Integration and Differentiation

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Lecture - 43
Integration and Differentiation

We were to discuss the relationship between the integration and differentiation today, for that we need to also revise one more thing while discussing the properties of the integrals. We had discussed this so-called domain division property namely.

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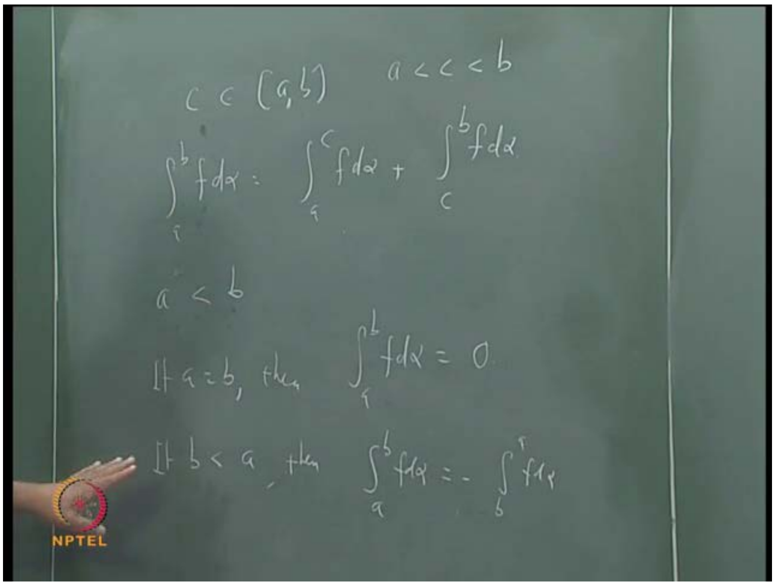

We had seen that if you take a b c in the integral a to b then integral a to b f d alpha is equal to integral a to c f d alpha plus integral c to b to f d alpha. What we want to do is that, we have we have proved this when c is in the interval a to b. What we want to is that something like this is true regardless of whatever is relative positions of a b and c and to do that first of all we have to make one more convention, okay? See, so far wherever we have discussed integral like this, we have always assumed that a is less than b, we have always assumed that a is less then b. Now, we want to remove that restriction because that is for some of the calculations that we will do subsequently, namely theta and b can be anywhere, that is b can be equal to a or b can be less than a also.

So, if a is equal to b we will take this interval to be 0, we will take this integral to be 0. So, that is the first thing, if a is less than b, in this case the integrate the integral is defined in the usual manner. If a is equal to b then integral a to b f d alpha, this we take as a 0 and the reason for this is obvious, because that interval has a length 0, all sub intervals also will have length 0. So, all the sums, upper sums, everything else will be 0.

So, it is quite make sense to take this value as 0. The question is what we do if b is less than a, if b is less than a what we can say is that what will do is the following we can always talk of integral from b to a. That is a well-defined thing in the previous case because b is less than a and then we will take that integral a to b. If it is negative of that by convention will take that, that is we will we will say that integral a in this case integral a to b f d alpha. We will take that as minus of integral from b to a f d alpha, this is a well-defined number because b is less than a, okay?

So, you take that integral and take the minus of that and take that as integral from a to b f d alpha, alright? Now, what is advantage of this convention? See, here what we can do is that for example, what is the meaning of say that c is in a b? It means a is less than c less then b, ok? In this particular we have proved this. Now, what I want to say that if we follow this convention, we can remove this restriction on a c and b and whatever be a b and c this equation will be always true. That is what we want it is a minor observation, but only thing is that for that the function should be integrable on some interval which contains all this three points, a c and b, okay? So, let us start with that observation.

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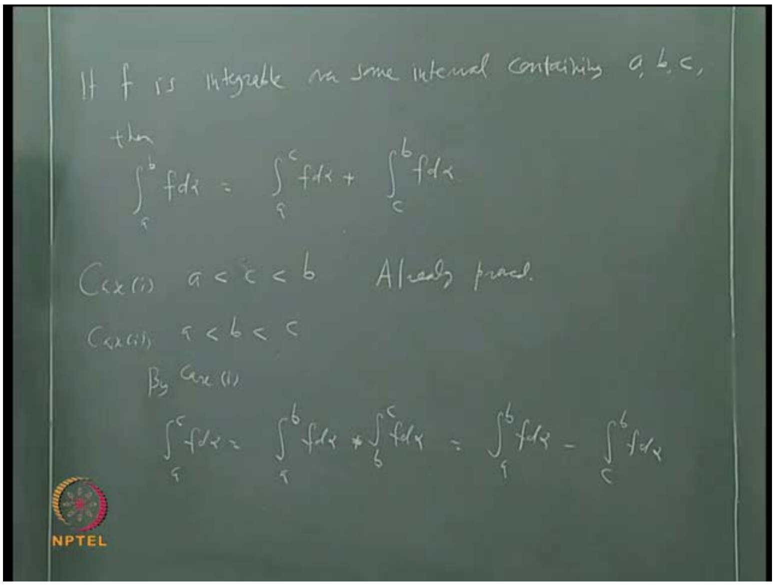

If it is integrable over some interval containing all these three points a b and c. Then if f is integrable over some interval, then it is integrable over all sub intervals that is something we have already seen. That is something we have seen well proving this so that I will not split once again, okay? So, that I want to select, so it will be integrable for all the intervals ab, bc, ac, ca and whatever it is, then integral a to b fd alpha is equal to integral say a to c fd alpha plus integral c to b fd alpha. Now, to prove that we will make various cases depending on relative positions of a b and c, let us let us take this case 1 as the normal case. Case 1 is a less then b, sorry a less then c less then b. This is the case we had already proved. So, that we did, so this already proved, okay?

What are the other possibilities? There are several other possibilities, we shall discuss one such possibility and a proof in all the other cases will be similar. So, for example this c may be outside. Suppose, this is the interval a b and let us say c maybe somewhere over here, that is let me call this as case 1, a less then b less then c. Then in this case also we want to say that this force, but you can see that what we can do is that we can use case 1, okay?

So, I can say that integral over a to c must be same as integral over a to b plus integral b to c, that is the something we can always say. So, by case 1 integral over a to c it is f d alpha, it is same as integral over a to b f d alpha plus integral over b to c f d, okay? But we want is in this integral from c to b, integral from c to b and here what we are getting is b to c, but c is bigger than b. So, we use this convention, here c whenever we take b less than a integral a to b f d alpha is minus integral b to a, so this integral b to c f d alpha, but minus integral c to b f d alpha.

So, using this will get integral a to b f d alpha minus integral c to b f d and that is of you bring that to left side. Bring that to left side, you will get integral a to b f d alpha is on the right hand side, and you will get integral a to b f d alpha is equal to integral a to c f d alpha plus integral c to b f d alpha. So, that is this even when c was here, even when c was here still this equation is true, okay? Once you understand this idea, you can then deal with all other cases. Suppose, c is less than a less than b, again you can similarly follow and all the other relative positions of a b and c you can take care of. Basically, you use this case 1 and this property that is why this convention is useful.

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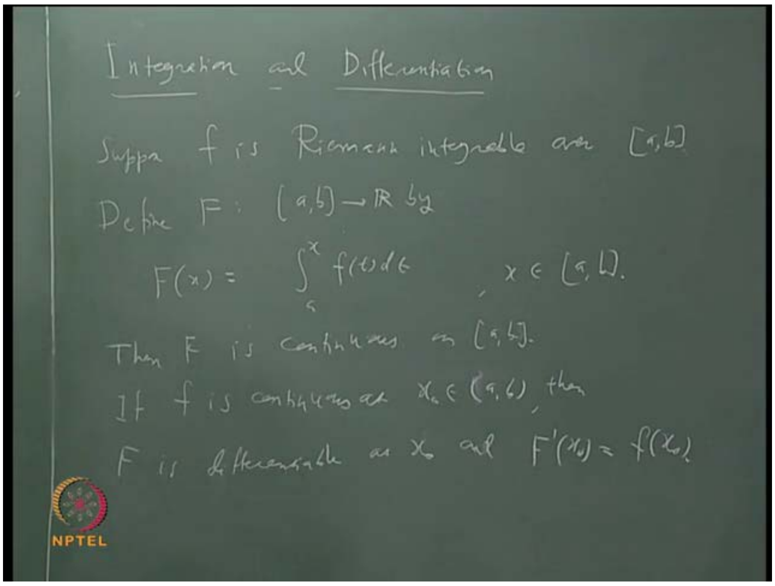

Now, let us go back to our discussion on the relationship between integration and differentiation and let me state what I have stated the last class once again. Suppose, f is Riemann integrable over a over a b and we define. So, again we use the fact that whenever f is Riemann integrable, it is integrable over all the sub integrals, okay? So, if you take a interval, if you take any x in a b it is also integrable over the interval a to x, okay? So, using that we defined big F from a b to R, by this big F at x is equal to integral a to x, integral a to x for small f, for long I should use some other variable small f at t dt, this is for x and a b, okay?

Then what we want to say is the following that this function big F continues. This function big F continues and if small f is continuous at some point, then at that point this big F is differentiable and is derivative as same as the value of the small f and that point. So, this is the first, then the f is continuous form a b. This is the first thing, in fact we can show that it is uniformly continuous, but you can notice that is not a big thing because any continuous function now closed at boundary interval is uniformly continuous. So, that is something you already know, alright? So, I could have stated that big F is uniformly continuous, but there is no point because that is we are not saying anything new by saying that, what more.

Next, what I want to say that at some point if say, if let us say, if f is continuous at some at some point say x not in. Let us take an open interval, then big F is differentiable at x not and derivative there is big F x naught is equal to small f at x naught. And in proving this we shall use all this whatever we have, because here we want to find the values of the integrals with respect to different positions of a b and c, alright? Have you heard of what is meant by Lipchitz condition and a function satisfying Lipchitz condition, and whenever function satisfies Lipchitz condition it is uniformly continuous, right? You know all these things, okay?

In fact, we can prove that this function big F satisfies Lipchitz condition and that will automatically imply that it is uniformly continuous, alright? So, since this small f is integrable we know that it must be possibly bounded in particular and so I can say that if f is bounded mod f is also bounded. And a kind of argument that we have used very often, we shall say that there exists some k valid.

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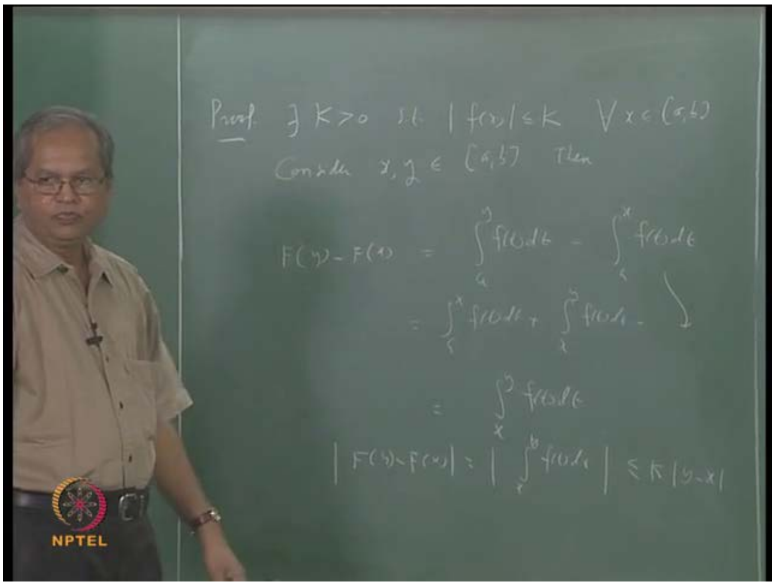

There exist some k bigger than 0, such that mod fx less, not equal to k for all x into a, okay? So, this something we shall use, okay? In fact this will be our constant. Now, is there any two points x y and a b? Then let us say first let us let us cut F y minus F x, Fy minus F x this is the thing, but according to our integral a to y ft dt minus integral a to x ft dt. Now, here is this is just we have used, because big F at any point axis integral a to x ft dt. Now, here we use this property, I won't write this as integral a to x plus integral x to y, right? You can see that it does not matter whether x is less than y or y is less then x, in all such cases we can write that is why we discussed those things earlier, okay?

So, I can say that integral a to y ft dt, this is same as integral a to x ft dt plus integral x to y ft dt and minus this integral a to x ft dt and what will be, this cancels with this. So, what we will get is integral x to y ft dt x to y ft dt, right? Now, you look at this mod fy minus fx, this is same as mod of x plus integral x to y ft dt, okay? And remember we have proved earlier that if mod fx is less, not equal to k then mod of the integral less, not equal to k times b minus a, a times the length of the interval, okay?

So, this must be less, not equal to k times. Whatever is the length of the interval, whether y minus x or x minus y both effects is less than y. It will be equal to k times x, if y is less then x it will be x naught equal to k times. So, in both cases I can say that this is less than or equal to small k times mod y minus x and this is true for every x y in a b. So, this shows that this big F satisfies this Lipchitz condition, with this Lipchitz constant k, right? So, this big F must be uniformly continuous or even otherwise you can use the epsilon delta definition. If you are given epsilon, you can choose delta as epsilon by k, okay?

So, when this mod fy minus x delta mod fy minus fx will be less than. So, it is not absolutely essential to recall the properties of Lipchitz continuous function, but that is what we are using here, is that clear? So, this part is easy that is showing that big F is continuous. Now, we shall go to the prove that big F is differentiable at those point where a small f is continuous. Now, this proof will become very easy if we assume that this small f is continuous everywhere in the interval if we are assume that small f is continuous everywhere in the interval, then the proof will become very easy.

Let us, just let me just indicate what will happen if small f is continuous. We also proved what is called mean value theorem. We also proved what is mean value theorem for the, then what does the mean value theorem say that, if you take integral a to b fx dx it will be, there will be some number. See in between such that it is same as f c into b minus a and that you can use for any interval. So, in particular one can use it for the interval x to y, here fy minus integral x to y ft dt. So, suppose the smallest were continuous I could have said that this is equal to this, equal to f at some point lying between x and y multiplied by y minus x.

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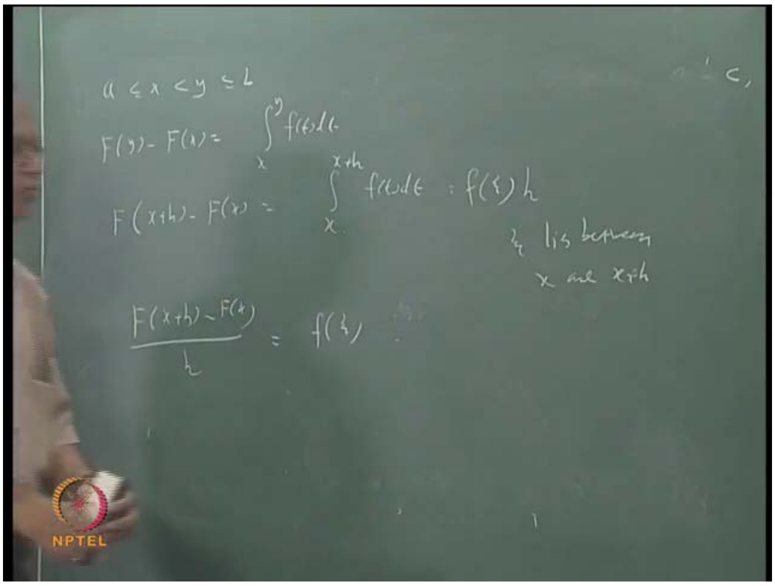

Suppose, it is supposed let us take the case when x is less then y. Suppose, a less not equal to x less that y less not equal to x. In this case I could have said that fy minus fx is integral x to y a h ft dt, think and by using mean value theorem. By using mean value theorem we could have instead of that, let us say ok this one is true. See, in order to look at the derivative, we look at f effect x plus h fh minus f x, right?

So, suppose we do that here, suppose I take y as x plus h, then f at x plus h minus f x that will be same as integral x to x plus h ft dt. And if this small f were continuous, if this small f is continuous function then you can write that this is same as f. At some point suppose, I call that point call that point such that this is same f at site, but if plan by this y minus x in this case f plus x minus h. So, site is into h well what is between x and x plus h between x and x plus h, okay? So, we would have got f of x plus h minus big F x divided by h, that is equal to f at where lies between x and x plus h, okay?

Now, if small f is continuous we can say that as h goes to 0 will go to x. So, the limit on the right hand side as h goes to 0 is fx. So, the limit of the left-hand side is also and that limit is same as small f at x. So, that would have proved the big F is differentiable and its derivative is same as small f before the interval and that is how the proof given in many of this undergraduate takes in calculus. That if you assume the smallest is continuous it is fairly easy to prove, but we are not assuming it here, we are assuming that f is just Riemann integrable. So, we need some extra work here and again this property we are proving just point wise that if f is continuous at just 1.6 not at that point big f is differentiable and derivative equals to the value of the small f at not, okay?

Now, let us see how that can be proved again. Let us take x naught and x such that both x naught and x naught plus h both should be in a b. So, consider let us comeback to this. Later we want to prove that this big F is differentiable and big F x prime is not equal to small f at x not, and we are given that small f is continuous at x naught. So, this information we have to use this information, now what is the meaning of say small f is continuous at x naught, let us use this epsilon delta definition of continuity. First of all it means that small f is continuous at naught.

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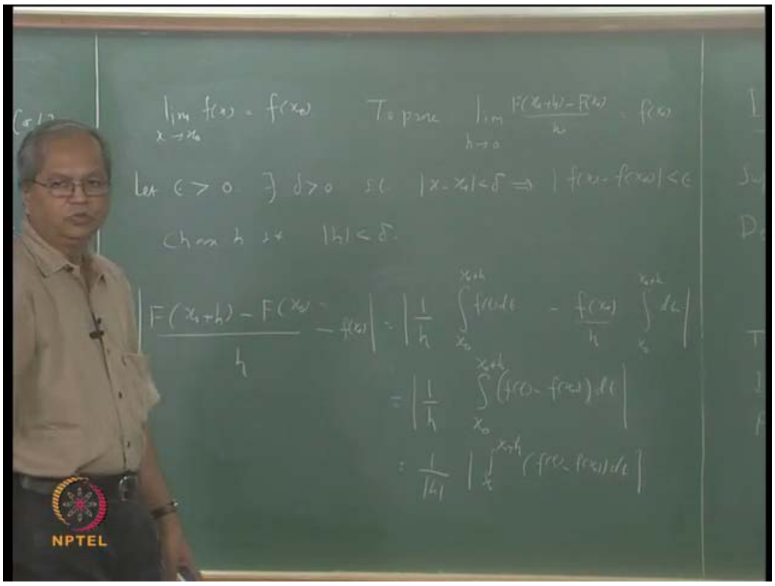

It means this limit of fx as x tends to x naught. This is equal to f at x naught and what we want to prove is this, what we want to is this that limit of big F at x not plus h minus big F x not divided by small h, limit of this as x tends to 0. This limit exists and that is equal to small f at x naught, this is what we want to prove, right? We are given that some limit exists and its value is something and we want to let some other limits exists and its valuation. So, will use this, so it is conveniently use this epsilon delta definition here. So, let us say that epsilon bigger than 0 be given epsilon bigger than 0 be 1. What is our m? We want to prove this, right? So far this epsilon we want to prove, we want to find a delta. We want to find a delta such that whenever this mod h is less than delta mod h is less than delta, the f of difference between this whole expression minus f of x naught that should be less than epsilon, that is our aim, right? But what do we know?

We know that there exists some delta such that whenever difference between x and x naught is less than delta. The difference between f x and f x naught is less than. Let us record that first that there exists delta bigger than 0 such that mod x minus x naught less than delta. Implies that is whenever mod x minus naught less than delta mod f x minus f x mod is less then epsilon or which again mean that whenever x belongs to the interval x mod minus delta to x mod plus delta this happens, okay?

Now, we will choose, see suppose we choose that choose h such that mod h is less then delta, choose h such that mod h is less then delta. So, we want to show that for this h the difference between this and this must be less than its epsilon, right? Let us write the value f of x naught plus h big f of x naught plus h minus big at f x naught, we can use this calculation here. Here, you this x naught plus h this as x naught, okay?

So, we have shown that there is nothing, but integral from x to y ft dt. So, this is nothing, but the integral from x naught to x naught plus h ft dt, okay? What we want is this divided by h, okay? So, this divided by h it will 1 by h into and what we want to show is that the difference between this and fx naught that is less than epsilon, all right? So, I will write that also here, I will just what the what is minus f at x naught. I will rewrite this minus f at x naught, what was this? It was 1 by h integral from x naught to x naught plus h ft dt and minus fx naught, right?

Now, there is a small observation here. See, suppose I just take this integral, integral from x naught to x naught plus h dt, what will be the value of that, that will be h. So, I can say this is same as f x naught by h into integral from x naught to x naught dt, if x naught is a constant, right? If f x naught is a constant, so if I want I can take integral sign. Also you can take integral sign inside, also we shall do, so doing that and simplify what will get is this whole thing will be 1 by h into integral x naught to x naught plus h ft dt ft minus fx not ft minus fx not minus dt, do you agree with this? Then what we want to show is that the absolute value of this whole thing is less than epsilon, okay?

So, will take the absolute value will take, so will take the absolute value. Absolute value of this is same as the absolute value of this and that is again same as the absolute value of this. And one can say further that this is equal to 1 by mod h into absolute value of this integral, integral x naught to x naught plus h ft minus fx naught dt, but we have proved that absolute value of the integral is less not or equal to integral of the absolute value, all right? That is something that we have already, so we can we can use that also.

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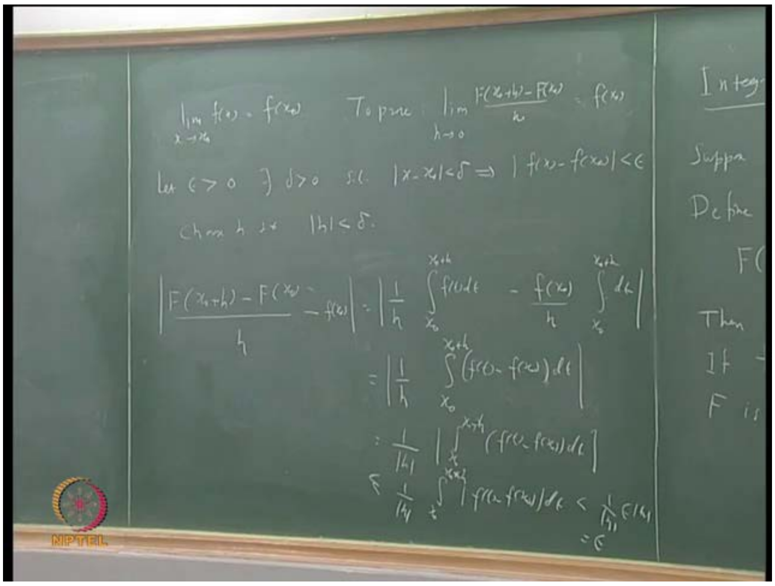

So, this is less or equal to integral 1 by mod h integral x naught to x naught plus h mod ft minus f x mod dt, okay? Now, the main question what can we say about this mod ft minus fx nod, see remember t lies between x naught and x naught plus h and mod h is less than delta. See, t is in the interval x naught and x naught plus h and mod h is less.

So, it is clear that t must lie between not minus delta to if it in fact t must lie between x naught minus delta to x naught plus delta. We have seen here that if in that case that means it is clear that mod t minus nod, in fact nod t minus x nod is less than mod h which is less than delta. So, mod ft minus of x dot must be less than epsilon. So, this must be less than epsilon and so the value of the whole integral must be epsilon into the length of the interval which is h h or mod h, depending on whatever.

So, this whole thing is less than epsilon times mod h divided by, this is less than 1 by mod g into epsilon times mod h, right? Or which is same as epsilon, right? This is what we wanted to show, right? There is given epsilon, just delta bigger than 0 such that whenever mod h is less than delta, whenever mod h is less than delta the difference between this whole expressions minus f x naught is less than epsilon, which is same as this saleslady, right? Okay?

Since, we did not assume that small f is continuous we have to do some extra work and similar thing you will also observe in something that we will produce next. Which is also called fundamental theorem of integral calculus because that is the one which relates, that is the theorem which shows that the process of integration and differentiation are in some sense in verses ((Refer Time: 31:47)).

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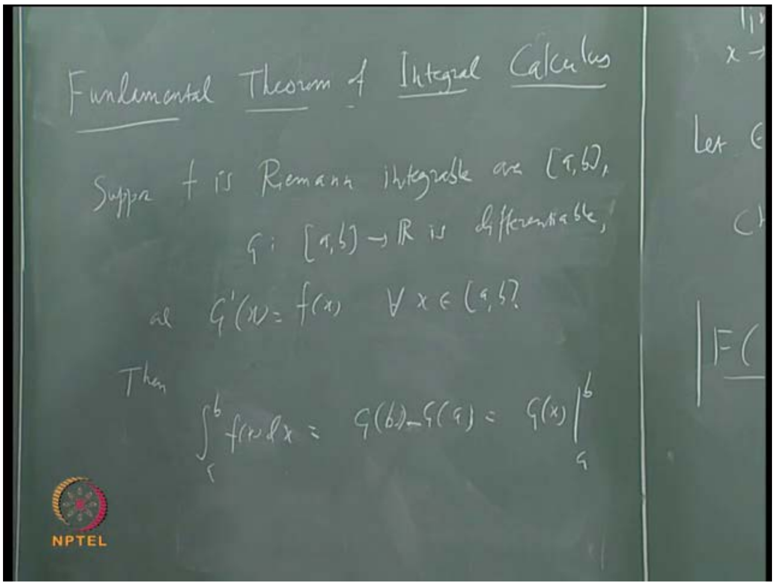

So, let us just state fundamental theorem of integral calculus, right? Again we start from a integrable function. Suppose, f is Riemann integrable over a b and we want to consider a function whose derivative, whose derivative is this given function, small f whose derivative is a given function small f. Normally I would have used the big F for that, but we have used that already for something else and we want to use that also in our proof more precise, and use some other. Suppose f is Riemann integrable over a b and let us say and g from a b to r is differentiable, g from a b to r is differentiable and g prime is equal to f. That is the derivative of g it is same as small f or if you want to write in full form, that is same as g prime is equal to f x for every x in a b, then the conclusion is this.

Then integral a to b f x d x is the thing, but g at b minus g at e and why it is called fundamental theorem. There are so many reasons, one reason has been saying right from the beginning that saying integrals will never use derivatives, but we observed that using the definition of the integrals very few can be integrated. So, to find the value of Riemann integrals, this is the most useful theorem and this what does it say? You find some function big G whose derivative is this function, that you started this such that function is you might call anti derivative or sometimes it may integral or anything like that. It is some function whose derivative is the function that you started with. And once you find the function the value of the integral is nothing but value of that function at b minus value of that function at a. So, we you can allow, you can use this notation which you normally use in calculus, this is the hyper G x between a and b G x between a and b, okay?

So, that use a method of finding the values of Riemann integrals, that is why fundamental theorem. Again here also if we assume that this function small f is continuous, then the proof will be very easy and that is how it is given in most of the books dealing with the undergraduate calculus. How the proof will be easy? Suppose, we assume that small f is continuous everywhere. If small f is continuous everywhere then we have proved, using this we can show that. That means this big F is also differentiable everywhere, big F is differentiable everywhere and then and its derivative is nothing but small f, but what does it mean?

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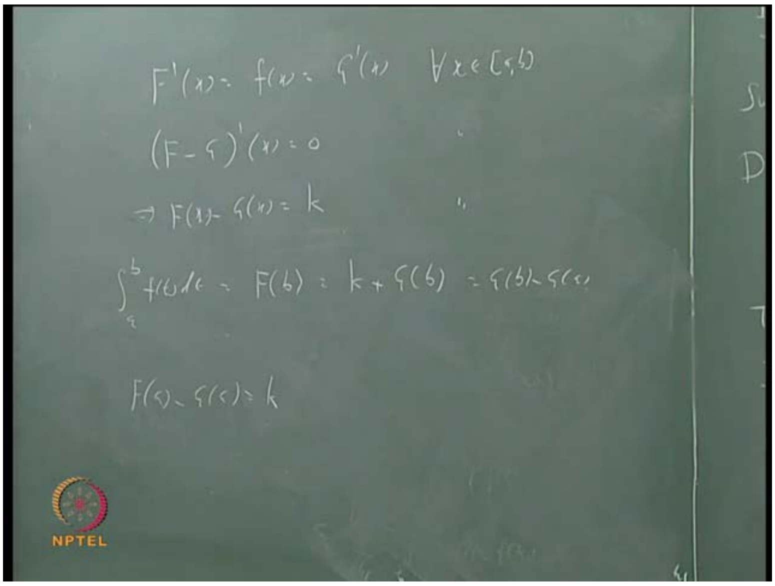

That is it will be in this big prime f x and that is same as big G prime at x for every x, okay? Another words it will mean that if you look at the difference big F minus G, it is derivative is 0 for every x in a b, right? We have already seen that the derivative of function in 0, the function must be constant again using mean value theorem. That means this will imply that big F prime G is equal to some constant. Suppose, I call that constant k, suppose I call that constant k or which is, which means that is this is same as saying that f x minus big G x is equal to k, for all x. So, in particular one can say that if you look at integral a to b f x ft dt, whether you call fx dx or ft dt that is same, ft dt this is same as b big F at b, right? Big f at b right, but at any point the difference between big F and big G is same as k, okay?

So, I can say that big F at b, suppose I apply for this for x equal to b you will get this as k plus G at b, right? Now, the only question is what is k, only question is what is k? To do that you apply, you take x is equal to a, okay? You take x equal to a, so it will also give that big F at a minus big G at a is equal to k, but what we know is that big F at a is 0. Look at the difference of big F at a, it is integral if x is equal to a, it is integral from a to a, that is how we started today's class. Big F at big F at a is 0 which is same as saying that k is equal to minus G a, that is same as I say that k is equal to minus G a. so, you put it back here, so you get this as G b minus G, okay?

So, again let me caution you that this is not the proof of this theorem, but if you assume the extra assumption, that small f is a continuous function, then the proof becomes very easy. And then that is how the proof is given in many undergraduate level books because they are you assume, always assume that small f continuous. But since we are not assuming that we will have to do some extra work and that extra will involve the several things that we saw about the Riemann sums. The relationship between integral as a limit of sum integral as a limit of sum that is what we have to use, okay? So, to do that let us consider a partition, let us consider some partition P.

Suppose, P is the partition with this property a equal to x naught less than x 1 less than 1 x n which is equal to b, then you look at this G b minus G a, G b minus G a in this notation it is same as G x n minus G x naught, okay? We can add and subtract in between terms, I can write this as G x n minus G x n minus 1, then plus G x n minus 1 minus G x n 2, etcetera, etcetera and coming to that I will take this last term as G x 1 minus G x naught. In other words I can write this as sigma i going from 1 to n, G x i minus G x i minus 1, all right?

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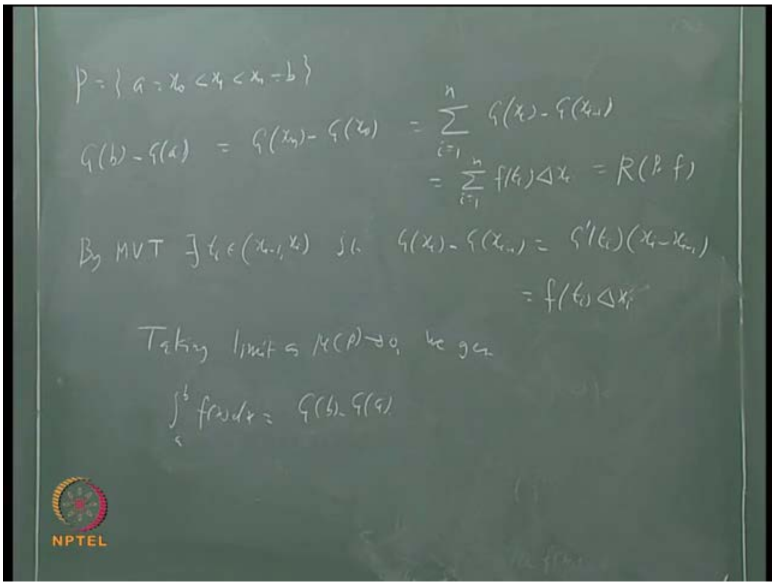

Now, we have assumed that this big G is differentiable on the whole interval. So, we can use we can apply mean value theorem to this big G. By mean value theorem what will happen, this you can say this G x i minus G x i minus 1, that will be same as G prime at some point in between a to b multiplied by x i minus x i minus 1, that point I will call t I, that point I will call t I, okay?

So, we can say that by mean value theorem, mean value theorem there exist t i in the interval xi minus 1 to xi, such that G at xi minus, G at xi minus 1 is same as G prime at ti multiplied by xi minus xi minus 1. But what is G prime at ti, look at here for each x g prime at x is fx g prime at ti is nothing but f at ti g prime at tis is noting but f at ti. So, this is nothing but small f at ti and this is what we have been calling delta xi, xi minus xi minus 1, that is nothing but delta xi, right? Okay?

So, what happens to this becomes sigma i going from 1 to n f at ti multiplied by delta xi f at ti multiplied by delta xi. And now what is this? It is nothing but Riemann sum for the function f. So, this is nothing but R P f, right? That is what we are doing R P f for this choice of the points ti, this choice of the points ti, but what we know that this function is Riemann integrable. And if the function f is Riemann integrable we know we have already shown that the limit of this Riemann sums as that of the partition goes to 0 exists and its value is same as integral a to b fx dx.

So, what we can say is that this tends to integral a to b fx dx as of p tend to 0. Whereas, left hand side is independent whatever the partition is, whatever is a partition this will remain the same. So, which means if the limit of this two must be equal, so we can say that taking limit as if, is that clear? Whatever I have done til now is true for every partition, whatever be the partition it does not depend. It does not depend itself this G b minus G a turns out to be a particular Riemann sum for every partition, okay?

Let us now take the limit as of the partition goes to be 0. So, that is same as integral a to b f x dx and hence it must be same as G b. So, taking limit and 0 we get integral a to b fx dx minus G b minus G a. So, that establish the fundamental theorem of integral calculus, we need to do one more thing now. See, till now we have been talking about the Riemann integrals as well as Riemann stieltjes integrals, whenever we take a function alpha we also want to establish whether those two are.

Of course, one thing we know that Riemann integral is a special case of Riemann stieltjes integral. But in there are some other cases also in which you can establish the connection between the Riemann integrals and or you can reduce the Riemann stieltjes integrals to the case of Riemann integrals. Remember, this why we want to do that because this fundamental theorem of integral calculus. We have now only proved for Riemann integrals, we have not anything similar for Riemann stieltjes integrals, okay? So, suppose we can reduce a Riemann stieltjes integral to Riemann integral, then one can use this fundamental theorem and find value of that, okay? So, that is the idea and the question is when we can do it?

Let me just give the conditions of all this theorem. Suppose, f is Riemann stieltjes integrable R a b alpha with respect to this function alpha which we always assume to be monotonically increasing all along. We assume that this function alpha is monotonically increasing, this means Riemann stieltjes integrable with respect the function alpha. So, suppose this function alpha has some additional properties then we can say that the two integral are same, okay?

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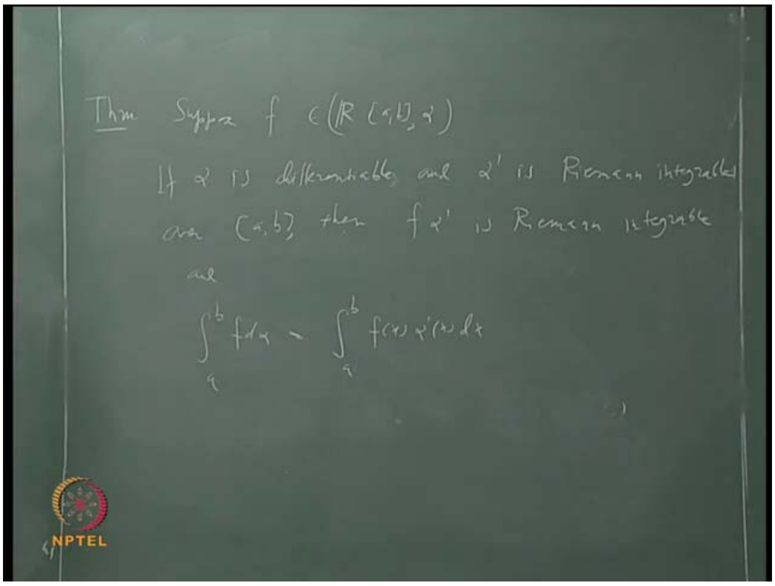

If alpha is differentiable and not just differentiable, this alpha prime is Riemann integrable. If alpha differentiable and if alpha prime is Riemann integrable, integrable over a b then what we can say is this integral a to b. Of course, I can also say that, then f into alpha prime is Riemann integrable, this is something I need not say separately. But it follows integrable and Riemann stieltjes integral of this function f coincides with the Riemann integrable of this function f into alpha prime or if you want to write in the symbols, what it means is this and integral a to b f d alpha, it is same as integral a to b fx alpha prime x dx, okay?

Now, so this says remember this what you have on your left hand side is the Riemann stieltjes integrable integral a to b fd alpha, and what you have on the right hand side is Riemann integral, but Riemann integral of different function, this function f alpha prime. So, what does the theorem say that if the alpha is differentiable and if the alpha and that derivative is Riemann integrable, then one can reduce Riemann stieltjes integral to a Riemann integral. And once we proved that we can if this function satisfies this condition, also if you can find some function G goes derivative is this. Then we can use fundamental theorem of calculus to evaluate this Riemann stieltjes integral also, that is the idea.

Now, I shall not give you a very detailed proof of this, while I will just give you the main idea because considerations involved are somewhat similar here. Let us again look at the partitions and see what is the corresponding Riemann stieltjes sum is, but before that is that first of all this part is clear, f alpha prime is Riemann integrable, f is integrable alpha prime is integrable. So, we have already shown that if the two functions are integrable the product is also integrable. So, f alpha prime is integrable that is clear and that is something that we have already shown, so only remains is why these two values are same. So, just look at one particular Riemann stieltjes sum.

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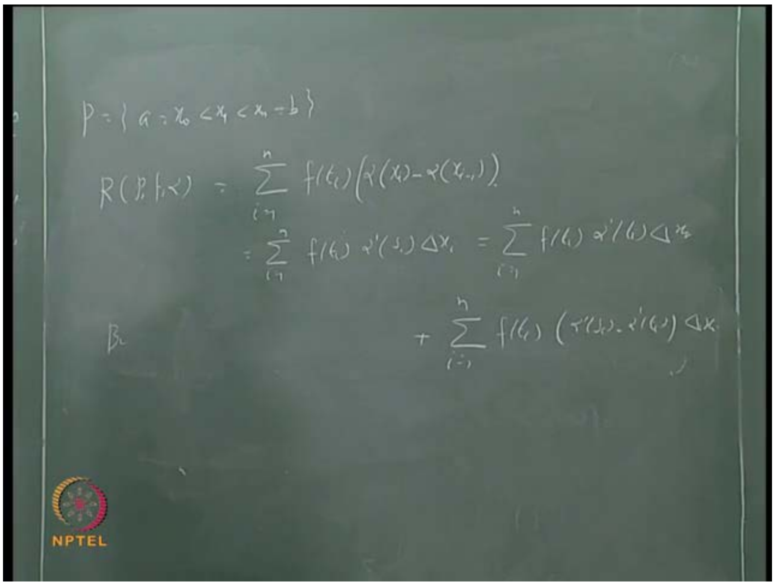

Suppose, I look at say R P f alpha, so what will be that? This will be sigma I going from 1 to n f at ti multiplied by delta alpha. I multiplied by delta alpha i. Now, what is delta alpha i? Delta alpha i, let me write the full form, it will be alpha at xi minus alpha at xi minus 1, all right? Now, we know that alpha is differentiable. So, we again use mean value theorem alpha is differentiable, so we again use mean value theorem and write this alpha xi minus alpha at xi minus 1 as alpha prime at some point in between those two and multiplied by xi minus xi minus 1. So, this will be only thing that we cannot see, that point is same as ti. It can be something different, it can be something different. So, suppose I call that point as si, so we can say that this is the argument by mean value theorem. There exists xi, sorry there exists si in xi minus 1 to xi, such that alpha xi minus alpha xi minus 1 is equal to alpha prime at si multiplied by xi minus xi minus 1, right? Okay?

So, we shall use that fact here, sigma i going from 1 to n, this is si multiplied by alpha prime at si multiplied by this xi minus xi minus 1. I can write that as delta xi, all right? Now, if this si were same as ti, I could have said that this is the thing, but the Riemann sum of corresponding to f alpha prime f alpha prime, but that is not the case and that is why we need some additional work, but what we will do? We will do the individual things in such a case will just add and subtract alpha prime ti.

So, we will write this as I will write this as sigma i going from 1 to n f ti multiplied by alpha prime ti multiplied by delta xi. Then plus sigma i going from 1 to n, what we do is this is alpha prime si minus alpha prime ti multiplied by delta xi, is that is that correct that is f ti. I have, I did not subtracted alpha prime ti, alpha prime si minus alpha prime ti and this since we have already shown that f alpha prime is Riemann integrable. You want to say that f alpha prime is Riemann integrable, this will go to this. This will go to integral a to b fx alpha prime x dx. This will go to fx alpha prime x dx and what remains is to show that this can be made arbitrary small, this difference can be made arbitrary small and I shall skip the proof of this because the time is over.

You can read the proof from this book, the though point is that alpha f alpha prime is Riemann integrable and if you write these things separately both turn out to be the Riemann sums of the same integral. So, the difference between them will be small, that is the essence of the argument, that is the essence of the argument. So, with that will stop here, as I said will not give the complete details of the proof, but this is the idea. And that is basically main idea is to add and subtract alpha prime ti, okay? Then show that whatever the difference is there from this to this, this difference can be made arbitrary small that is ready. We will stop here, all right?