Real Analysis
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Lecture - 41
Integrable Functions (continued)

So, we were discussing various theorems about the integrability of the functions namely given two integrable functions what can we say about the sum of those two functions etcetera. So, let us list a similar properties or similar theorems or as been done by Rudin. We will just take one theorem, which includes all of those.

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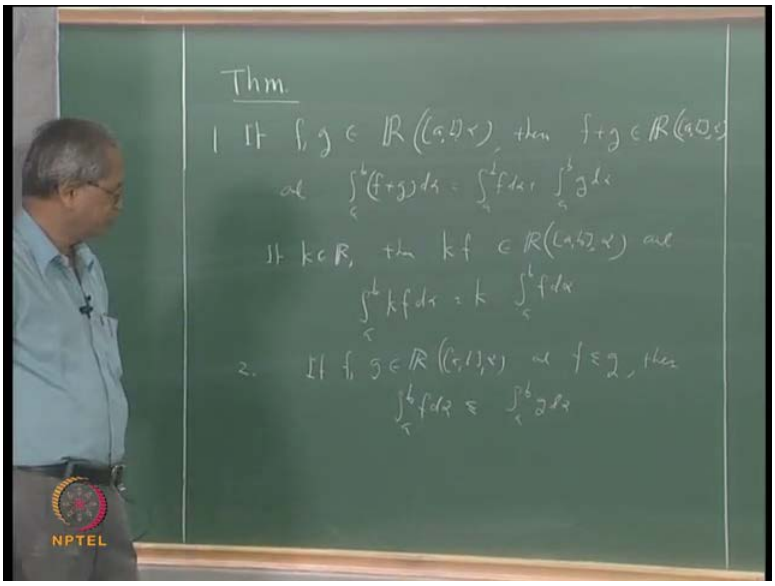

So, first we had said that if f and g are integrable R a d alpha. Then f plus g is also integrable and integral a to b f plus g d alpha, its same as integral a to b f d alpha plus integral a to b g d alpha. Also, in this itself I will say that if K is a real number. Then K times f is integrable and we can say similar thing about the integral of K times f. So, that is integral a to b K f d alpha that is same as K times integral a to b f d alpha. As I mentioned earlier this whole thing can be said in just one single sentence. If you use terminology of linear algebra namely that this is a vector space. It is set of all integrable function is a vector space and the map, which takes a function to its integral is a linear functional on that vector space.

We can say something more we have we have seen this two properties yesterday and it is proof also. There other properties, which are very similar to this and the proofs are also very similar. So, we will simply state we will not into very d too many integers to the proof. For example again, suppose we take say two functions two such things. Let us say we have two functions if f and g belongs to this R a b alpha. Suppose, f is less than or equal to g f is less than or equals to g means what? At each x in a b f x is less than or equal to g x.

Then integral of f is less than or equal to integral of that is what we should expect. Then integral of a b f d alpha is less than or equal to integral of a b g d alpha. Another way of expressing the same thing is that this is a positive functional, that is this defines a partial order on this vector space. This function respects that partial order if is less not equal to g integral f is less than or equal to integral of g. You can see that the proof will be very straight forward of this. Just compare the upper and lower sums with respect to f and g, you can say that for every partition U P f alpha will be less than or equal to U P g alpha. That is easy to see, because of this f less than or equal to g, it will follow from there, then that this will follow.

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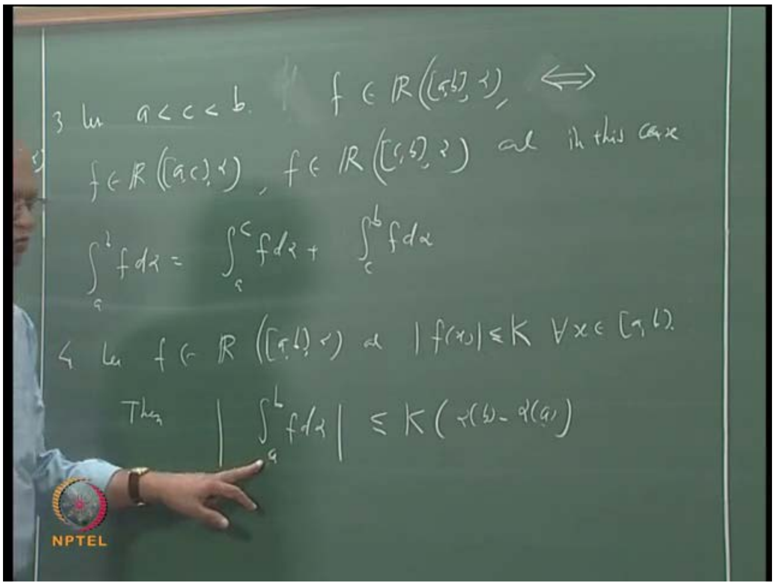

So, in fact once you understand the proof of this all other proofs are similar, that is why we shall not spend too much time in discussing those proofs. Another thing is that, which is again a very well know property of the integrals. That if you take some point in the interval. Suppose, I call that point C, let us say take let a less than c less than b.

Then if f is integrable over a b that is f belongs to R a b alpha. Then f is also integrable over a, c, it is also integral over c d and the corresponding integrals. Their sum is same as integral over m that is then f is integrable over this interval a c also f is integrable over the intervals c b. Integral a to b f d alpha is equal to integral a to c f d alpha plus integral c to b f d alpha. Only thing that is required to prove this is that the partitions that you are considering should contain this point c partition, that you are covering should contain. If it does not contain fine you const add that point c that will verifyment and use the property of the refinement of the upper and lower sums. That is the only thing.

So, once the partition contains the point c, then you can see that the upper sum, which corresponds to this integral will be sum of the upper some, which corresponds to this integral, that is all is the observation here. So, that is all that you need to need to prove this as well as this is infinite converse is also true. In fact I should say that if f is integrable over a both a a c and c b. Then it is integral over a b and then also this follows. So, instead of writing if this happens then this happens, one can also use this if and only if write that is this way is also true, that way is also true. In this case we can say that it is whenever when ever any one of this is true. In this case it is true all, then one more thing we can say that.

Suppose, we find some number, which is let us say let f belongs to R a b alpha. Suppose, f is integrable and suppose K is some number. See, once we say that f is integrable it means it is bounded, we have only talked about integrablity of bounded functions. So, if f is bounded mod f is also bounded function. So, we can find some number such that mod f t is less than or equal to that number for each t in a b. So, let us say f is integrable and mod of f x is less than or equal to K for all x in a b. Then this absolute value of integral a to b f d alpha this is less than or equal to K times alpha b minus alpha a. This less than or equal to K times alpha b minus alpha a, well I mean this is also fairly easy to see.

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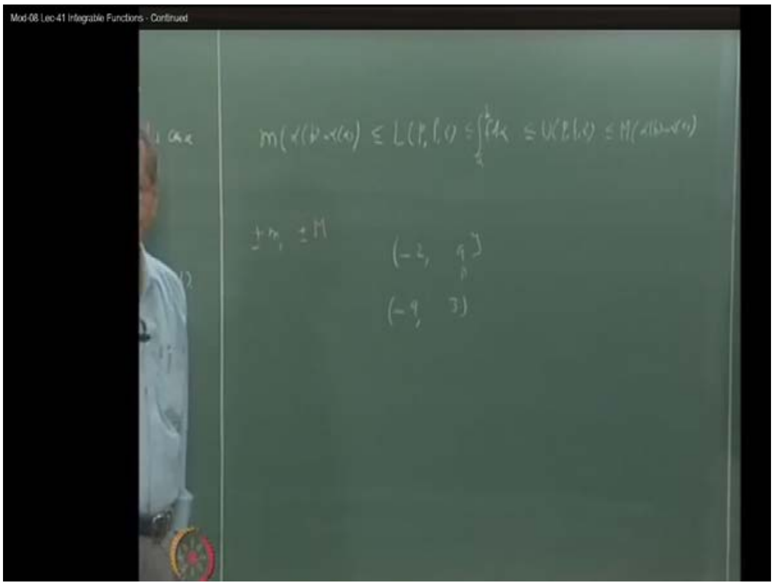

Remember, we have we used shown this all ready that is small m into alpha b minus alpha a. For any partition this is less not equal to l P f alpha and this is less than or equal to U P f alpha. This is less not equal to big M into alpha b minus alpha K i, this is what we have shown for any partition p. We know that when the integral exists that should lie between these two values.

So, that means this should be less not equal to integral a to b f d alpha. Now, what is this number K? It is it has a noted mod f x is less than or equal to K. What is this number big M and small m? We know that f x is less not equal to this big M. So, depending on if M is depending on whether m is positive or negative. Similarly, depending on whether this m is positive or negative, this K can be one of this infinite. You can see this K can be one of this four values plus or minus m or plus or minus m big M, do you understand this?

For example, suppose f x lies between let us say f x lies between let us say minus 2 and 9. Let us say that means what small m is minus 2 and big M is 9, but if that is the case, then mod f x is less than or equal to 9 mod f x is less than or equal to 9. On the other hand, so 9 is nothing but m, on the other hand suppose this something like this. Let us say it is between minus 9 to 3, in that case big M is 3 small m is minus 9, but then you can take K as 9. Because, mod f x will be less than or equal to 9. So, K will be 9, so this K will be one of this four values plus or minus 9 or plus or minus big M. We all ready know this the integral a to b f d f is less than or equal to this.

So, in this is also less no you can say this is less than or equal to K into alpha b minus alpha a. This is also bigger than or equal to minus K into alpha b minus alpha a and that will prove this all. Now, so far we have said so many things about taking various functions f g etcetera. Now, we will say similar things about this function alpha that is the function with respect to which we are taking the integrals. Suppose, we take two functions alpha 1 and alpha 2, remember here this alpha we always take protonically increasing function.

So, what can we say about something like, say integral f d with respect to alpha 1 plus alpha 2 etcetera. So, when one can ask one can expect that should also be the sum of the two corresponding integrals. So, what we would say is this that, suppose alpha 1 and alpha 2 are monotonically increasing in a b, then okay. Let us say it is in a b, if f is integrable with respect to alpha 1 and also with respect to alpha 2.

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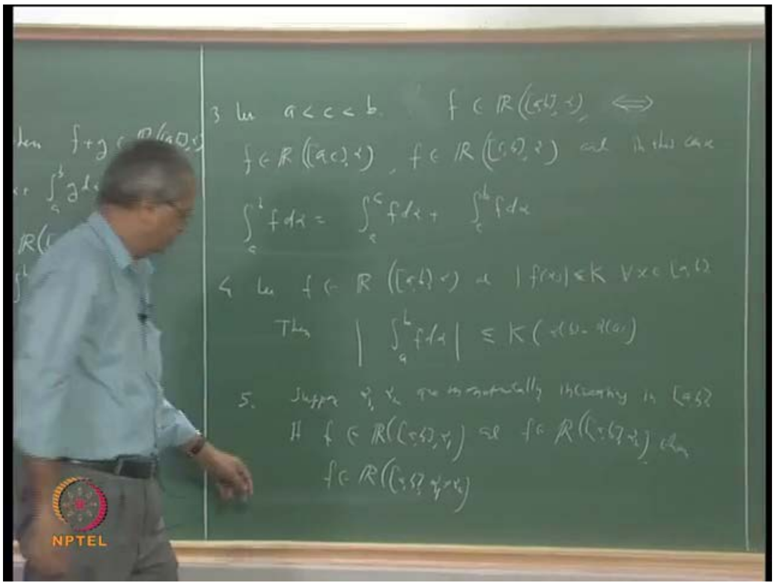

Then f is integrable with respect to alpha 1 plus alpha 2. So, if f is let us use symbols if f belongs to R a b alpha 1 and f belongs to R a b alpha 2. Then f belongs to R a b alpha 1 plus alpha 2 that is not R, we also want to say something about the integrals. So, I will continue here and integral over a to b f d alpha 1 plus alpha 2.

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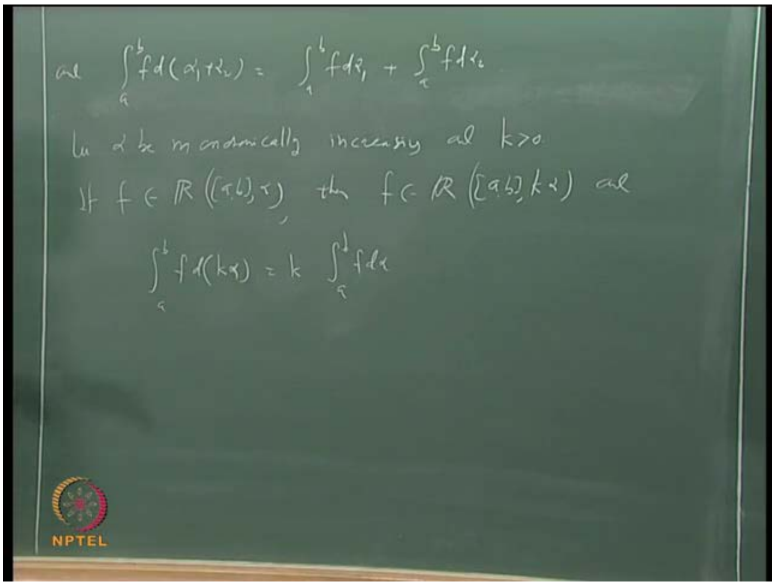

It is same as integral a to b f d alpha 1 plus integral a to b f d alpha 2. Again, see only thing that you have to notice here is that whenever alpha 1 and alpha 2 are monotonically increasing, alpha 1 plus alpha 2 is also monotonically increasing. So, this makes sense, so if you compare the upper sums or the lower sums. So, if you take the upper sum with respect to alpha 1 plus alpha 2, that is going to b a sum of the upper some with respect to alpha 1 and upper sum with respect to alpha 2. So, it will be essentially exactly the proof like this similar to similar to this. Now, let us also look at analogue of this here, we have taken K times f. So, similarly when we can think of K times alpha, suppose we take K times alpha there is slight problem here.

If alpha is monotonically increasing K times alpha need not be monotonically increasing. So, what is the restriction that we require for that? K should be positive. So, then with that restriction it is of course, if K is equal to 0 the whole thing is trivial. So, you take a positive number K, then we can say that if f is integrable with respect to alpha. If you take K as a positive number then it is also integrable with respect to K times alpha, a similar relation will hold. So, one can say that. So, let alpha be monotonically increasing and K strictly bigger than 0. Then what we want to say that if f is integrable with respect to alpha, then f is integrable with respect to K times alpha.

You have similar relationship between the integrals integral a to b f d K alpha. This is same as K times integral a to b f d alpha. That means the constant K will go outside the integral side, whether it is K times f or K times alpha. Only thing is that if it is K times alpha you have to have this restriction K must be positive constant. Of course, this restriction is because till now, we have considering integrals only with respect to monotonically increasing functions.

Suppose, we remove that restriction, suppose we have find some way of considering the integral with respect to other kinds of alphas. Then this restriction also can be removed, but that we shall discuss sometime later. For the time being we shall stick to monotonically increasing functions is. Again, here also there is nothing much in the proof.

You have to just look at the corresponding upper and lower sums. So, as I said earlier since the proofs of all these other things are very similar to what we have done here. We shall not go to details of those proofs. Now, it leads to some other natural questions, we have taken the some f plus g, what about the product? Suppose, we know that f n g are integrable, then can we say that the product is also integrable. Then why a product, one can ask several such questions. There are several other ways of comparing the two functions or making, constructing a new functions from a given function f. So, in which case we can say that a new function will also be integrable. So, what we can do is that we shall prove one good theorem in that respect. Several such cases like product etcetera will turn out as a corollary.

Basically, what we want to prove is that if you compose with a continuous function. Suppose, you take an integrable function and if you compose it with a continuous function, then composition also will be integrable. Once we prove that all these other things about the product etcetera, will follow from that. So, let me just go back here and state that theorem. So, let us say that we have start with this, let us say f is integrable with as usual alpha is 1 and f is integrable with this respect to alpha. Let f belongs to this.

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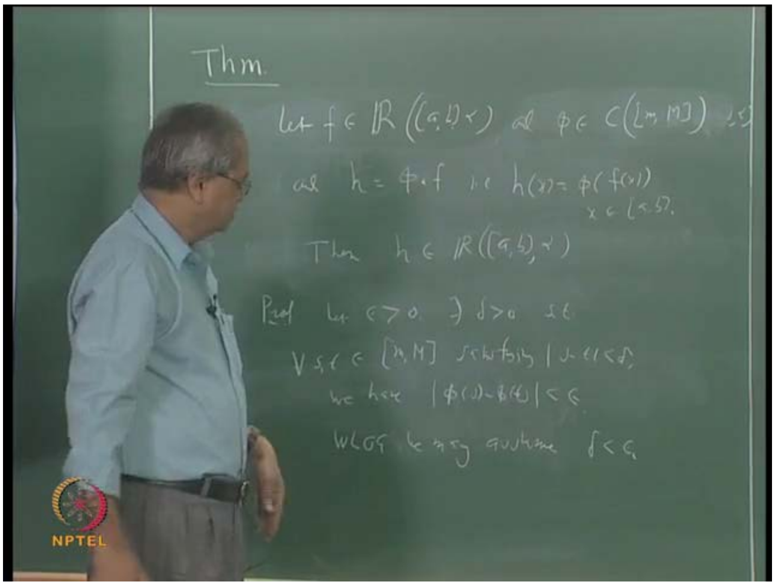

So, we want to compose this f with some continuous function. As you know, if you want to compose a function that function must be defined. The domain of that function should at least contain the range of this, but we know that if we use the notation, that is range of this is contained in the interval small m to big M. That is what we have seen that small m is a infimum and big M is a supremum. So, if any function is defined on the interval small m to big M, then we can talk of composition. So, let f belongs to R a b and phi belongs to see c will use the as U n set of all continuous function on this interval c, small m to big M.

Suppose, we give some name to this composition we call h as h s phi composed with f h as phi composed with f. That is if you for your satisfaction let us write the full form that is what we mean is h at x is equal to phi of f x, for each x in a b. Then h is that h is remain integrable not then h is with respect to the function. Of course, in this case we cannot give any formula for the integral of h in terms of integral of f and phi etcetera. We cannot say anything like integral of h is something like phi composite with integral of f and any such thing right though. Of course, the value of the integral will also lie between small m and big M multiplied by alpha b minus alpha a etcetera, but we cannot say any such thing about the values of that unless.

For example, here for case in all such cases apart from saying, that the certain functions are integrable will also related. That values of the integrals, but this is a different theorem here. We cannot say anything about, what any relationship about integral of h integral of f, but that is okay, when we just want to say that it is integrable. Now, to look at the proof of this let us start with this first of all this function phi is continuous on this interval small. This is a compact set and every continuous function on a compact set is uniformly continuous. Something that we have been using again and again in the theory of integration. So, we shall use that here also and what is it that we want to show? We want to show that this h is integrable.

We shall use the standard technique to solve that, but given epsilon you find a partition. Such that the difference between upper and lower sum is less than epsilon some constant multiple of epsilon that. So, let epsilon be bigger than 0, so our final aim is this that we want to find the partition. Such that with respect to that partition upper sum of h minus lower sum of h should be less than epsilon. As I said some constant multiple of epsilon, but on the other hand this phi is continuous. So, corresponding to this epsilon there will exist delta we can say that there exist delta bigger than 0. Such that if you take any two numbers in this interval whose difference is less than delta. Then the difference between the corresponding values of h is less than epsilon, there exists delta.

Such that for all is n t in this interval small m to big M satisfy mod s minus t less than delta. We have mod h s minus h t is less than epsilon phi s minus phi. You are right,phi s minus phi t phi s minus phi t is less than epsilon. Then what we also know is that in all such argument if some delta works for this. If some delta satisfies this then any numbers smaller than that will also have the same property.

We cannot, suppose we have chosen one delta, which satisfies this relation that is whenever mod s minus t less than delta. Then mod phi s minus phi t less than epsilon instead of delta. Suppose, I replace it by delta by 2 or delta by 3 or any numbers follow. Than that delta still this property will be true. So, that is why I can in particular assume that this delta is smaller than epsilon, do you agree with this? That is something we made in subsequent calculations.

So, I will assume that delta is less than epsilon. So, as usual this such a thing is called without loss of, we assume we may assume delta is less than epsilon. Till now we have only used the continuity of this function phi. Whatever we have done here it follows nearly by the continuity of the function phi, but there is one more thing that we know.

Namely, that we have assumed that f is integrable that something, we have not used yet, f is integrable. So, that is something we have not used yet.

So, we can see that because, f is integrable given any positive number we can find a partition. Such that difference between upper and lower sum, with respect to that partition is less than that positive number. So, we can say that since f is integrable, since f is integrable, there exists a partition P. Such that what should happen? if you take this U P f alpha minus l P f alpha? This should be less than whatever I take epsilon delta or whatever positive number.

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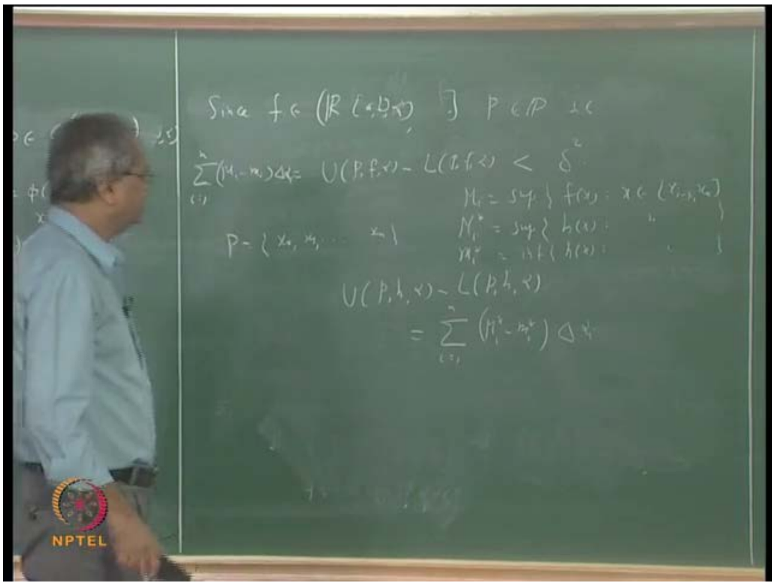

So, here I will take that as delta square, why delta square? That will become clear to you when we proceed with proof. Because, we actually we are not interested in this difference we want to show that U P h alpha minus l P h alpha is less than epsilon. So, this is just a preparation for that. Remember one more thing, this is nothing but sigma if the partition P is this P is less than x naught. So, P is equal to x naught x 1 x n etcetera then this is nothing but big M i minus small m i into delta alpha i going from one to n.

Now, what are we are interested? We are in interested in similar difference between that is we are into not into U P f alpha and U P f alpha minus l P h alpha. That is what we want to show that that is less than, but what is that? Let us just write the value of that U P U P h alpha minus l P h alpha, what thing with this? We will have to define the new numbers corresponding the function h. We will define the new numbers corresponding to the function this M i this big M i and small m i correspond to the function f. Let us let me write here big M i is the what is that supremum of f x, where x belongs to x i mins 1 2 x i. So, what we want is similar number for the function h. So, suppose I call that number M i star.

That is I will write supremum of h x for x belongs into this same interval. Similarly, small m i star is infimum of i will use sum infimum of h x for same x belonging to x i minus into x i. So, using this notation U P h alpha minus l P h alpha that will be nothing but sigma i going from 1 to n big M i star minus small m i star, big M i star minus small m i star multiplied by delta alpha i. Though, we want to show that our aim finally, is to show that this sum is this difference U P f will this difference, it is nothing but this sum that can be made arbitrary spot. That is less than epsilon or epsilon multiplied by some constant etcetera.

Now, to do that again we will have to use what we have done here. That is we have seen that, this if for this epsilon there exist a delta bigger than 0. Such that whenever mod s minus t is less than delta mod phi is minus phi t is less than epsilon. Now, what is h? h is phi composed with f remember, let us say let us just see that is see. Let us just picturize this, suppose this is this the interval a b, this is the given interval.

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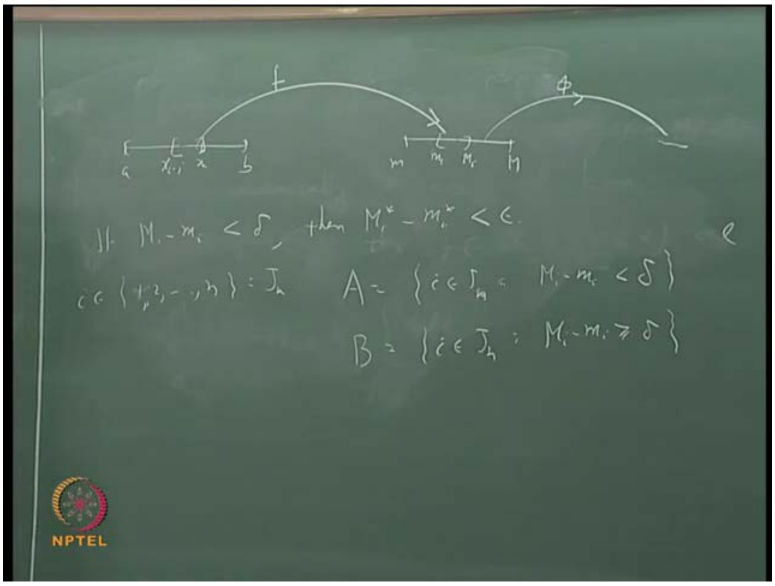

Suppose, this is the interval small m to big M f goes from here to here, f goes from here to here. Then h is defined on here it may go to somewhere does not matter h is defined here. Now, what we know is that if the difference between any two values here is less than delta then the difference between the corresponding values. There will be less than epsilon not h this is phi, this phi. In other words if I know that this big M i minus small m i is less than delta

Then for a corresponding value Big m i star minus small m i star will be less than epsilon big M i star minus small m i star will be less than epsilon. Because, what is the meaning of that? Big M i minus small m i is nothing but the corresponding, because this is the value that f we take. So, just tell me whether agree with this if big M i minus small m i is less than delta then big M i star minus small m i star is less than epsilon.

Just tell me whether you agree with this, what is big M i? Big M i is the supremum of f x when x lies between x i minus see just as I done this interval a to b. Suppose, I take the interval x i minus 1 2 x I, that is some interval. Is it clear that this image of this interval under f will be small m i to big M i. Because, small m i is the minimum of x when x very strong x i.

So, f x when x varies between x i minus 1 2 x i f x will vary between small m i and big M i. Now, we know that this difference is less than delta, so phi of that. Suppose, we take any that is if this difference is less than delta phi m i minus phi small m i, that will be less than epsilon. For that not only phi m i minus phi m i for any two values that lie in this interval corresponding difference phi s minus phi t is less than epsilon. That is how we took this delta, this only one problem here this may not happen for all i. Because, we do not know anything about the function f this may not happen for all i. So, we have to split this indices that is this set, there are n such intervals. So, we have to split this indices those i for which this happens and those i's for which this does not happen.

So, suppose we take this set or i 1 2 etcetera up to n there are n such intervals let us say i belongs to any of this, then what I will say is that you can use some time to this sets. Suppose, I call this set as j suffix n will i call this set as j suffix n. So, I will take one set a as the set of all those i in this j suffix n. Such that this big M i minus small m i is less than delta. For those i I know that big M i star minus small m i star is less than epsilon. Then the remaining set, suppose I call this b for b this does not happen for v this does not happen. So, one can say that I will say sigma i belonging to j n this does not happen means what? Big M i minus small m i is bigger than or equal to delta big M i minus small m i is bigger than or equal to delta.

So, now come back to this sum here this U P h alpha minus l P h this is sigma i going from 1 to n this m i star is delta alpha etcetera. I will split this sum into 2 i belongs to a and i belongs to b. That is what I will say is this is same as sigma i belongs to a big M i star minus small m i star into delta alpha i.

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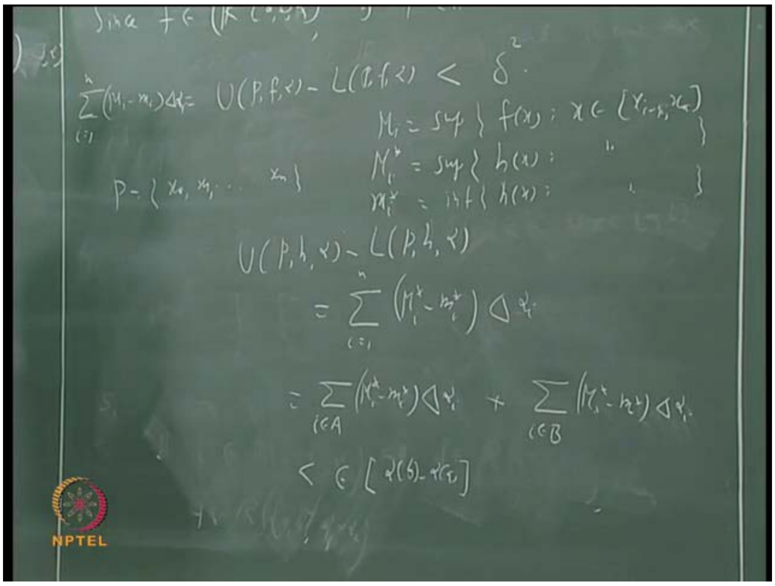

Remember, A union B is j n every i from one to n has to either belong to A or to B. Every i has to either belong to A or B some i going from 1 to n is same as sum of those indices, for which i is in a plus sigma i belongs to b big M i star minus small m i star into delta alpha I, do you agree? Now, out of these two parts I know that at least for this part this big M i star minus small m i star is less than epsilon. So, we can say that this is less than epsilon into epsilon into sigma i belonging to a delta alpha i and sigma i belonging to a delta alpha i. You can say sigma i belong less than or equal to sigma i belongs to j and delta alpha. So, which will be ultimately less than or equal to alpha b minus alpha a.

So, this part is less than epsilon into alpha b minus alpha a. Now, the question remains what do we say about this? Now, here you can easily say the problem we cannot say anything about big M i star minus. It is not less than epsilon, but since we cannot say anything about this. What we will do is that we will have a control over this claim sigma delta alpha. Let us see how does that follow we can see one more thing, one thing here that is again look at this. Look at this equation here sigma i going from 1 to n m i minus m i delta a. Let me write this equation once again here, what is the equation i will write in the reverse way.

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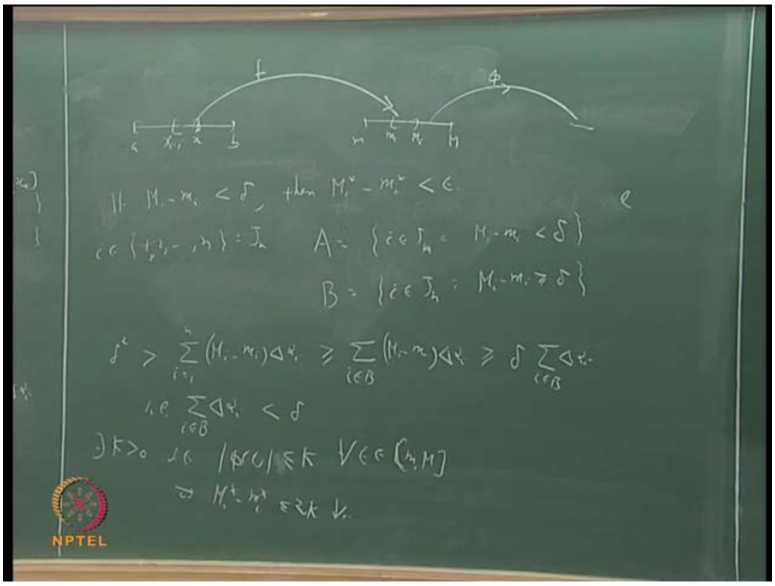

This delta square is bigger than sigma i going from 1 to n big M i minus small m i into delta alpha i. That is the like I just written this equation once again, that is all not equation inequality as we taken this inequality once again. Now, do you agree that this, suppose I take this sum i going from 1 to n that has to be bigger than or equal to sigma i belonging to b big M i minus small m i delta alpha i. Because, you are just taking few of this terms ignoring those terms, which belong to for those terms, which corresponds to i belonging to a, but what we know if i belongs to b? What we know about this big M i minus small m i that is bigger than or equal to delta. That is bigger than or equal to delta that means i can each of this sum is bigger than or equal to delta times this delta alpha i.

So, what we can say is this is bigger than or equal to delta times sigma delta alpha i for i belonging to big M. So, what does this prove that delta times sigma delta alpha i is less than delta square, which means this sigma delta alpha i is less than delta. It is sigma delta i that is sigma delta alpha, of course it is not for i going from 1 to n. It is only for i belonging to be that is for those intervals, which corresponding to i belonging to b. If you take the sigma delta alpha i for i belonging to b, that sum means less than delta that sum is less than delta. Now, you will understand why would 2 delta square here. Now, we understand 2 delta square here, of course delta to the anything bigger than the power 1 would have done, but this is all right.

So, if I now look at the sum I do not know anything about this m i star minus m i star, but I know this sigma delta phi that is less than delta. We all ready assumed that delta is less than epsilon, we all ready assumed that delta. So, only thing remains is what do we do about this m i star minus small m i star, but you can say that phi is a after all what are this m i star and big M i star. Those are the values of this h x, but h x is phi of f x h x is phi of f x. So, phi is a continuous function phi is a continuous function, so continuous function defined on this closed mod intervals. So, phi is bounded, so as you have done that I can say that always take some number key. Such that mod phi x is less than or equal to K in their interval small m to big M.

So we can say that there exists K bigger than 0 such that mod phi t is less than or equal mod phi t is less than or equal to this big K, for all t in the interval small m to big M. Then what can you say about this m i star minus big mi star minus small m i star do you agree that this will be less than or equal to 2 K. Each of them is less than or equal to see m i star. Remember, again look at the definition what was M i star? M i star is supremum of h x. What is h x? h x is phi of f x. So, phi and we know that for any x mod phi ah for any t mod phi t is showing particular phi of f x is mod phi of x less than or equal to K for anything.

So, in particular m i is mod m i is less than or equal to K mod small m i star is less than or equal to K mod small m i star is less than or equal to K. Hence, we can say that this, therefore this means that big M i star minus small m i star is less than or equal to 2 K for this is true for all i not necessarily for i in b, but for i in A R etcetera.

Basically, what we say is that since mod phi t is less than or equal to K. If you take any two values of t, then the difference or sum or whatever will be less than or equal to 2. So, we will use that here, so this second part is less than 2 K times delta, because each of this is less than 2 K less than or equal to 2 K. So, that 2 K will come outside, so that 2 K times sigma delta alpha i going to be all right.

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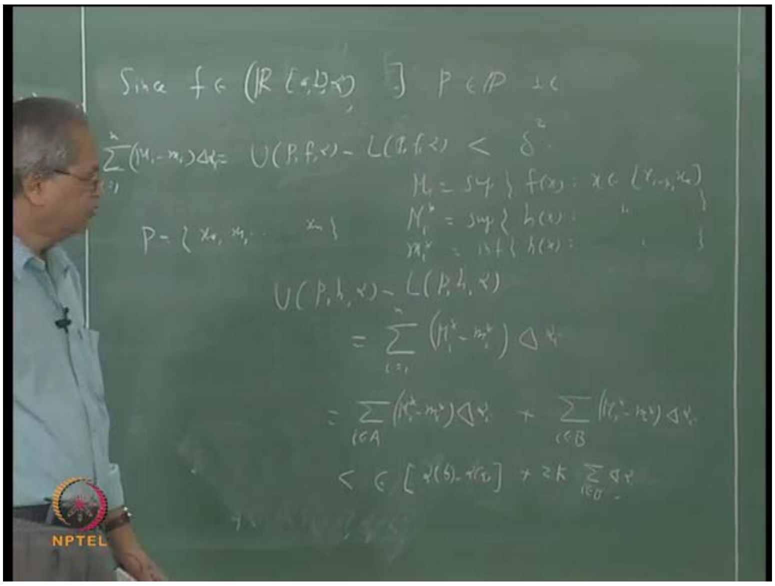

I will write it one more statement to this is less than 2 K times sigma i in b delta alpha i. We have all ready shown that this part sigma delta alpha i is less than delta we sigma. We have already chosen delta to be less than epsilon. So, in fact here itself I would have said that sigma i belong to be this part is less than epsilon.

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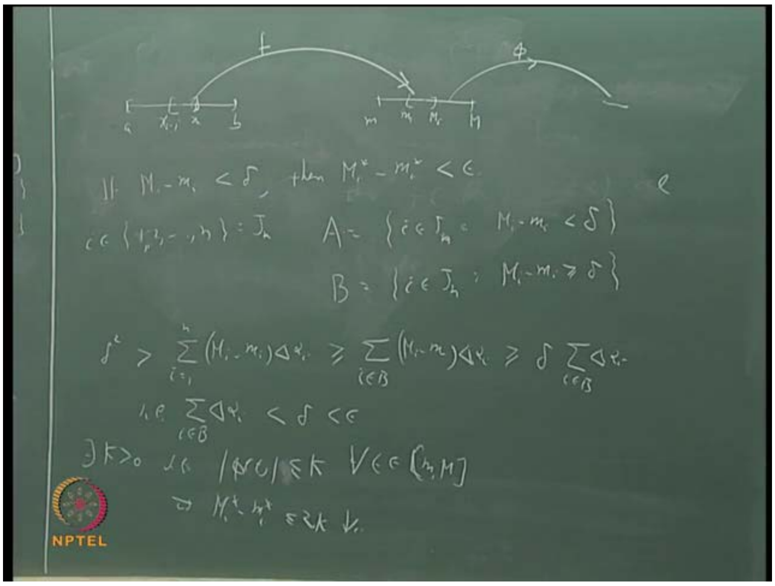

So, what does it mean? That this whole thing is less than epsilon into alpha b minus alpha a plus two times K.

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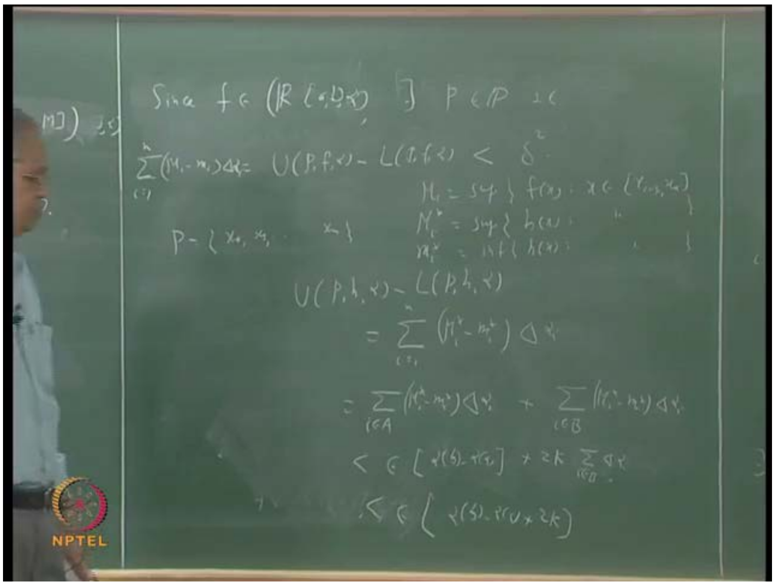

So, let us again take, what did we show? We started with an epsilon bigger than 0, corresponding to that we produced a partition P. Such that with respect to that partition P U P h alpha minus n P h alpha minus n P h alpha is less than epsilon into this number. Remember, this number does not depend on anything. It is alpha b minus alpha a plus 2 K, 2 K is the just number it depend on the function phi. It does not depend on the partition.

So, this is just the absolute constant multiplied by epsilon. So, that shows that the function is h intergrable. Of course there are so many things that are involved in this proof. Mainly, for example choosing this partition with this kind of property. Then observing that the sum of that those delta alpha a i becomes less than delta etcetera. So, this construction is involved.

So, what I suggest to you how close look at this proof once again on your own afterwards. If you have still any doubts we will discuss those things. Now, I said that once you prove this you can take care of many other things, each for product functions etcetera. Let us see how that can be done. Now, I will write it is as corollary there are several things one is this.

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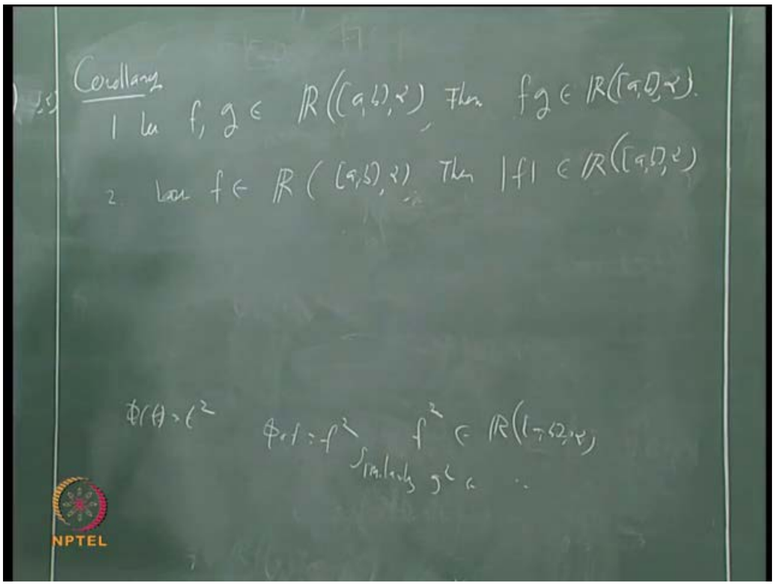

Suppose, we take two functions let f g belonging to this R a b alpha that is f and g both are integrable. Then earlier I had remarked that this is a vector space. Now, when you prove this it will also mean that a product of such elements is also there. So, not only it is a vector space it is also a ring. Such things are called algebras a vector space on which a product is also defined. That product should also satisfy some extra actions with respect to the vector space operations. Then that is called an algebra, but let us not go into those things. Let us see they how this can be proved. In fact if you have first look you will say that that is non relation should between this and this.

Then you cannot get f g by given two functions f and g you cannot get that product by composing with some function, but that is not how we proceed also, what we do is as follows. Suppose, I take phi of t is equal to t square and compose this phi with f, what will be the result? It will be f square. It is if you take the function phi t is equal to t square and then phi, then phi composed with f is nothing but f square. Because, phi of f t is nothing but f t square, so that is same as f square.

That is how we will define the function f square now, but this is a particular function phi t. Whatever is the interval phi t is equal to t square is a is a continuous function, which means if f is integrable f square is integrable. So, this means that f square is integrable f square belong to this R a b alpha, but then in the same way I can say g square is also integrable. If you compose phi with g you will get g square. So, similarly, g square also belongs to this.

So, what we say is that whenever a function is integrable it is square is also integrable. Then f plus g square is also integrable, we also we all ready shown that f is if f is integrable. This is integrable then f plus g whole square f plus g is also integrable. Hence, f plus g square is also integrable. So, you can either say from here that this you can take this minus f square minus g square is also integrable.

That is nothing but two times f g, so this fine. So, this divided by 2, so all that we are using here is that, if a function is integrable it is square is also integrable. We all ready proved that f plus g is integrable. So, f plus g is square is integrable and f square and g square is integrable anywhere. So, this is also integrable, again we have all ready shown that the sum of the two functions is integrable and all that.

So, this follows one more thing in a similar fashion. Let f b integrable then mod f is also integrable. Now, what is the obvious way to show that? you just take phi t is equal to mod t you just take phi t is equal to mod t, that is the continuous function. Also, phi compose with f is nothing but mod f phi composite the things, but in this case, we can say something more about this integral of mod f, that is following and we can say this. If you look at the absolute value of the integral that is integral a to b f d alpha.

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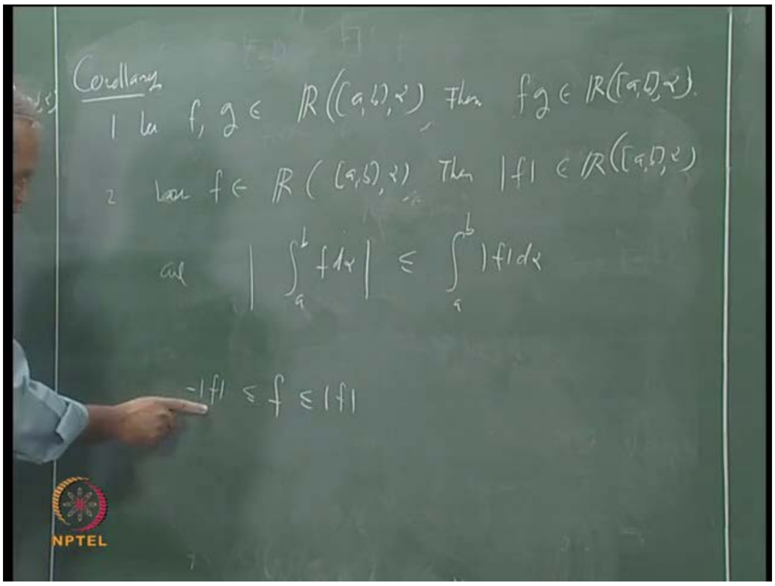

Then that is less than or equal to integral a to b mod f d alpha, but this is this part is trivial. Once you understand that mod f is integrable. Remember, we have all ready proved that if you take two functions f is less than or equal to g. Then integral of f is less than or equal to integral of g.

We just use that fact for example, we know this that f is less than or equal to mod f. Whatever be x f x is less than or equal to mod f and similarly, it is bigger than or equal to minus mod f. So, integral of f will be less than or equal to integral of mod f. It will be bigger than or equal to integral of minus mod f. So, that shows this right integral of f plus or minus integral of f b alpha is less than or equal to integral mod f.

So, for example this shows that integral f is less than or equal to integral of mod f. This part shows that integral of minus f is also less than or equal to integral of mod f. So, that is and whichever is the bigger is the absolute value. So, this is less than or equal to integral mod f. So, we will stop with that, tomorrow we shall see the integral as a limit of a sum. Till now we have been only discussing the integrals in terms of upper sums and lower sums. Though, we have introduced this remark stages sums, we have made any use of those till now that relations, we shall see tomorrow.