Real Analysis
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Lecture - 42
Integration as a Limit of Sum

Let us take a quick review of whatever we have done so far about integration we started with the definition of what is meant by saying that a function is Riemann integer able or Riemann styles integer able. Then, we saw that using definition we can decide about the integrality of a very few functions, so we come up with a criteria in terms of partitions how to decide whether a function is integer able or not. Then, using that we proved several theorems about the integrality of functions namely when the function is continuous it is integer able.

When it is monotonic also it is integer able and things like that and also theorems about the sums products etcetera of the integer able functions. If you remember, we began with saying that you also learn integral for the first time as some kind of a reverse or inverse process of differentiation, but if you if you have seen our discussion so far we have never refer to differentiation anywhere. There was no reference to differentiation integral we have define totally independently, so in order to establish that connection we require one more thing namely to view integral as a limit of sum. So, that is what we shall do today and then we will come back come to that connection between integral and differentiation derivatives.

So, we first look at integral as a limit of a sum limit of a sum, let us recall once again that we were defined what is called Riemann styles sum that is s p f alpha. As usual, f is a function bounded function alpha is a monotonically increasing function in some interval a b and what was this defined as this was defined as sigma i going from 1 to n f at t i and multiplied by delta alpha i right. Till now, we have not referred to this sum at all really speaking whatever we have discussed about integrity and all we discussed only in terms of this.

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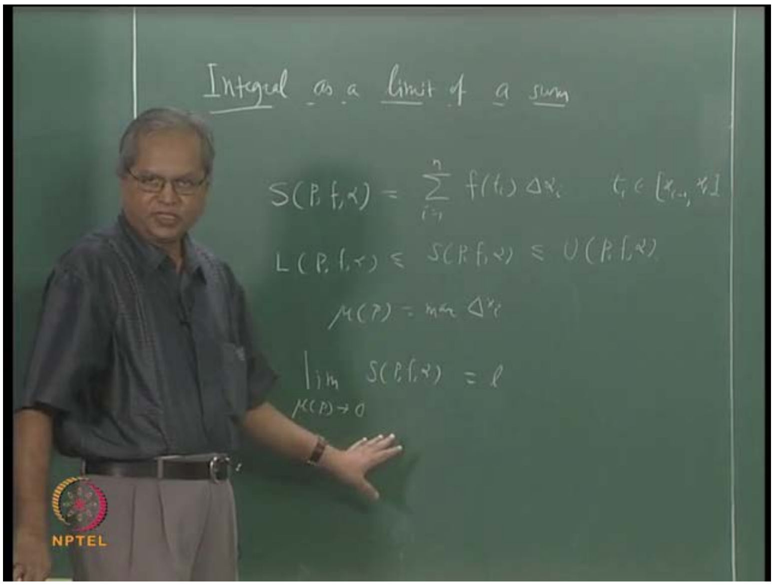

So, call lower sums and upper sums, but we have mentioned one thing that relation between this and that and that is this is l p f, alpha is always less than equal to s p f alpha and that is less than or equal to u p f alpha. Now, what we want to define is what is meant by saying that the limit of this Riemann styles some exist, what is the value of that. Let us recall once again that this point t i is any point in the interval x i minus 1 to x i and let me again recall this definition.

We have defined what is meant by mesh of the partition p mesh of the partition p that was maximum value o d delta x i maximum value of delta x i that is maximum of the length of this some interval. So, what we want to talk about now is that limit of this Riemann styles sum as this mesh of p tends to 0 that is this is what we want to talk limit of s p f alpha as the mesh of the partition tends to 0.

First of all, we shall define what is meant by this what is meant by saying that this limit exist and let us say it is value is let us say it is value is l what is meant by saying this as usual we give this definition in terms of this epsilon delta etcetera. So, what we what we would expect is given any epsilon bigger than 0 this should exist some delta such that whenever the mesh of the partition is less than delta the difference between the Riemann sum and this number l that should be less than epsilon.

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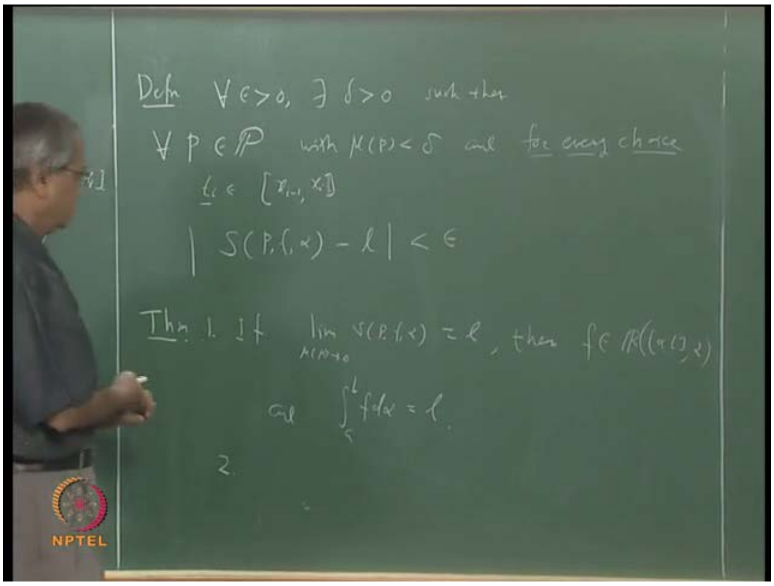

So, let us re write that definition so definition ok for every epsilon bigger than 0, there exist delta bigger than 0 such that what should happen for every partition p with mesh of p less than delta. For every partition p that every p in this remember this script p we have been denoting as the set of all partitions on a b. So, every partition p with mesh of p less than delta and one more thing for every choice of p i in this some interval x i minus 1 to x i.

For every choice t i if this some interval x i minus 1 to x i what must happen is the difference between this number l and this Riemann styles some that should be less than epsilon minus l. This should be less than epsilon if this happens we will say that limit of this s p f alpha as u p tends to 0 exists and it is value is equal to l and all that is written in this symbol. Now, if you notice that we have not defined the integrals in this fashion so far if you remember our definition of integral was we define the upper integral as the indium of the upper sums and lower integral as the supreme of the lower sums.

Whenever those two coincide we said that that is the common value that is what we called integral where it is. So, what we should expect or what act that if not for all functions at least certain one of the conditions if this limit exist then the function should be integer able and the value of the integral should be same as this. On the other hand, if the function is integer able, then this limit should also exist and its value should be same as the value of the integral.

So, this is what we want to want to if not for all functions at least under some extra conditions. So, let us let us state that is a theorem, so as far as the first thing is concerned we do not require at that is for one way we do not require any additional condition and that is the following. If this limit exist, then the function is Riemann stylus integer able and this limit exist means this happens remember here this is important for every choice t i for every choice t i.

This should happen that is corresponding Riemann stylus sum minus l that should be less than epsilon. So, we will say that if that is the that is first statement if limit as u p tends to 0 s p f alpha equal to l that means the limit exist and it is equal to l. Then, f is integer able f is Riemann stylus integer able and the value of the integral it is same as this limit the value of the integral is same as this limit and integral a to b f d alpha is equal to l second part is the converse.

For the converse, we require some additional condition for the converse I will come to converse, later let us first see how we prove this let us tell the first part to show that something is integer able. We know what we did is given any epsilon we should produce a partition p such that difference between a corresponding upper and lower sum is less than epsilon, so let us start with that.

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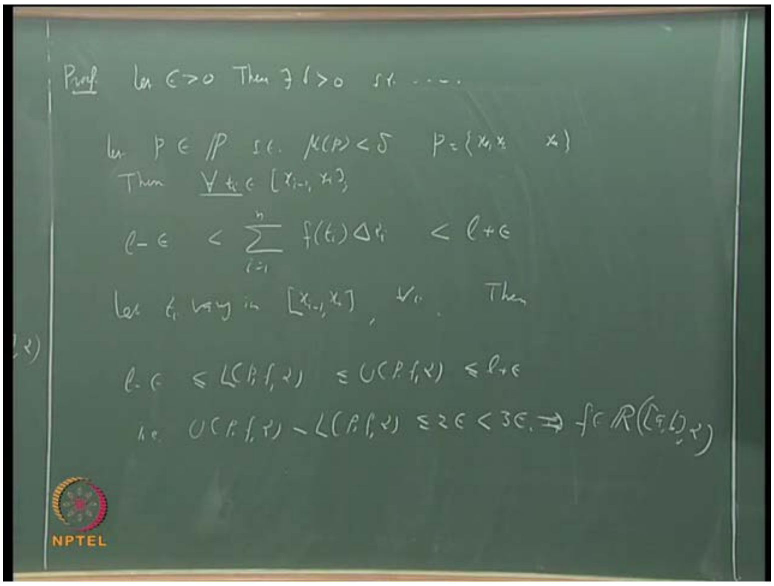

So, let epsilon be bigger than 0 all right we know that this limit exist, so we applied this definition, so for this epsilon that should exist some delta such that this should happen whatever we have written here this. So, then there exist delta bigger than 0 such that whatever I have written here such that for that if you take any partition with mesh of p less then delta. For every t i etcetera, this difference between the s p f alpha minus l is less than epsilon, choose one such partition choose one such partition.

Then, for that partition that is choose a partition let us let p be a partition with mesh of p less than delta p partition such that mesh of p is less than delta then for these particular partition. The difference between this Riemann stylus sum and l should be less than epsilon or which is same as p that this should lie between l minus epsilon to l plus epsilon. Let us write it like that that is let such that, then for every t i, suppose this partition is p let us say this partition is x naught x one etcetera x i.

Then, for every t i in x i minus 1 to x i what we must how is that let me write this Riemann stylus sigma i going from 1 to i. Let it this time I shall write in the full form instead of using this notation s p f alpha I shall write it as f p i in to delta alpha i. This should be less than l minus l plus epsilon and should be bigger than l minus epsilon, now the important point is this that this must happen for every t i this must happen for every t i in the interval x i minus one to x i.

So, suppose I will write t i vary in the interval x i minus one to x i, suppose I will write t i vary, then I can take the supreme over all such t i here and say that sigma m i in to delta alpha i is less than l plus epsilon which is nothing but upper sum. Similarly, if I will write t i variant, take the infimum take the infimum of the corresponding term, then I will get the lower sum. Then, 1 can I can say lower sum is less than l minus, sorry bigger than l minus epsilon, so what we can say is this basically what I am saying is this let t i vary in the interval x i minus 1 to x i for all i that is for each term. For each i you take all possible values of t i in the interval x i minus 1 to x i and all of them must be less than this l plus epsilon.

So, their supreme also must be in fact only thing that will happen there when we take the supreme this inequality may not remain strict. Instead of less than, you may have to replace it by less than or equal to instead of less than you may have to replace it by less than or equal. Similarly, then what we can say is this then what will get is l minus epsilon this is this will bigger than less than or equal to l p f alpha.

This is less than or equal to u p f alpha and this is less than or equal to l plus epsilon, but which is same as saying which is same as saying that u p f alpha minus l p f alpha. If you take the difference that difference must be less than or equal to the difference between these two l plus epsilon and l minus epsilon and that is 2 epsilon. So, we can say that that is u p f alpha minus l p f alpha less than or equal to 2 epsilon.

Suppose, I had started right from here with rest of epsilon, suppose I had taken epsilon by 2 or epsilon by 3 you would have got here less than epsilon. So, that is why we say that any caution multiple of epsilon is given any epsilon, if you produce a partition such that u p f alpha minus l p f alpha should be less here you have less than or equal, but does not matter. I can say this is less than 3 epsilon, so what does this shown it shows that the function is integer able it shows. So, because given any epsilon we have produce a partition p such that for that partition u p f alpha minus l p f alpha is less than three epsilon so that shows that that shows that f is integer able this implies that f.

Now, what remains we also need to show this we also need to show this all right we know that for every partition this number. Whenever the whenever the function is integer able for every partition this number prescribe between an upper sum and lower sum. So, this number is somewhere here less than or equal to integral a to b f d alpha right that is integral a to b f d alpha is bigger than or equal to l minus epsilon and less than or equal to l plus epsilon.

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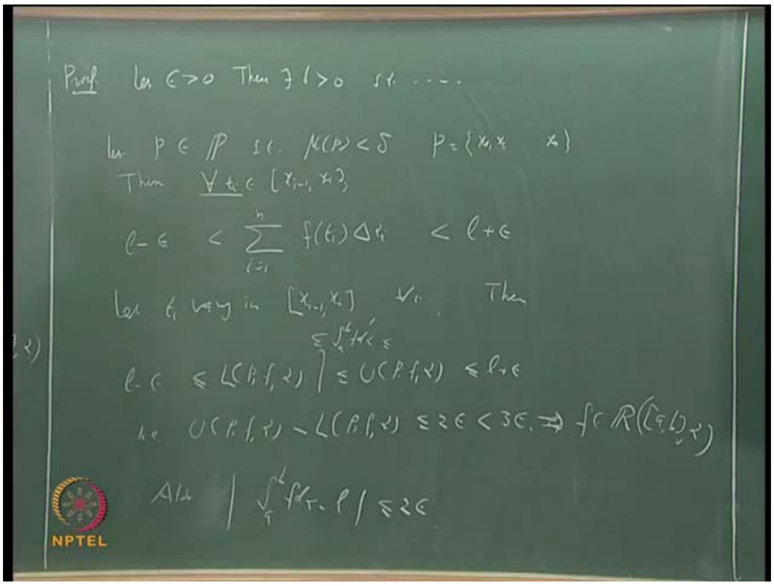

So, what we can say that means what can we say the difference between that number and l difference must be less than less than or equal to 2 epsilons. So, that is also we can say that also integral a to b f d alpha minus l is less than or equal to 2 epsilon, but epsilon was arbitrary. So, if this has to happen for every epsilon this is any way a non negative number it substitute value. So, the this is the standard technique when ever for example, when whenever we want to said something is 0, one way is to show that show that it is less than every positive epsilon less than or less than or equal to every positive epsilon.

So, that shows that these two must be equal this two must be equal and which is same as saying this. So, we have proved it whenever this limit exist the function must be integer able and value of the integer able must be same as this limit. Now, obvious question is the converse, so suppose we know that a function integer able can we say that this limit exist and whether it equals the value of the integral. Now, there as I said earlier we cannot make that kind of a general statement for example, what we would have would have like to say is that if f is integer able.

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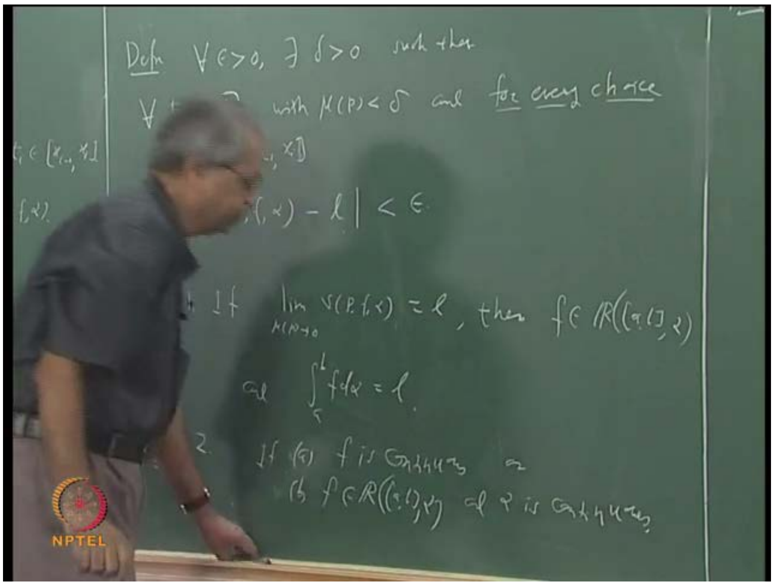

Then, this we need to exist then it is equal to this integer able, we need to put some extra condition. There are various possibilities, one possible condition is that if f is integer able sorry if f is f has be integer able any way to taco the converse, but in addition if f is continuous. Then, we can prove it this way or else if f is not continuous then f is integer able and alpha is continuous. So, at least one of the function f or alpha should be continuous, then we can prove the converse also, let us just so if f is continuous.

Let me split the conditions in two different way, remember what when we talk of a converse what we have think that if is integer able. Then, this limit exist that is what we want to prove, but I want to say that we need some additional condition here the first possible condition. If f is continuous, I do not have to say separately there that is integer able, we already prove it whenever it is continuous it is it integer able or f is integer able f r a b alpha and alpha is continuous and alpha is continuous.

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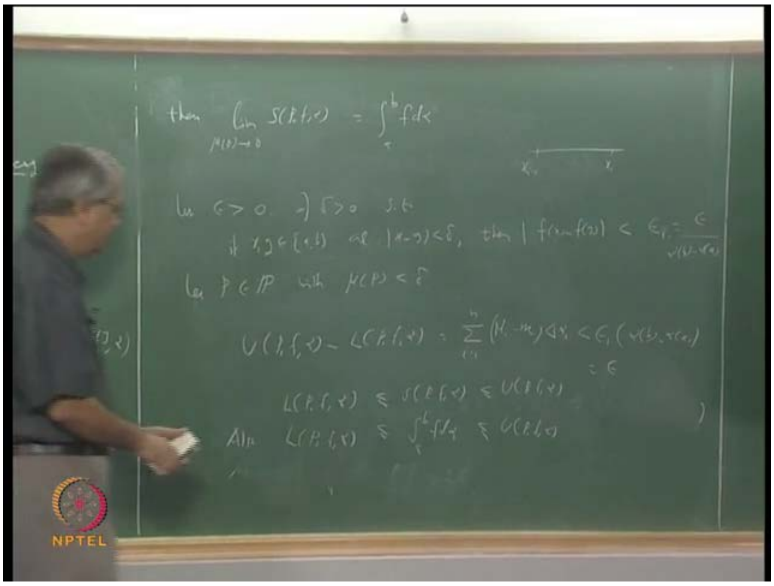

Then, this limit exist that is then I said limit s p f alpha limit s u p tends to zero this limit exist and it is value is same as integral a to b f d alpha. That is one of this two conditions as to satisfied either f is continuous or f is just integer able, but alpha is continuous out of these two condition if f is continuous this fairly easy to prove. In fact you can say the bulk of this steps in the proof we have already discussed what is the first f first f is that if f is continuous.

We know that since it is a closed bounded interval it is a compaction it is uniformly continuous every continuous function is uniformly continuous on a compaction that is something that we will we will use. Then, let us look at this proof let us say that let epsilon b bigger than 0, then to show that this limit exist what we have to show that there exist some delta there exist some delta. If you take any partition with mesh of p less than delta the difference between this and this should be less than epsilon.

That is that is what we need to show all, but since because of uniform convicted there exist there exist delta bigger than 0 such that if such that if x and y belong to a b and mod x minus y is less than delta. Then, mod f x minus f y is less than instead of epsilon we will take something multiplied by epsilon, here I will for the time being i shall call that number epsilon 1, I will call that number epsilon 1.

We shall choose that epsilon one later depending on what we require all right now then what we do what we can see from here in fact this is how we proceeded with the proof since the difference between mod f x minus 0. One can say that take any partition with mesh of p less than delta take any partition with mesh of p less than delta. So, let p belonging p with mesh of p less then delta, then what we want to show is that for such a partition p for such a partition p we difference between the Riemann stylus sum.

This number alpha that is less than epsilon, but what we know about this is the following that if you look at for this partition u p f alpha minus l p f alpha. Then, u p f alpha minus l p f alpha that is nothing but sigma i going from 1 to n or big m i minus small m i into delta alpha i and M i m minus m I see that. Suppose, now you take x and y in the interval x i minus one to x i then since the mesh of the partition is less than delta mod x minus y will be less than delta.

There is look at this i th interval x i minus 1 to x i and suppose x and y both are in that interval then the difference between f x and f y must be less than epsilon 1 which is say m for this must happen for every x y which lie in this interval. So, by taking supreme and few we can say that difference between m i big m i minus small m i must be less than that number epsilon 1 must be less than epsilon 1. So, we will take that epsilon one outside a summation sign so it is epsilon one in to sigma delta alpha i.

Something which we will done very often, so this is less than epsilon one in to alpha b minus alpha a. Now, suppose I want this whole thing to be let us say less than epsilon or epsilon by 2 or whatever it is, I can take this epsilon one accordingly. For example, see if I take this epsilon 1 as epsilon divided by alpha b minus alpha a, then this will be nothing but, epsilon that means u p f alpha minus l p f alpha is less than epsilon, but this not what we actually want to prove. You can say from this it is follows that f is integer able but, that is something we have already proved that is not our idea our idea is that we want to prove this limit exist.

Now, what we notice is the following that this l p f alpha this, so you take any Riemann sum with respect to this partition. If you any Riemann sum with respect and for take for every choice of t i you can set l p f alpha less than or equal to s p f alpha and less than or equal to u p f alpha. This is something that we know always whatever be the t i and what is one more thing that we know that this number also lies between l p f alpha and u p f alpha. That is also less than or equal to integral a to b f d alpha and less than or equal to u p f alpha, but what do we know we know that the difference between this and this is less than epsilon.

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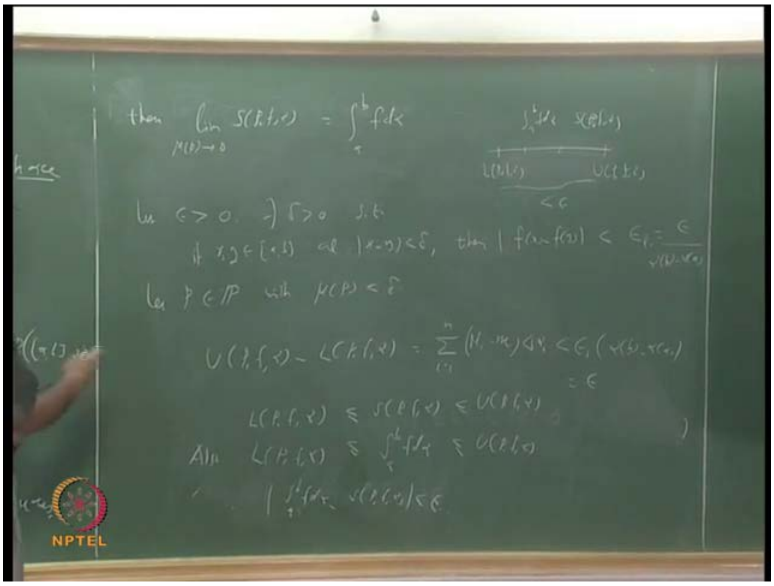

Just try to picturize what is this, let us say this is l p f alpha you have this u p f alpha some were here. This distance is less than epsilon that is the difference between u p f alpha minus l p f alpha that is less than epsilon. Both of this numbers integral a to b f b alpha and this s p f alpha both are them are lying in this interval. Both of those number integral s p f alpha, sorry integral f d alpha and this Riemann styles some their lying in this interval whose length is less than epsilon.

So, does it follow for a here the difference between them must be less than epsilon, therefore, integral a to b f b alpha minus s d f alpha that is less than epsilon. This is what we wanted to this is we wanted prove it see what we have proved we proved given epsilon there existed delta such that if you take any partition with mesh of p less than delta then for that partition. The difference between these two numbers is less than epsilon and that is how we defined by saying that limit of that is how we defined this that limit of s p f alpha as u b tends to 0 is equal to l.

So, in this case the proof is somewhat easy here the proof is slightly involved because you are using the continuity of alpha instead of f and you have to also use the integrality of f something similar to what are the proves that we discuss in other class. That is you do not have controller over all the all the entries in a sum you have controller over some entries. So, there you do estimate there particular manner and the remaining terms you estimate in some other way using the bounded ness of the function etcetera and that is how will have do here also the problem is here that is because f is integer able.

You can give any epsilon we can always find a partition such that for that partition the difference between u p f alpha and l p f alpha is less than epsilon, but a point is to show that this limit exist. We have to show that for every partition whose mesh is less than delta that same thing here happens, but the mesh suppose we knew that that partition is refinement of this.

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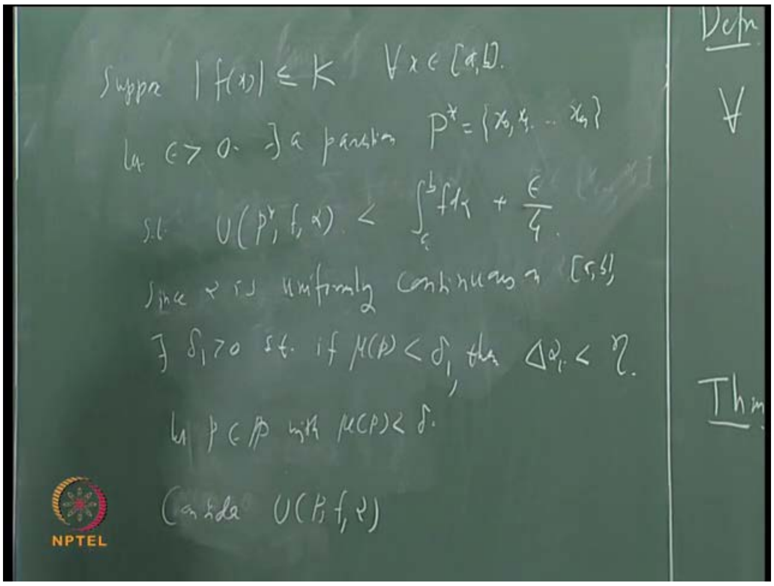

Then, we could have said that earlier but, we could have we could have we could have said that using our earlier property is both refinements, but the mesh is less than mesh. Some other partition does not mean that the partition is refinement because it may not contain those points that is why we need to do some extra work here. Let us now take this part what are we assuming now that f is he f is integer able an alpha is continuous all since we here we did not use the bounded ness of f.

Let us assume that suppose mod f x is less than or equal to sum number k or every x in a b This we can always because we assume that f is Riemann integer able f f is integer able. So, once the function is integer able we are taking integrality only about the bounded functions. Let epsilon be bigger than 0, again we have to produce delta in in that fashion so first of all let us say that since will first use the integrality of f since f is integer able. So, integral a to b f d alpha is also infimum of u p f alpha u p f alpha right so using this we can say that there exist there exist a partition p.

Suppose, I call that partition p star, there exist a partition p star such that u p star f alpha is less than integral a to b f d alpha plus let us say epsilon by 4 something. Similarly, we had done in one of proved yesterday see this is infimum, so anything bigger than that is not an upper bound. They should exist some partition such that upper sum with respect to that partition is less than integral a less than this number integral a to b f d alpha plus epsilon by 4. Suppose, this partition p star as some intervals let us say p star is x naught x one x two x one more thing since we have assume that alpha is continuous.

Since, we have assume that alpha is continuous alpha is also uniformly continuous in a b and hence what we can say is that there will all always exist some delta one such that such that the difference between x and y is less than 1. Difference between corresponding alpha x minus alpha y is less than what ever number we want, so we can we can also say that since alpha is uniformly continuous 1 a b. There exist for the time being I will call that number delta 1 there exist delta 1 bigger than 0 such that see again. We shall use the similar argument here see here what did we use that whenever mod x minus y is less than delta then mad f x minus F i is less than what ever number that that we want there.

So, suppose we want apply to this alpha what we can say is that if this length of this some interval is small then the corresponding value delta alpha i should be small that is the that is the idea. There exist delta one such that such that if mesh of the partition p is less than delta one then delta alpha i is less than i will again I want some number which depends on epsilon. For the time being I will call that number eta, this eta we shall we shall decide later for the time being I will call that number eta then we take such partition we take such partition.

So, consider such p let p let belonging to p with mesh of p less than delta and here the argument is as follows and consider this u p f alpha consider this u p f alpha. Now, what we do here is that we compare this u p f alpha with this u p star f alpha, but the thing is if we knew that p is refinement of p star I could have immediately said that u p f alpha is also less than this, but that is something that we do not know. What we know is that mesh of p is less than delta, so this delta, so we have control over this delta alpha i, we have control over this delta alpha i and we do not have control over this. Now, let me just try to explain to you can write the details that the how we are going to think of this is the following.

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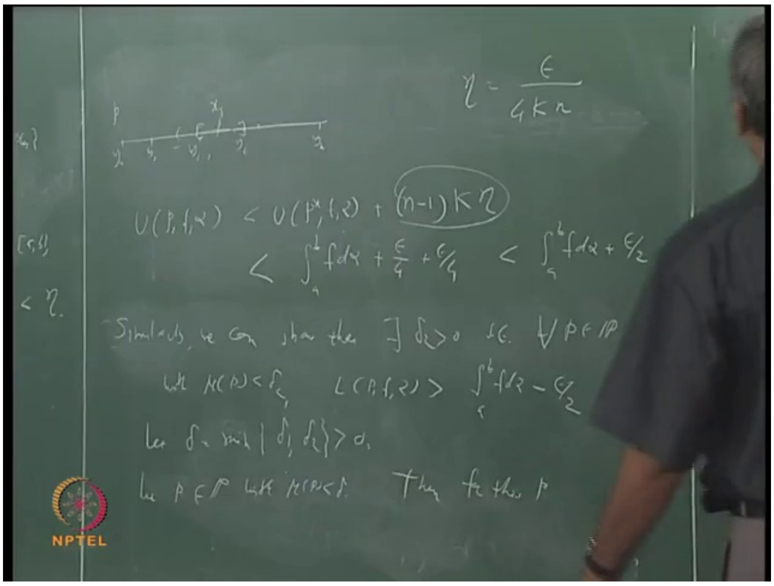

Suppose, this is the partition p let us say I call this instead of x naught x 1 x n let us say p is y naught y 1 etcetera y n what is the idea here there are possibilities. Suppose, you take any part any numbers say y i minus 1 to i, this interval may contain one of the x i as a interior point that is one possibility that is x i minus that is y i minus 1 to i may contain 1 of x 1 of the x size. That is one possibility, what is other possibility that y i minus y i may not contain any of the points.

If we does not contain any of the points x i x i it means it is sub interval of one of the intervals here. Suppose, y i minus this interval contains then of this x says there it must is contain in one of the sub intervals there if that is the case then whatever contribution. This will do to that upper sum that that contribution is already there here or that is less that is less than or that is whatever is a supreme over this sub interval will be less than or equal to the supreme over whatever is the interval. It is coming here and that term is already here, so if you take the contribution to the upper sum from those kinds of intervals then that will be less than this u p star f alpha.

That will be less than this u p star f alpha, so I can say that u p f alpha not the entire u p f alpha that is contribution to u p alpha. Due to those intervals which do not contentment due to those contents which do not content any of the excess that part of the sum that will be less than u p star f alpha this slightly involved. So, just try to understand what remains that those intervals those intervals which contents one of the exits those intervals which contain one of these exists. In case of those intervals I cannot say that we cannot compare those with this step, but we can say what the maximum is.

We cannot say what the maximum possible contribution due to those intervals those are two things you can say there are at the most n minus one such intervals is. So, the partition p contents this points n plus 1 that x naught and x n are n points, so remaining points are n minus 1 points. So, and the must n minus one of this sub intervals can have one of this exist as interior points, so we will we will estimate what is the maximum possible contribution from those intervals.

So, since there are n minus one such intervals we will say what is the maximum possible contribution from each interval and multiply that by n minus one multiply that by n minus 1. So, I will say this plus n minus 1 n minus 1 in to what is the maximum possible contribution to this interval, you have take the supreme over that some interval where it is a supreme over that some interval. We do not know how it is related to this this supreme, but we certainly know that it must be less than or equal to this number k because what f x is less than or equal to k for all x in a b.

So, obviously it must be less than or equal to this number k, so we can say n minus 1 in to k and then because of this delta 1. We know that whenever mesh of p is less than delta one each of this delta that is length of each of this some interval is less than eta length of each of this some interval is less than eta. So, this must be less than eta that is yes did you follow the first part that is those intervals see you take any interval correspond partition p interval is y i minus 1 to y i.

There are two possibilities either this interval contains one of this x j s or it does not contain suppose it does not contain there it means this interval is contain of this intervals x j minus one to x f from here. Hence, the supreme of that some interval will be less than or equal to the corresponding supreme over the corresponding interval there.

So, that term is already coming here in this u p star f alpha, so whatever is a contribution due to those intervals some total of all those over those some intervals will be less than less than or equal to u p star f alpha. So, we need to see only the remaining intervals what is the remaining intervals remaining intervals are the once which contains x j is as x j is as one of the interior points, but how many intervals can be there because there are only n minus 1 points here.

There are only n minus x naught and x n cannot be interior point x naught is a and x n is b so there are only n minus 1 points here. So, there are only n minus 1 such intervals then supreme over each such interval must be less than or equal to this number. This is supreme or the whole interval and then multiplied by the length of that sub interval, but the length not length delta value that is delta y i minus delta y i minus 1, but we have chosen this small delta one such away.

Then, that number must be less than eta that is that is before we use the continuity of alpha that is where we have use the continuity of alpha. Hence, the uniform continuity of alpha there is we if the mesh is less than this delta one then delta at y i minus delta at y i minus one that must be less than this number eta. So, the contribution due to the remaining n minus for intervals must be less than this, now at this point we can make a choice for this eta I said that I have I will not try eta 1. I will write now we will we want to choose eta is we know that this is less than this interregnal a to b.

This first term is less than integral a to b f d alpha plus epsilon by four and I will choose eta in such a way that this number also becomes less than epsilon by 4, so choose for example, eta is equal to epsilon divide by 4 in to 4 k n. Let us say n minus 1 will do, but we can choose something bigger than that also, suppose we choose eta is equal to epsilon by four k n. Then, this whole thing will be less than interregnal a to b f d alpha plus epsilon by 2.

Now, we have more or less completed main idea of that this proves just recall what have you proved we have proved that given epsilon bigger than 0 given epsilon bigger than 0. There exist a number delta 1 such that every partition which may show p this should be less than delta 1 mesh of p less than delta 1 upper sum of that partition is less than integral a to b f d alpha plus epsilon by 2.

We make a similar statement about lower sums we can say that, similarly we can show we can show that there exist delta 2 bigger than 0 such that for every partition p with mesh of p less than delta 2. We must have l p f alpha should be bigger than in integral in to b f d alpha minus epsilon by 2 what is to be done after this is obvious choose delta to be minimum of these two delta 1 and delta 2.

So, let delta to the minimum of delta 1 delta 2, so this is also positive number, then if you take any partition with mesh less than delta then both of things should happen, so let p belonging to split b with mesh of p less than delta. Then then for this partition p u p f alpha should be less than integral a to b epsilon by 2 and similarly, l p f alpha plus b bigger than integral a to b f t alpha minus epsilon by 2, so we can say that then for this p then for this p.

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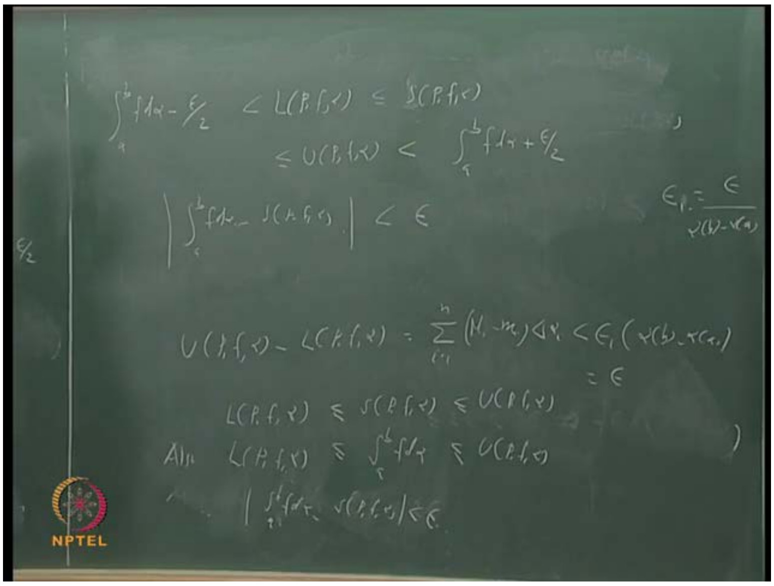

Let us start from here integral a to b f d alpha minus epsilon by 2 this should be less than l p f alpha and we know that any remark in the sum is bigger than or equal to lower sum. That is what we have we have seen already this l p f alpha must be less than or equal to s p f alpha and then this is less than or equal to u p f alpha. This is integral a to b f d alpha plus epsilon by 2, so what did we prove that is just the whole summary this start given epsilon will there whether 0 we have given epsilon bigger than 0.

We have got a delta such that whenever mesh of this partition is less than delta all the sums of that partition are related by this in equalities. We only need actually this is less than s p f alpha this less that right that is this is same as saying that integral a to b f d alpha minus s p f alpha that is substitute value of this is in fact less than epsilon by 2. The different, sorry the difference that is less than epsilon the difference between in this 2 less than epsilon.

That is what we wanted to prove, so let we again recall what we proved that if the limit of Riemann sum exist as mesh p goes to 0. Then, the function is Riemann styles integer able and the value of the integrals same as the limit of the sum conversely. We would have like to prove that whenever the function is Riemann stylus integer able then the limit of some also exist and equal to the value of the integral.

We have not proved it in general you have shown some extra condition either f should be continuous or f is integer able, but, alpha is continuous k in if either a if at least one of this two conditions is satisfied. Then, we can assigned a converse also namely that the Riemann stylus limit of the Riemann stylus some exist and value the limit is same as the value of the interregnal.

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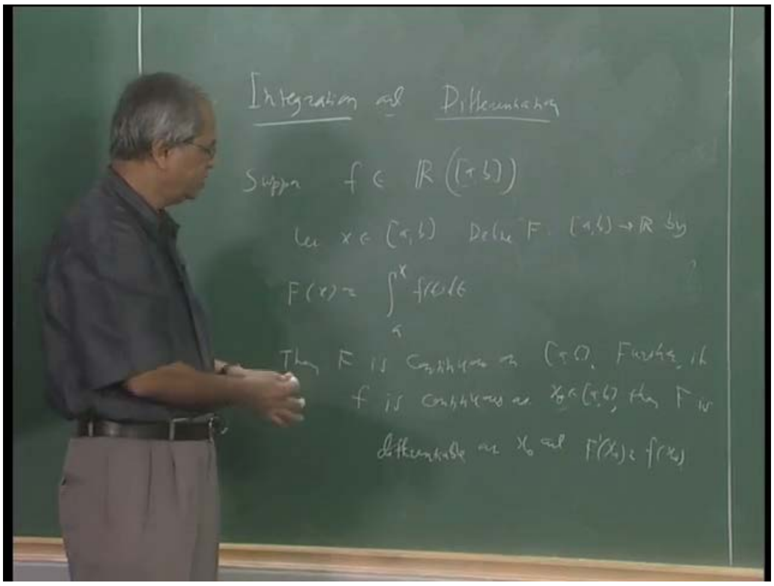

Now, let us come by to the let us come to what I centroid the beginning the relationship between integration and differentiation. So, let us start with Riemann integral functions first. So, suppose f is rebel integer able function, then given any such f we have seen that f is rebel integer able on the interval a b. Then, if you take any number c in the interval a to b it is also integer able in the interval a to c it is also integral in the interval a to c.

So, what I can do is that I can say that you take any x that x belonging to a b, then the function is also integer able in the interval a to x in the interval a to x. So, consider that interval integral a to x and let us consider f t d t integral a to x f t d t, then call this function big f of x.

So, define big f of a p 2 r 1 by this then these function big f is differentiable this function big f is first of all this function big f is continuous. Always this function big f is continuous always and it is differentiable if the small f is continuous if it whichever point this small f is continuous and that point.

This point this big f is differentiable and value of the derivative it is same as the value of the function small f I will just write that and then proof we shall discuss in next class then f is continuous on a b. So, next what you have to say that further if small f is continuous at x naught at some point in x naught in a b. Then, big f is differentiable at x naught at x naught and the value of the derivative that is big f lying at x naught is equal to small f at x naught that is whichever function f is that is small f is continuous.

That point the function big f is differentiable and it is value is value derivative is same as small f. So, this is a theorem which connects the ideas of integration and differentiation and it in certain way it shows that integration and differentiations are in some sense reverse process of each other, we shall discuss the proof in the next class.