of writing it again and again function of bonded, let us use some notation for this. I shall denote that said by B V of in A V B V for bonded variation, B V for bonded variation, B V of A V.

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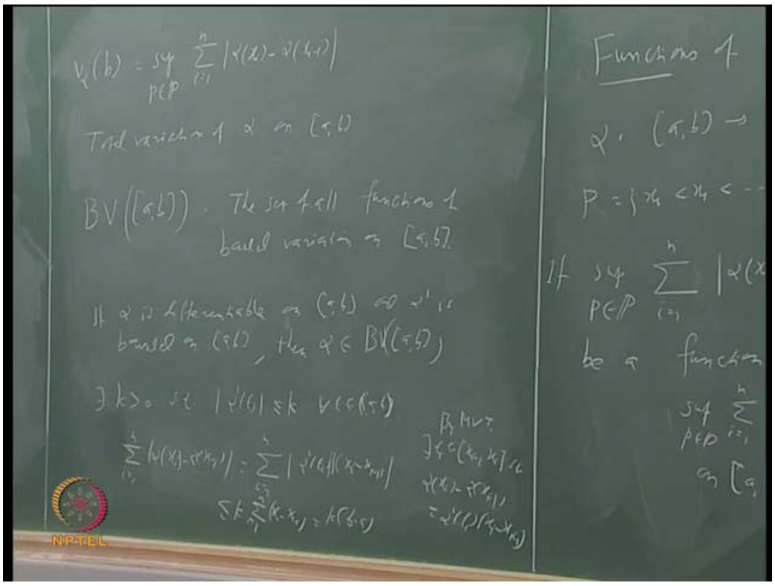

This is the set of all functions of bonded variation, all right? So, what you have seen so far is that, so all monotonic functions belong to this class, all monotonic functions belong to this class. There is one more class in which it is very easy to show that those also belong to this class. Suppose, alpha is differentiable, suppose alpha differentiable and it's derivative is bounded on a b, all right?

Suppose, alpha differentiable and it is derivative is bounded on a b then also alpha is of total variation, then also alpha is total bonded variation. Let us say here if alpha is differentiable on a b and with bonded derivative this and alpha prime is bounded on a b. See, this always happen if alpha prime is also continuous, all right? So, if alpha is continuously differentiable, this will be a functions satisfying property then alpha belongs to b and how does this follow? I will simply say in one word by mean value theorem. If alpha is differentiable then you apply mean value theorem to the interval x i minus 1 to xi and you can write this alpha xi minus alpha xi minus 1 as alpha prime at some point lying between the two, multiplied by xi minus xi 1, right? So, what we can say is that, okay?

Since, a alpha prime is bounded on a b, we can there exist some number case such that mod alpha prime is less not or equal to k for all. We can say that there exists k bigger than 0 such that mod alpha prime t is less not equal to k for all t a b or over interval a b. Then what we can do is consider this sigma i going from 1 to n alpha x i minus alpha x i minus 1 absolute value of this. This is same as sigma i going from 1 to n absolute value alpha prime t i into xi minus xi minus 1 for some ti lie between xi minus to xi.

That is by mean value theorem because we can say by mean this happens, because by mean value theorem there exists ti in xi minus 1 to xi such that alpha xi minus alpha xi minus 1. This is same as alpha prime ti into xi minus xi minus 1, but we know that mod alpha prime ti is less not equal to k, right? So, each of these I can write this as mod alpha prime ti into mod xi minus mod xi minus 1. Now, mod xi minus xi minus 1 is same as xi minus xi minus 1.

So, this whole thing is less nor equal to k times sigma i going from 1 to n into this xi minus xi minus 1 and we know that this sum is nothing but b minus a. This whole thing is less not equal to k into b minus a, where k is this term. So, if a function has a bounded derivative of a function is differentiable and derivative is bounded than that is of bounded variation, that is of bounded variation.

Of course, not every continuous function is of bounded variation, one can give examples of a function which is continuous, but not of bounded variation bonded variation and I will come that bit later on. Anyway that also since that is not very important point that we want to make, I shall tell you when I take this exercise take this value function which we have discussed several times.

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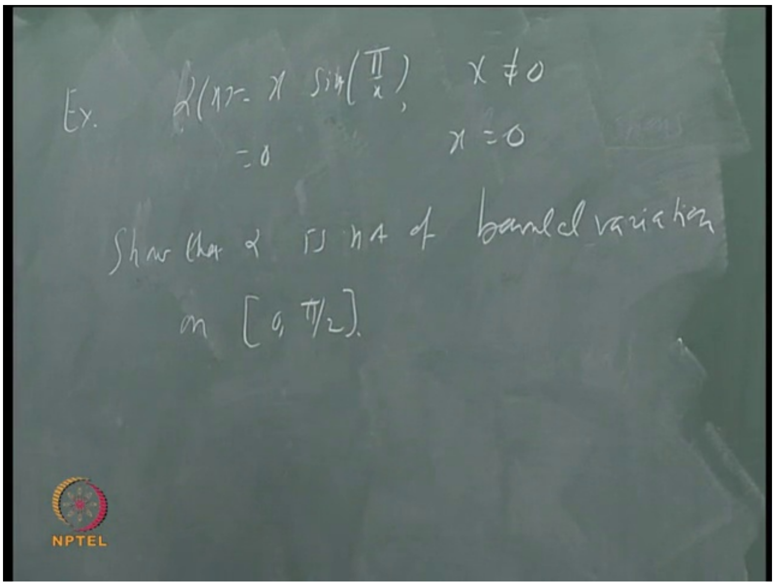

Let us say alpha x is equal to x sign 1 by x for x not equal to 0 and 0 for x is equal to 0. We have discussed this function several times, it is and we have shown that it is continuous at x is equal to 0 and x not equal to 0. There is no problem, so this is this a continuous function. So, that alpha is not of bonded variation on some intervals 0. Let us say on 0 to pi by 2 some calculations will be little easy, if instead you take 1 by x instead of pi by x to choose those points in the partition. Somehow those things become easy if you choose instead of taking 1. Of course, this is not different just to make the matter little simpler, okay?

Now, coming back to this class we have what we have seen that it contains all monotonic functions. It contains all those functions whose derivatives are bounded and there are several such functions that we know. For example, if the derivative is also continuous then it will be a function of bonded variation. So, all polynomials functions like sin x, cos x, e power x many of the well-known functions are functions of bonded variation, though many of them are not monotonic or monotonically increasing, okay?

So, this is a much bigger class, another property of this class is that one can show that if just is it has a property similar to the class of continuous functions, namely this is also a vector space. That means if you take two functions of bonded variations, then there sum is also a bonded variation. For example, suppose alpha and beta is of bonded variation then alpha plus beta is also a bonded variation and again that is very easy to see from the definition. Instead of alpha suppose you take alpha plus beta here, then this will be split into some alpha plus beta of xi, that is alpha and this also alpha and beta of xi minus 1.

So, this will be split into sum of alpha xi minus alpha xi minus 1 and plus beta xi minus beta xi. So, alpha plus beta will also be of bonded variation, not only that we can say that total variation of alpha plus beta will be less, not equal to total variation of alpha plus total variation of beta. All that is fairly easy to prove and similarly, for some k times alpha, if alpha is of bonded variation k times alpha is also a bonded variation etcetera. So, let me just state that property here.

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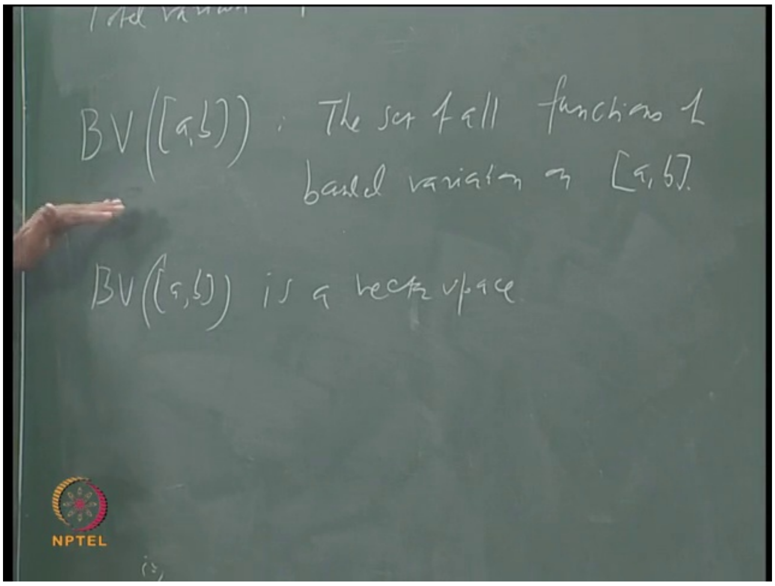

I will simply say this that B V of a b is a vector space. Now, once I say that this set functions upon variation is a vector space, it means if you take any two functions in that there sum difference whatever it is, is also again of bonded variation, okay? And as I said earlier this is not true of monotonic functions, if you take two monotonic functions their summary not be monotonic or difference need not be monotonic, but it will be of bonded variation, okay?

Suppose, it takes it two monotonic functions and if you take their difference, that may not be of increasing or decreasing anything, but it will be of, but it will be of bonded variation. But it will be of bonded variation, but what is more interesting and what is not, obvious is the following. We can further show that if you take any function of bonded variation, then it is a difference of two monotonically increasing functions, right? This is what we can show that is every suppose, you take any solution of bonded variation. Suppose, alpha then I can find two functions alpha 1 and alpha 2, such that alpha 1 and alpha 2 both are monotonically increasing and alpha is alpha 1 minus alpha 2, right? Now, again I will just what is involved in that and will not go into too many details proof, but before going into that idea let me again remark of small things. Here we have defined what is meant by, it is a function of bonded variation for real valued functions, okay?

Suppose, instead of that I take a vector valued function, instead of that I take a vector valued function. Instead of that I take the function R k, I take the function R k then instead of this absolute value, here it will be changed to not it will be changed to the not, right? And then so that is so, that is the only banner modification here and this is a concept which is very useful in practice, especially when you talk of functions taking values in R 2 or R 3, okay? If you know that any function which takes value, suppose alpha from a b to R k is let us say continuous function, then that is called a curved arc. For example, if you take a continuous function, let me instead of alpha use some other notation.

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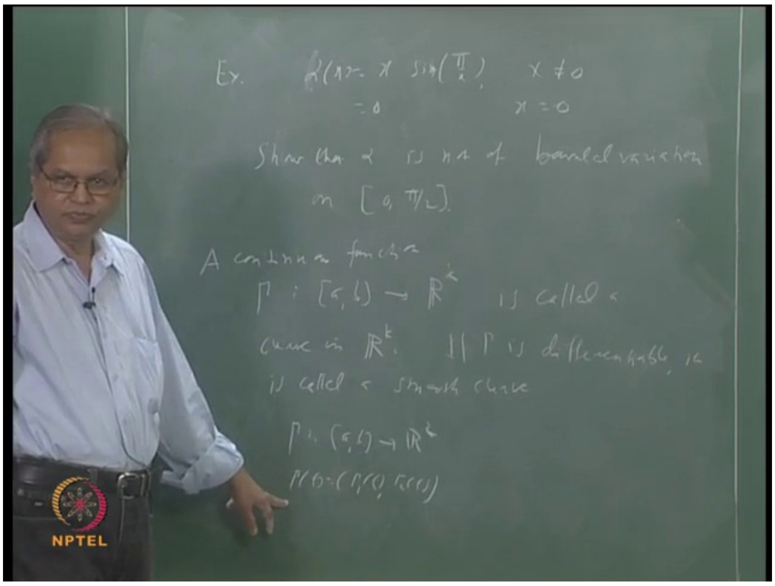

Suppose, I take a function gamma from a b to let us say R 2, in fact you can take any R k a h. So, a continuous function gamma from a b to R k a h, this is called a curve arc and if this gamma is differentiable it is called a smooth curve. If gamma is differentiable it is called a smooth curve and here we must emphasise that in this kind of analysis by a curve, we mean the function. Function means its domain co-domain, etcetera is specified in the usual language, geometric language. When you talk of a curve, you simply mean the set of points which make that curve or which is nothing but the range of gamma. In usual language by curve you mean the range of gamma, but that is not our idea here.

In particular for example, if you take gamma from a b to R 2 or R 3 then this gamma will lead to, if you take to coordinate wise. For example, if gamma is a function from a b to R

2 then this corresponding function gamma 1, gamma 2 or gamma 1, gamma 2, gamma 3 which represents the various coordinates. And those gamma 1, gamma 2 or gamma 3 those what are formally called paramedic equations of the curve.

For example, if you suppose let us let us take the case in let us say the gamma is from a b to R 2, then what it means is that you have this gamma of t is equal to gamma 1 t and gamma 2 t. When t varies in a b you will get various values in R 2, that will form occur that will form occur and if gamma is differentiable. Then this curve has a tangent and that is why it is called a smooth curve and in particular if gamma a is equal to gamma b, that will be called a close curve etcetera, all right? Now, where does this issue total variation comes or bonded variation comes into picture? It is that one can easily show that this number what we have called total variation, that the difference is nothing but the length of this curve, right? For example, if you take gamma from a v to R 2 let us say.

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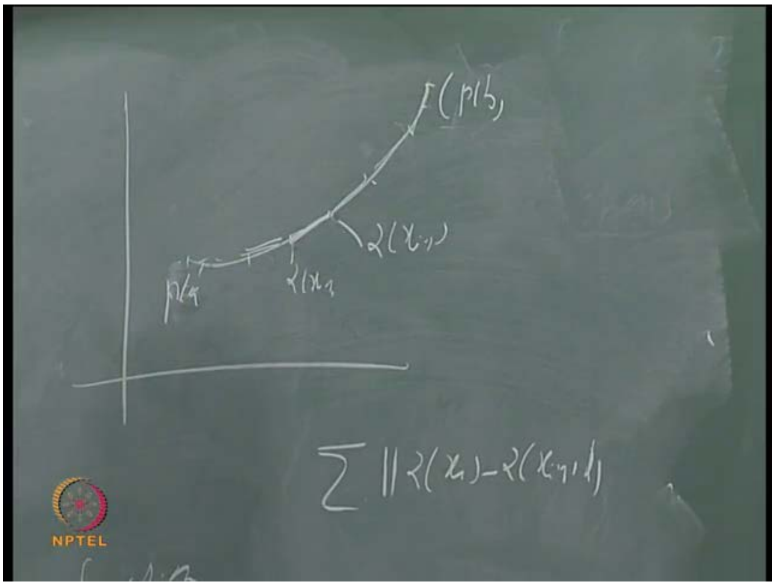

Suppose let us say that this is the range of R 2 and this represents to curve for this corresponds, let us say t equal to a. Let us say we will call this as gamma of a and this is of gamma of b, this interval a b is somewhere here because this represents the map from a b to R 2. We cannot represent them on the same plate, we will have to discuss for difference. So, for each t in a b you will have some point here, you will have some point here and so suppose you partition this interval x 1, x naught, etcetera. What will happen? You will get this corresponding chord here, okay?

What are these the worst? Now, norm of alpha of xi minus alpha of xi minus 1, suppose this corresponds to alpha of xi, xi minus 1 and this is alpha of xi minus 1. Then this number is nothing but the distance between these two points are nothing but the length of this chord length of this chord. So, suppose you take the sum overall such intervals, what you are going to get is the sum of the length of these chords, right? That is nothing but an approximation approximate value of the length of the curve, approximate value of the length of the curve.

To get the exact value of the length of the curve what you will do? You have to take the finer and finer partition so that of the partition goes to be 0 and take this supremum. To take the supremum or what you call total variation, so that total variation is nothing but the length of this curve. So, that is where the origin of this whole idea of functions of bonded variation and concepts like total variations etcetera, all right? I think that is about the total variation or of vector valued functions. Let us go back to real valued functions, it should be clear to you that if a function is of bonded variation on a b then it is also a bonded variation on all, sum all sub intervals all sub intervals. So, what I can do is that I can take let us say I take some function alpha which is a bonded variation.

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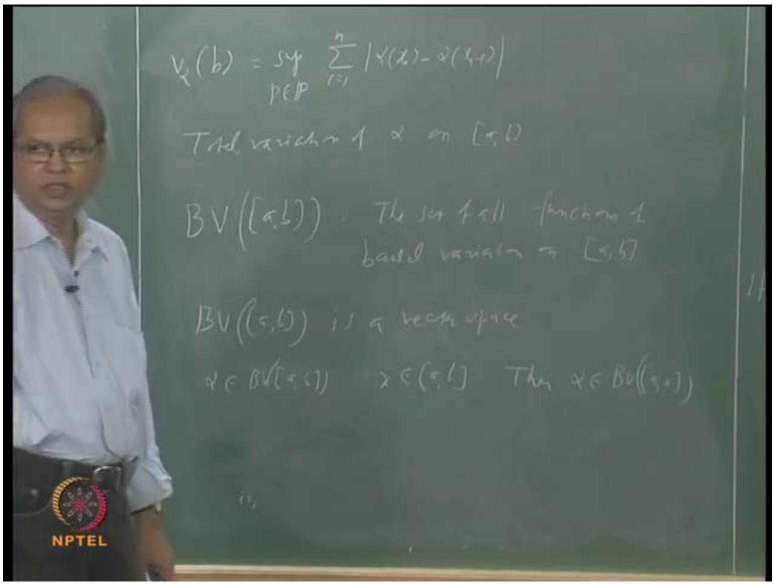

So, let us say alpha belongs to B V of a b and I will, this time I will take alpha as a real valued function, then we can talk off. Then we can say that suppose I take x in a b, x any number in a b. Then if I take a interval from a to x, if I take a interval from a to x, then it is of bonded variation in that interval also, right? Then alpha belongs, that is alpha is of bonded variation in the interval a to x also. So, what I can do now is that I can think of total variation of alpha. In this interval a to x instead of taking this whole interval a to b, I will take this interval a to x. Then for each x you have one such number, so that will give a new function from on a b. So, let us let us define that and that is a function which I will denote by V alpha.

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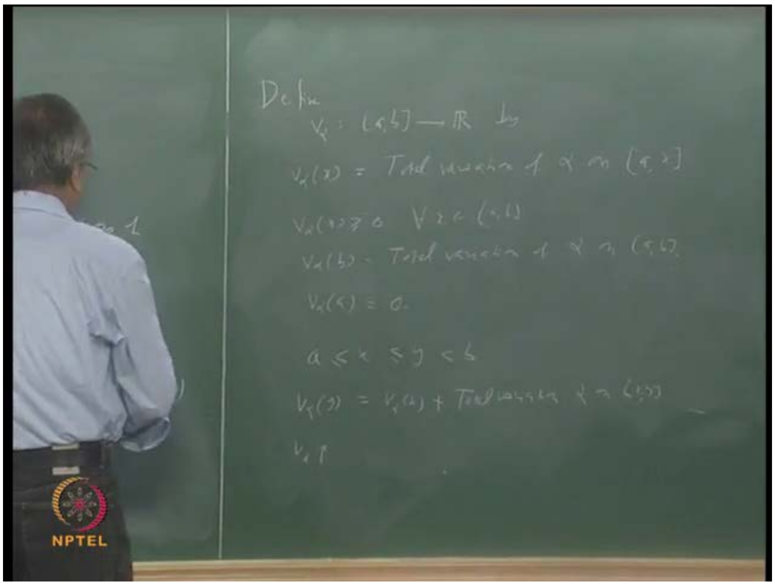

That is define V alpha from a b to R by V alpha of x, V alpha of x is equal to what is the total variation of alpha on the interval a to x. Total variation of alpha on interval a to x it means it is again the supremum of those sum alpha xi alpha xi minus 1, etcetera but the only thing is this time the partition is taken for the interval a to x and not over the whole interval a to b. It is obvious that this will be function of non-negative values because all numbers involved are absolute values.

So, this is in fact our this number is bigger and now you have understand why I have called this b alpha of V, right? b alpha of v is nothing but total variation on the whole interval a to b, right? So, this is a function which takes non-negative values. So, those are from a b to r, but you can actually a b to R plus a b to R plus. So, we in other word what I want to say that V alpha x is bigger to 0 for all x in a b. Of course, there are a and then V alpha of b is nothing but this total variation of alpha on the whole interval in. And similarly, what is the alpha of a is obvious that the alpha on the whole interval a b, okay?

Similarly, what is V alpha of a, is it obvious that V alpha a is 0, right? All right then let me state something which is not immediately obvious, but not very difficult also. That is suppose you take this a less than x less than y less than b. Of course, this can be less than or nor equal to also, all right? Then suppose I consider V alpha of x or let us say V alpha of y and V alpha of x, what I want to say is that V alpha of y will be V alpha of x plus some non-negative number. Always plus some non-negative number, always and what will be that nonnegative number? You can say that will be a total variation of alpha on the interval x to y because if you take any partition from a to x and something from x to y, that will be a partition from a to y.

So, taking supremum section you can it is not very difficult to, this V alpha of y is equal to V alpha of x. Let me just write this one total variation of alpha because you are not introducing any notation for this total variation of alpha on the interval x to y. So, it is obvious that this will be a nonnegative real numbers, again it is a total variation on some interval which are non-negative real number. So, does it follow from here that V alpha x is less not equal to V alpha of y, right? When does that happen? When x is less not equal to y, okay?

So, what we have proved that when x is less not equal to y, V alpha of x is not equal to v alpha of y. So, what does it mean? It mean it is monotonically increasing function, it is a monotonically function on the interval. So, we have what we have proved is this that is v alpha is a monotonically increasing function. Then what we can also prove is that I can write this, we have said that V alpha is monotonically increasing function. So, in particular it is of bonded variation, right? So, alpha plus V alpha, alpha minus V alpha those are also bonded variation, those are also bonded variation. So, what we will do next is the following.

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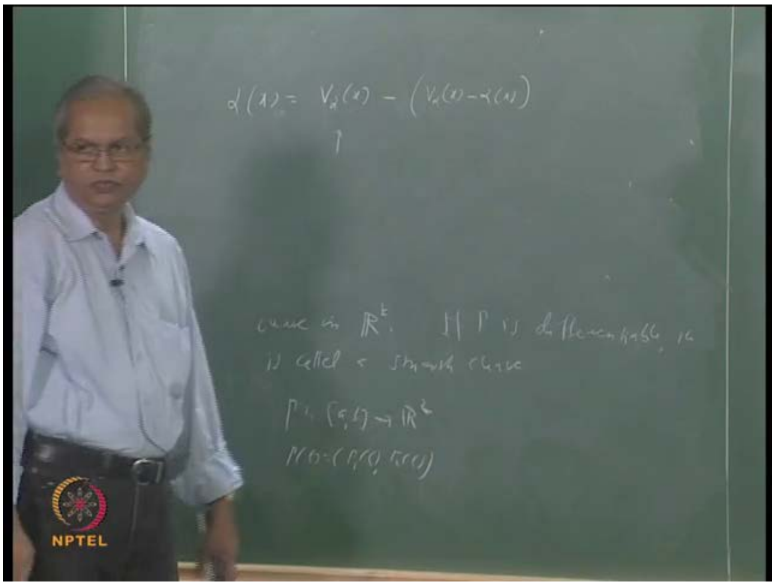

I shall write this alpha of let us say alpha of x as V alpha of x minus V alpha x minus alpha of x, all right? What we have shown is this is monotonically increasing, V alpha of x. What we have not shown here is that this is monotonically increasing, but that is true. It can be shown that V alpha x minus alpha x is also a monotonically increasing function. In fact it is also clear that this is also, this will also be bigger than 0, always alpha x is less nor equal to total variation of alpha from a to x, is this clear?

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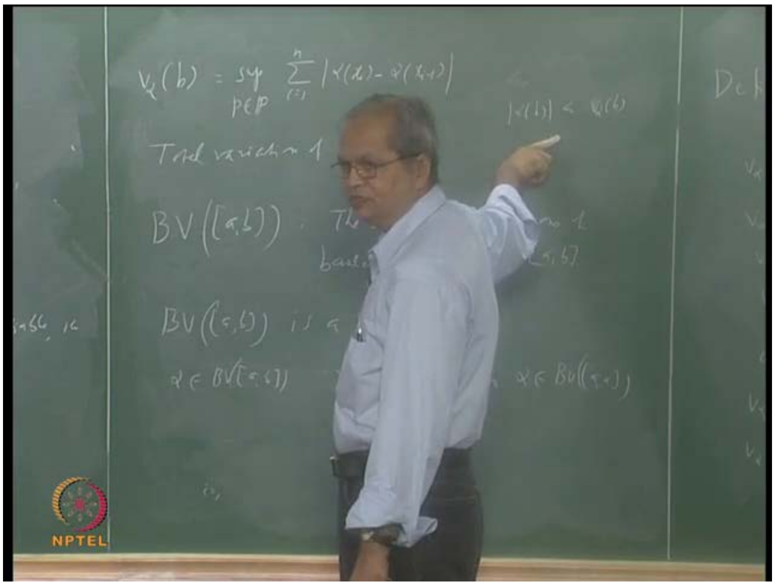

That V alpha b will always be less nor equal to at least mod alpha of V, okay? Not not this, other way mod of alpha b will be always less then nor equal to V alpha of b. That is obvious because this number alpha b will appear in all this alpha, xi minus alpha, xi minus 1, etcetera, okay? Similarly, one can show that this is also a similarly this alpha x will be less nor equal to V alpha x for x, all right? There is only one thing that is required, we have not said anything about the value of alpha at a. We have not written anything about the value and we know that the alpha at a is 0. So, if alpha at is different from 0, we will not be able to say this, okay?

So, let us assume this additional or you take this as alpha x minus alpha a, but let us just assume that alpha of a is equal to 0. Then this is also true and we can also show that this is also a monotonically increasing function. So, what does this mean that every function of bonded variation can be represented as a difference of two monotonically increasing function. That is an important result every function about variation can be represented as a difference of two monotonically increasing functions.

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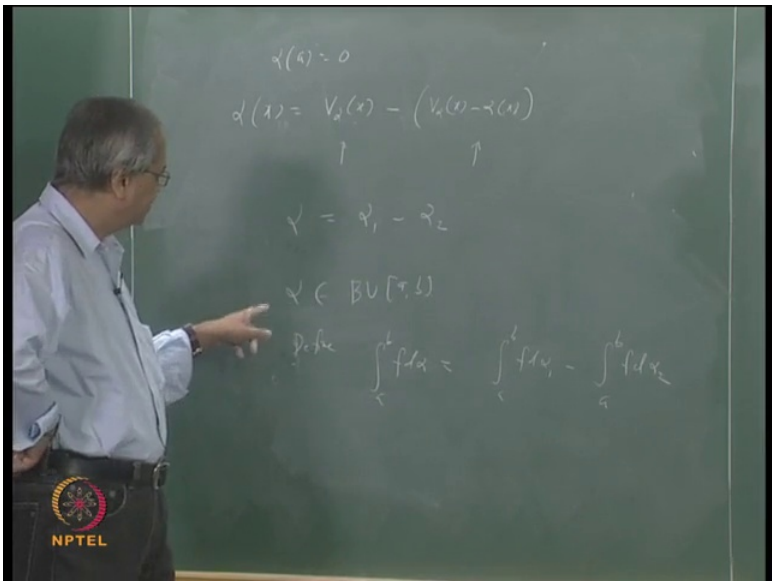

So, suppose that is I can always write alpha as alpha 1 minus alpha 2, where alpha 1 and alpha 2 both are increasing. Of course, this is of course, this decomposition is not unique. For example, I can also write this is something like alpha 1 plus 2 minus alpha 2 plus 2. Still this will be 2, but there will always exist at least two functions such that the given function is a difference of two monotonically increasing functions, okay?

Now, let us come back to our original question. What was the original question? That so far we have discussed stieltjes integrals, only with respect to monotonically increasing functions and can we extend that class and what is that class? That classes is this, that is you can integrate with respect to any function which is a bonded variation and what is and how do, how one does that the idea is.

Suppose, I take alpha from or suppose you take alpha is a bonded variation in and then you take and suppose you take alpha is of bonded function on a b and then you take. And suppose f is a bonded function will define integral, then we alpha can be written as alpha 1 minus alpha 2, where alpha 1, alpha 2 both are monotonically increasing and we can talk of integral f with respect to alpha 1 integral of f with respect to alpha 2, okay? So, you the define integral of f with respect to alpha f, integral f d alpha 1 minus integral f d alpha 2, okay? So, that is the idea, define integral a to b f d alpha is equal to integral a to be f d alpha 1 minus integral a to b f d alpha 2, okay?

There is a slight problem here, but once you understand a problem its solution is also clear. I said just now this function alpha 1 and alpha 2 are not unique, alpha alpha will also same as beta 1 minus beta 2 alpha may also be same as beta 1 minus beta 2. And then this will be become integral f d beta 1 minus integral f d beta 2 and integral f d alpha 1 may not be same as integral f d beta 1.

What we have observed here that even if that is the case, the difference between beta 1 and alpha 1, beta 2 minus alpha 2 will be a constant. Beta 1 can only be of the form 1 plus k and then beta 2 can only be of alpha 2 plus k and then like this difference that constant value will get cancelled, that constant and will get. So, what we cancelled? That constant value will get cancelled.

So, what we can, so to summarise the whole thing if you take a function of bonded variation, we can represent that as a difference of two monotonically increasing functions. And if the given function is integrable with respect to both of these functions, alpha 1 and alpha 2 then we can define integral with respect to alpha as the difference of those two integrals, okay?

Then we can carry on from here with various things about integrals of this type and again the many things which we have said about the earlier integrals we will follow every time. We have to just split up into these two integrals and apply to each of these. That is the idea which shall see what all things, we can do with this kind of things in the again as I said this is useful because especially for example something like integration indignation by part, etcetera. We want to also take integral alpha d f and there this is more useful, I think we will carry on the discussion in the next class.