Real Analysis
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Lecture - 04
Countable Sets

Well, we were discussing finite and countable sets in the last class let us recall again basic definition.

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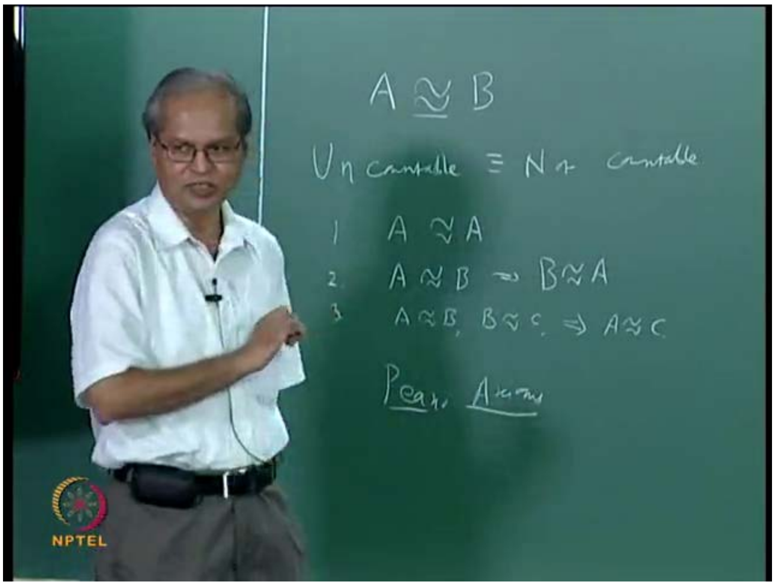

Namely, that we had said that two sets A and B are numerically equivalent and we had used this notation A is numerically equivalent to B if there is bisection between A and B. Then we had said that A is finite, if A is numerically equivalent to some segment 1, 2, 3 and etcetera and A is countable. If A is numerically equivalent to the there is countably infinite if it is numerically equivalent to n countable infinity is same. As I said some book called a numerable and countable means finite or countably infinite and, similarly of course uncountable there is something I did not mentioned last time.

Uncountable this simply means not countable that means neither are finite nor countably infinite or to be more precise it means that there is no bisection between that set and set of all natural numbers that is that is the meaning of uncountable. Now, the whole questions is how it is one show that a certain set is a finite or countable or uncountable or anything like that and let us now see some techniques of that. In this class let me again remind you that in the last class we showed it every infinite set has a countably infinite subset and using that we should that every infinite set is numerically equivalent to a proper subset.

So, this is a property we distinguishes between finite infinite sets, no finite set can be numerically equivalent to a subset of it whereas every infinite set is numerically equivalent to a proper subset of it. Before proceeding further, let us first observe let this relation is basically an equivalence relation that is something that is fairly easy to see. So, we will see this thing that is first of all A is numerically equivalent to itself that is clear you can take the identity map. Secondly if A is numerically equivalent to B then B is numerically equivalent to A that is also easy to see if f is a bisection from A to B you can consider the it is inverse function f inverse that it will be a map from B to e B to A and that will also be a bisection.

Similarly, the so called transitivity if A is numerically equivalent to B and B is numerically equivalent to C these two things should imply that A is numerically equivalent to C. How can this last thing we proved just because then account that is if is A max from A to B which is bisection and if g is a map from B to C, you consider g compose with if that will be a map from A to C and that will be a bisection. Now, this is of course a very straight forward thing to see, but what is the great use of this it is the fully that if you have shown some already that some set is finite.

If you know that there is bisection between A and B it will give a automatically that B is also finite, similarly if A is countable then B also will be countable or if A is not countable un countable then B also is uncountable. So, you can use the information about the sets that you have already proved using this fact we will see how this can be done. Now, to first of all let us get certain some working list of countable sets, we already know off course that N is countable. Countable means, in this case countably infinity and then let us comes to prove certain things about countable sets we need some properties of this set of all natural numbers N.

Now, how does one prove properties of natural numbers N, I mean if you want to proceed very logically you have to start from the definition of N. How N, how this null set of all natural numbers of N is constructed and stander of way of that is following what is called is Peano actions it is called Peano actions. This is a method of constructing the set of all natural numbers, but that will be very time consuming and we shall not go in to that kind of instruction of natural numbers. If you are interested I will give you some references where you can find these things, but there are two properties of natural numbers which we shall be using very often and, so let me list those properties.

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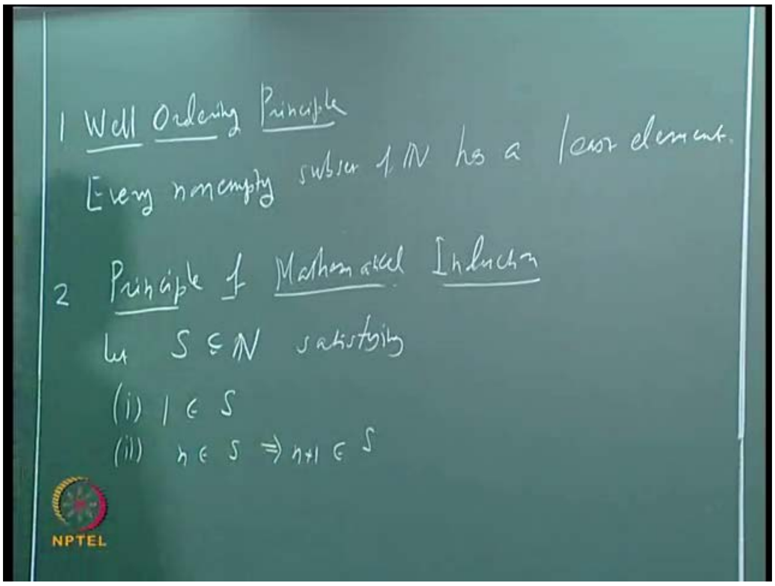

First is for this called Well ordering principle, the principal is very easy to state it simply says that if you take any non empty subset of N then it has a least element. So, every non empty subset of N, N has a least element, least element means what that element is less than or equals to every under element in that set. Of course using this property when we can show that least element is unique, the second property is what is called principle of mathematical induction.

Mathematical of course these are principle which you have used right from your school days to give various proofs, now what does these principles say, of course you would have used in some form let us, let us see in what form we are going to use it. The principle says this suppose you have subset of N suppose S is a subset of N, subset of N and it satisfies these two properties for other property, the first property is that one belongs to S. Second property is that if some natural numbers m belongs to S then m plus 1 also belongs to S, so there is a second property that is n belongs to s implies n plus 1 that is if n belongs to S then n plus 1 belongs to S.

So, if you consider a set satisfying these two properties then what should happen that is set is must be set of all natural numbers then conclusion is this, then S is equal to N any set satisfying these two properties must be the set of all natural numbers. This is something you have used several times improving so many things may not be in these particular form for this slot and we shall be using these two properties of natural numbers.

These two improving that several sets are countable since that is going to we use again and again I have mentioned it again you can take this as an exercise. Actually, these two principles are equivalent what it means that well already in principle implies this and principle of mathematical induction implies this. So, I will give that to you as an exercise it is an interesting exercise to try, show that, show that 1 is equivalent to 2.

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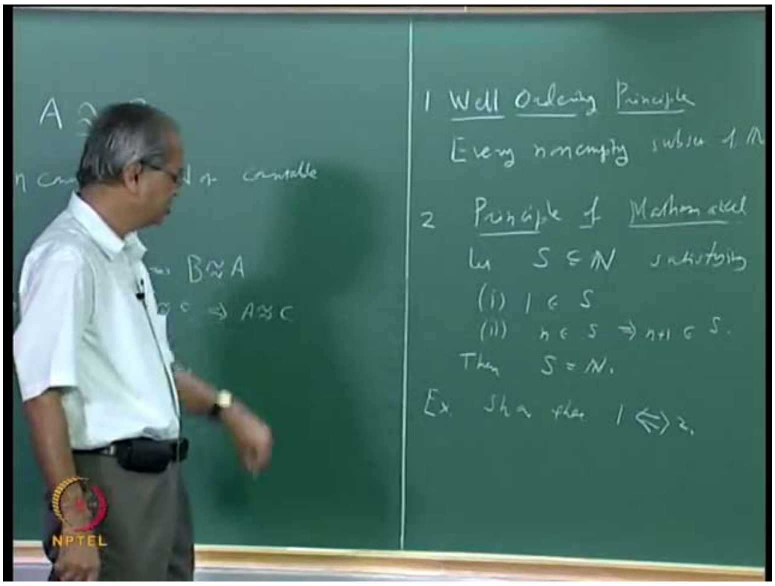

That means if you assume well are in principle you can prove this, on the other hand if you assume principle of mathematical induction then you can prove well ordering principle I will not discuss that proof. Here, it is try it on your own if you get held up we shall discuss, here interesting exercise 1 its 1. Now, how we are going to use this as I said we shall use it to show that several sets are several familiar sets are countable to begin with let us just start with this.

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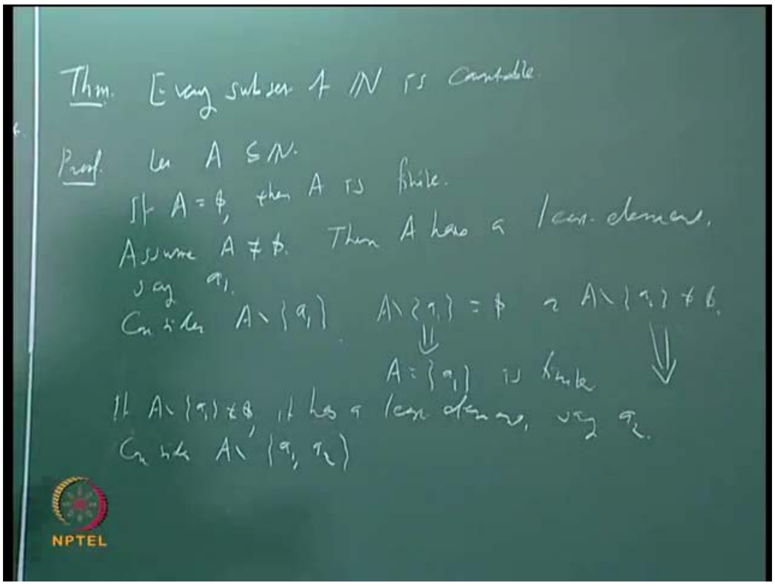

By the way why is this called well ordering principle, the reason is following suppose you let you take a non empty set let us say A a is a non empty set then it has a least element. Suppose you call that element let us say a 1 you remove that element even then a minus, even either it is the whole of a or it is still non empty if that is the case that remaining set also has a least element. You can call it A 2 then again do the same thing look at a 1 a 2 either that is the whole of a or a minus a 1 a 2 is again non empty, if it is non empty then that also has a least element you can call it a 3.

So, this way you can order the element of a, you can order the elements of A and that is why this is called well ordering principle and this is inside the principle we shall be using in some of the proofs. So, in fact in these particular proof which let us what is what does this theorem say it is that every subset of a not, not a every subset of n is countable we already know that n its self is countable. So, let us see how this can be this can be proved, so let A be a subset of m if A is empty there is nothing to be proved right, so if A is empty then a is finite nothing to be proved.

Finite means countable because over deferral are countable means finite or countable infinite, so assume A is non empty once you say A is non empty I can invoke this, I can invoke this. So, if it is non empty it has a least element then A has a least element I do exactly the thing that I mentioned just few minutes before that least element I will call a 1. So, that least element a has a least element suppose I call that element a 1 all right now consider A minus a 1, consider a minus this equal to a 1 then again there are two possibilities either A minus a 1 is empty or non empty. So, A minus a 1 is empty or A minus a 1 is non empty if this happens if a minus a 1 is empty what does it mean, it means A is just inviter a 1 it is a it has only one element it is, it is a fine set.

So, this implies that is singular a 1 is finite what does this imply what will this be if it is non empty you again apply the same principle, here by this that will have a least element and I will decide to call that least element it. So, if a minus a 1 is non empty it has a least element, it has a least element and suppose I decide to call that least element a 2 what is to be done after that you concentrate a minus a 1 a 2. So, consider A minus a 1 a 2, consider A minus a 1 a 2, now I will not write everything in the board.

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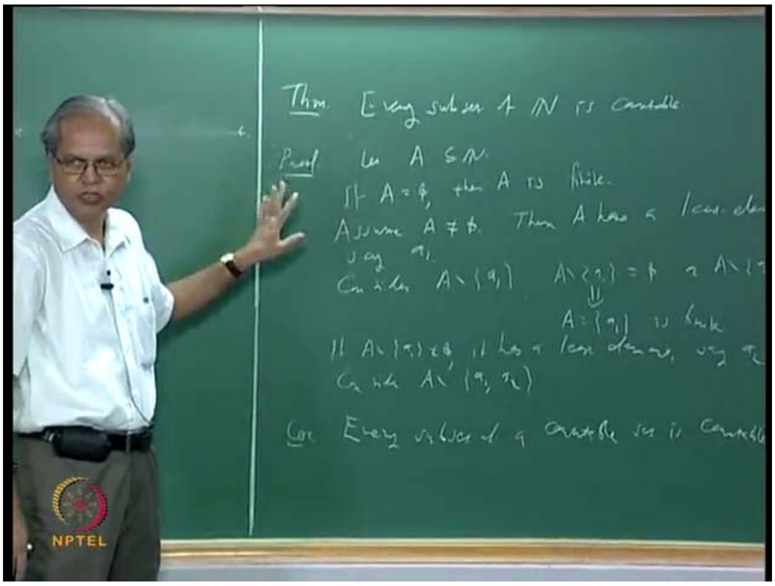

Let us assure A minus a 1, so what are the possibilities here either this is finite or infinite if it is finite the again discuss the same. Then A is just a 1 a 2 it is a finite set consisting of two elements if not again consider least element call that element a 3 and proceed. So, this tells you technique of how to proceed with this suppose we go on like this what are the possibilities there are two possibilities that is after n steps this may stop.

Then that is A minus a 1, a 2, a n will be empty in which case a is same as a 1, a 2 n, so then it is a finite set it is, it is said one are corresponds with the 1, 2, n. If this does not end that means it every stage n, A minus a 1, a 2, a n is non empty you can consider least element of that set call it a n plus 1 and proceed in which case A will be equal to a 1, a 2, a n infinite set and that will be in bisection with n. So, it is countably infinite that is clear in either case either this procedure, if this procedure ends after a finite number of set it will be a finite set consider of an elements.

Otherwise, it is an infinite set and that infinite set is 1 1 correspondence with the set of all natural numbers that will be countable infinite set. So, it is either a finite or countable infinite which is basically that it is countable that is clear, now let us come back to this principle which I had seen. Once you prove that certain set is countable anything which is in 1 1 correspondence with that will also be countable. So, using that fact we can prove the following, I will call it as a corollary, now that we have proved it every subset of n is countable does it follow from, here at every subset of countable set is countable, here that is what I said just now.

If you prove something for a 1 set, you can say the same thing for the any other set which is in 1 1 correspondence with that, which is numerically equivalent to it. So, I will say it is corollary of that is every subset of a countable set is countable, now in order to show that a set is countable we have seen that there are two ways. One is that you can show that it is in 1 1 correspondence with set of all natural numbers.

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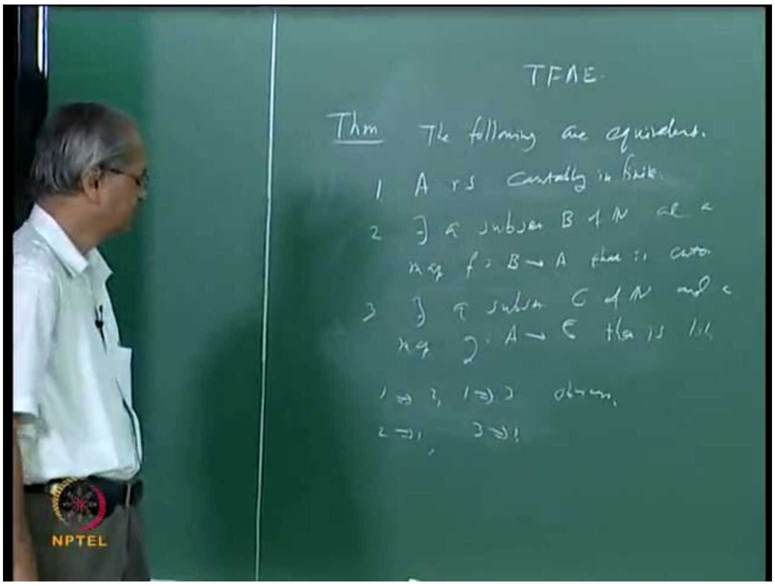

Let me state that, here I will only consider the case of count ably infinite set if a set is finite I mean there is nothing must to be proved. So, consider the set a then I want to say the following things are equivalent, the following statement are equivalent in fact this is something we are going to use very often. So, there is again a very notation used for this it is this TFAE, the following are equivalent we will just use this TFAE to say that following are equal. Since we are going to use it again and again we will use this standard notation, it is fairly common it is used fairly often also and what is that first I want to say that a is countably infinite.

Then there exists a subset B of N and a map F from B to A that is onto all right and similarly we can say that there exists a subset let us say C of N and map. So, they are called g from A to C a map g from A to C that is one word, what is it mean that these three are equivalent that is and how what is the practical use of this theorem there if I want to show that a set is countable. We have seen earlier that by definition I should say there exists 1 1 correspondence from A to N what this says is that in fact one can show something less.

For example, it is enough to show that there is a map from A to some subset of natural numbers which is 1 1, that is enough there is you can show g there is a map g from A to C not this C this will be in complex number A to C that is 1 1. You can find a subset of natural numbers and find a map from that subset to A which is on 2 that is sufficient, that is sufficient. Now, first of all it is clear that 1 implies 2 and 1 implies 3 you can just take the subset as the set of all natural numbers itself, you can just take this B is equal to N and then there is a map from N to A which is 1 1 and on 2.

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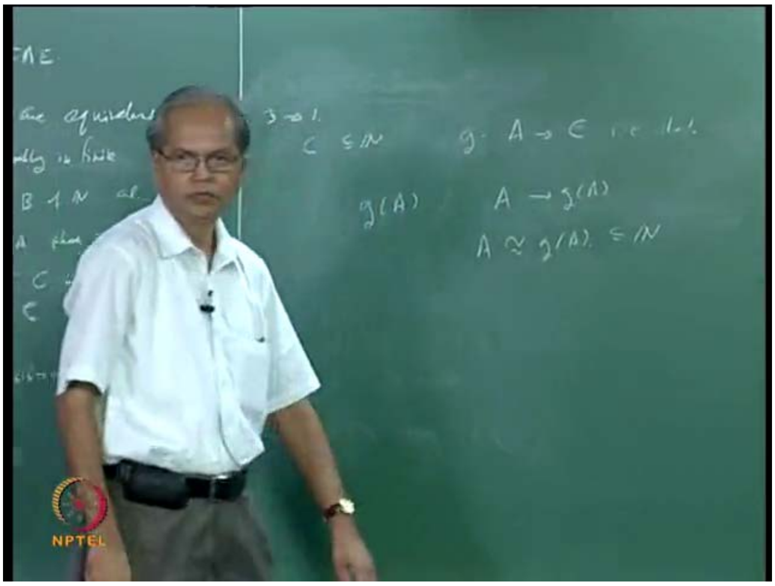

So, in particular on two similarly, here you can take this C as N, you can take this C as N, so real issue is to prove that 2 implies 1 and 3 implies 1 to show that, see let we just 1 implies 2, 1 implies 3 obviously clear. So, the issue is to prove 2 implies and 3 implies 1 all, so we need to prove 2 implies and 3 implies 1.

Since 3 implies 1 is easer take that first what does this say that there exist a subset C of N, there exist a subset C of N and map g from A to C that is one word, that is there exist subset C of N and A, A back g from C to A that is 1 1. Only thing is that this not may not be on 2, only thing is it this map g from C to A may not be on 2, but does not matter see instead of I can, I can consider no I think I that not C to A, A to C, sorry A to, but one we can say is that even if it is not on 2 N. We, can consider this g of A, you can consist g of A then A to look at the map A to g of A, look at the map A to g of A we already know that it is 1 1.

Since we are not there to taking only the elements of g of A that is on 2 we are taking only elements from A to g of A that is on 2. So, A and, so what it means is that a is numerically equivalent to g of A, A is numerically equivalent to g of A all right is this g of A subset of N, subset of N. Now, look at this theorem every subset of N is countable we have already show every subset of N is countable, so in particular this g of A is countable and hence countably infinite because it is not finite.

So, it is infinite, so it is countably infinite that shows that A is countable infinite fine, so in a similar way you can show that 2 implies 1, we was what we can think of is the map which takes if from B to A that is this, clear. First of all 3 implies 1 it is it is enough to just consider a subset of all natural numbers and show that there is a map from A to C that is 1 1. Similarly, it is enough to consider subset of B of N and show that there exist some map B to A that is 1 2, see what you can do is that, you can see the mat again similarly as we followed, here the map from B to A is on 2, it is not 1 1.

But, what we can do even if it is not 1 1 from B to A, I can think of a subset B of A and map from B of A which is both on 2 as well as 1 1 and in that subset discountable. How to consider that map from B to A which is 1 1 and on 2 either you can take, either you can use action of choose or you can use Bill Ordering principle. That is you can take, that is you look at an element A consider its inverse image in B that will be some non empty set if that is a subset containing of just one element there is no problem. If it contain many more elements you choose the least element of that and do which for a each element in A that why you will get a subset of B which is in 1 1.

Correspondence with A that will be a subset of natural number I have given you the main idea you can develop the proof with this wanted. Now, we just consider of one more extension of this instead of considering subset B of N that is what I want to do is that I want to replace this N. But, any countable infinite set instead if taking N, I can say that there exist subset B of some countable set and A map B from A, from B to A witch is on 2.

So, again by the principle which we thought of just now this is okay, similarly I can consider any other subset from A to any subset C which is instead of taking subset C of N, I can take any countable set. Say that there exist some map from A to C that is 1 1 that will be enough, that is enough and this is a principle that we are going to use to show that several well known sets are countable or not countable. In fact, this will be your state that countable, so first of all let us just take an illustration on this.

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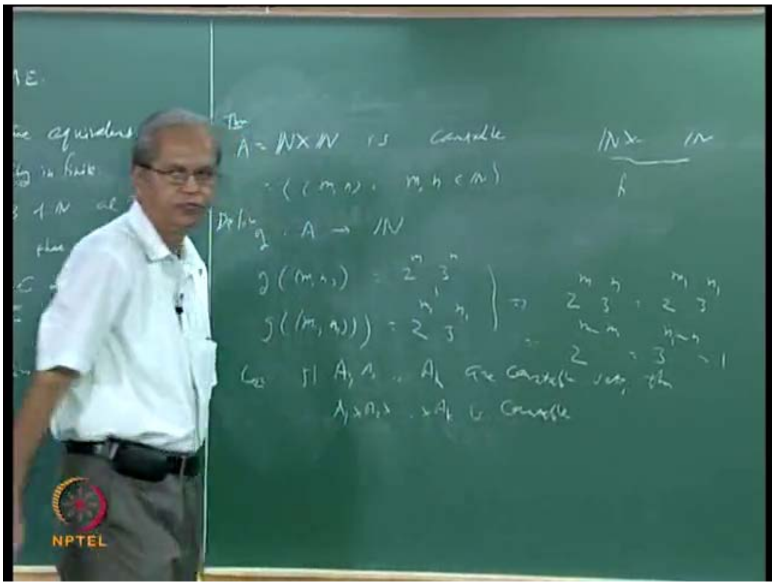

So, I want to say that N cross N is countable and of course since it is easy to say it is infinite we shall show it is countably infinite N cross N is count ably infinite I will use this 3 if I have able to say find subset of natural numbers and a map. So, suppose I called this A is N cross N if I am able to find a subset of natural numbers and a map g from this set A to the subset C which is 1 1 that is enough. What are the elements N cross N yes order pairs of a natural numbers order pairs of, so N cross N this is the set of all the elements of the form m, n where m and n both are natural numbers.

So, I consider a map it is say g from A in fact I will take this sets, here in its C as, so I will take a map g from A to m as follows we define g from A to N as follows g of this m, n g of this m, n I will take this as say 2 to the power m into 3 to the power n. So, this is a map from A to N that is it 1 1 see how does one show that something is 1 1 suppose you take some other ray m 1 n 1 then g of m 1 n 1 is 2 2 the power m 3 to the power m 1 3 2 the power n 1.

So, if let us say we consider a g of m 1 n 1 this will be 2 to the power m 1 multiplied by 3 to the power n 1 suppose these two are equal g of m n is g of m 1 n 1. Then that means 2 to the power m into 3 to the power n is same as 2 to the power m 1 into 3 to the power n 1, will it follow from that that m is m n is same as m 1 n ,1 that is suppose these two are equal. Then that will imply that 2 to the power m into 3 to the power n is same as 2 to the power m 1 into 3 to the power n 1.

So, what I can say from here is that this will imply that 2 to the power m minus m 1 is same as 3 to the power n 1 minus n. But, some power of 2 is equal to the some other power, some power of 3, so when can this happen this only one way that is can happen that is that must be 1 that is if this is to happen then this must be equal to 1.

That means n minus m 1 is 0 and n 1 minus m is 0 which is same as say that m is equal to m 1 n is equal to n 1, so does it prove whatever you wanted to say you have the map from N cross N into n which is 1 1. So, that is, this is, so that N cross N is found that is that was our 3, that was our 3, what if I take N cross N cross N, I can say I can take a map 2 the power m 3 to power m. So, let us some 5 to power of something and what is the Bose general, Bose general statement I can take a finite number of times countable products, finite number of times countable products that is N cross N cross N finite number of times.

In other words, a product of N cross N cross N taken finite number of times is countable set and all that what is, so peculiar this 2 and 3 it is just that get they are prime, it is just that get they are prime. You can just choose a finite number of prime and construct a map like that, so a product of a quotation product of n taken with itself of finite number of times is accountable set yes, no. We will come to that little bit late as far as finite, so f quotation product of a finite number of copies of n is countable.

So, this is the theorem, here it is coronary is that I can replace n by any other countable set in other words if I, what I want to say is that suppose if A 1, A 2 let say A k are countable sets are countable sets then A 1 cross A 2 cross A 3 is countable. In other words, you take a finite family of countable sets and then take their quotation product then that is countable. Does it follow from here all that we are doing is replacing N by any countable set each of this A 1, A 2, A k is equivalent to N, so replacing N by, now let us look at the union.

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In union we can take even a countable finite family, so let us say that that in A i, i belonging to let us say N be a family of countable sets and let A be there union A B equal to union of A i, i belonging to N then A is countable. Then A is countable then or in words, it should take a countable family of countable sets, then their union is countable that is union of countable family of countable sets is countable, that is what we say in words.

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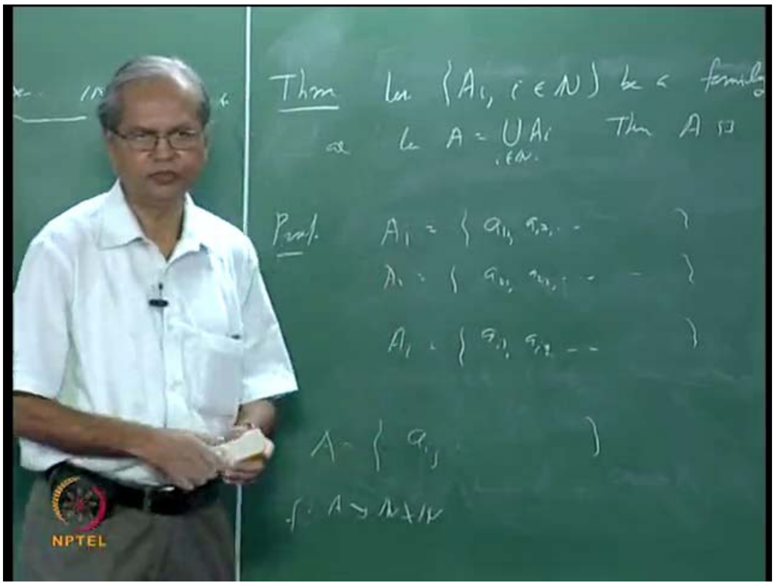

The proof is very simple we can view this union A as a subset of N cross N we can, we can view this union A as a subset that is in other words we can establish A 1 1 correspondence between A and a subset on N cross N. Since we have shown that N cross N is countable very subset of it is countable and hence that is countable and how does one show that let us look at the set A 1 we know that A 1 is countable. So, it is finite or countably infinite, so it is either finite or it is in verbal correspond with N, so suppose it is finite I can write it is element as let us say a 1 1, a 1 2 etcetera.

If it is finite, it will stop somewhere if it is infinite it will continue it will be in 1 1 the correspondence with N and similarly for A 2, I can right elements of a 2, a 2 1, a 2 2, etcetera. If it is finite it will stop at some stage A 2 and otherwise it will continue and do it for any i. So, A i will be a i 1, a i 2 etcetera, does it more or less that is the proof, now A is union of all this a i s, so I can write a as nothing, but the same of all elements. That is from a i j because if you look at all of them all elements can be expressed as A i s and can you see an obvious map from this into this N cross N.

For example, a map F from A to N cross N, what is the map A is equal to i, j, so that map is 1 1, so A is in 1 1 correspondence with some subset of N cross N and hence A is countable. We do not have to show that is on 2 we already seen that it is essential where used in this principal thing, so let us again recall what did we proved so far that if you that any.

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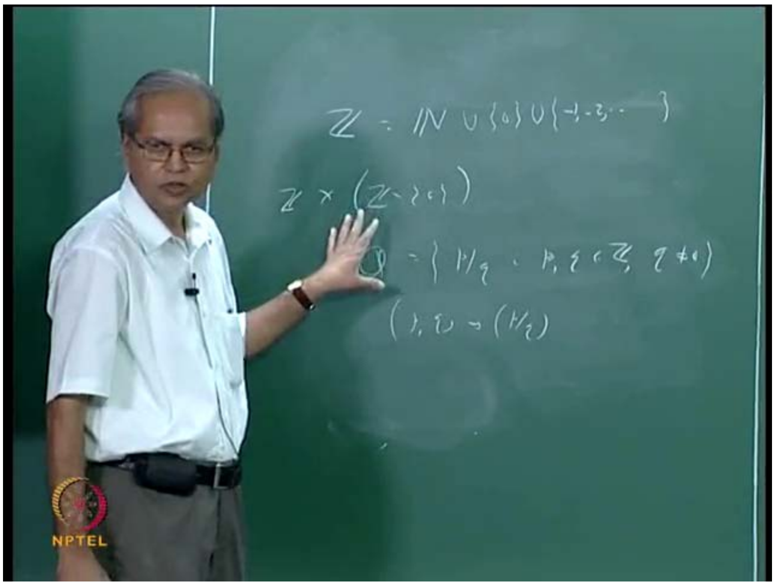

Now, let us go some familiar sets, go to set Z, Z is the set of all integers what are the elements of this they are all integers, so what are, so they are let us say first is set of all natural numbers. So, what are the etcetera elements other than this 0, this I said this union 0 union minus 1 minus 2 etc this is anyway a countable set we know already this is a finite set what about this is in verbal correspondence with N it is countable N it is. So, this is finite union of countable sets, this is finite union of countable sets, so it is countable. Now, we have got see how did we begin we wanted to discuss how to show that certain set is countable or finite.

So, after proving these theorems we have, now found one more technique see what were the technique showed so for either you can that it as 1 1 correspondence with N or with some subset of N or with some set which you have already shown to be countable. Now, we have got one more technique namely suppose you are able to show it is a union of countable sets either finite union, or countable union of countable sets that is also countable. So, this is the one more technique to show that some set is countable you can show that it is union of countable sets either finite union or countable, in this case we have shown that it is a finite union of countable sets.

So, Z is countable, so Z minus equal to 0 that is also countable, now suppose that I take the product cross product, Z cross this that is also countable set. Now, can you see that obliviously this is look at set of all rational numbers set of all rational numbers though what is that set, by definition it is the set of all elements of the form p by q where p q are integers and q not 0. That describes all rational numbers, that describes all rational numbers does it follow that this is countable because we have seen that this there is obviously relation between this and this set, there is one problem.

Here, if a presentation p by q is not unique 2 by 3 is same as 4 by 6 that is same as 8 by 1 2 that is not the problem, we can see that once, so that means you have a map which goes from, here to here. I can say that I can take the map p q goes to, p q goes to p by q that is a map from, here this to q this to q that map is 1 2, but may not be 1 1 that map that map is 1 2. But, may not be 1 1 because as I said just, now this 1 3, 2 4 they will all hold same rational number, but we have seen that is the problem you all look into statement 2 of our theorem if you a map which is from a countable set which is on 2.

But, not necessarily 1 1, still this set is countable, in other words we can q, we can say that q is in 1 1 correspondence with some subset of this, q is in 1 1 correspondence with some subset of this. Since, we already show that this is countable all it is subset are countable, so the set of all rational numbers are countable it is clear, now before proceeding further I will give you one exercise.

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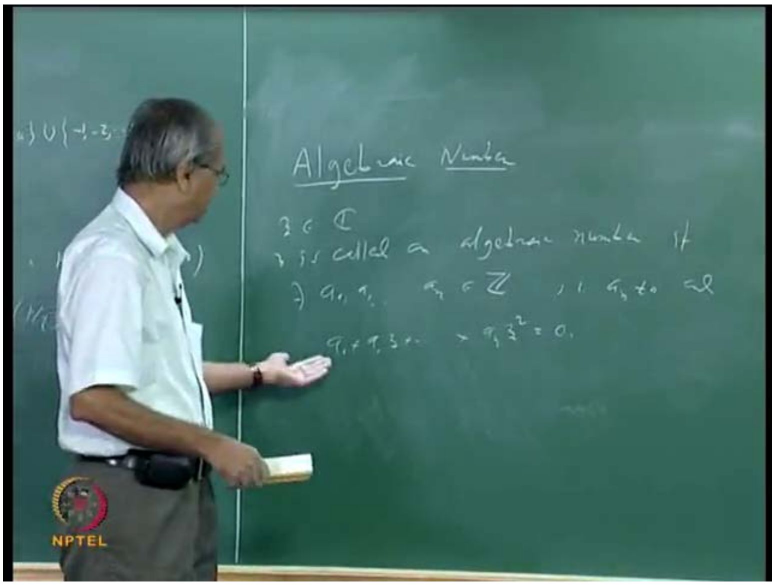

Let us, to take the considered excursive we need the definition of what is called algebraic number have you heard of this earlier what is meant by algebraic number does not matter, we will definite it. The number is said to be algebraic number if it a root of a polynomial which integer coefficient, so it can be a really number or complex number. So, it is called, so let us say that Z is a complex number, it is set of so Z is called an algebraic number, algebraic number.

If Z is root of polynomial, if Z is a root of polynomial with integer coefficients, with integer coefficients which is same as the that means polynomial means what polynomial is a polynomial is a first of the form of a naught plus a 1 z plus a, a 2 z etc all that. Now, what we want is that is all this a naught a 1 a 2 plus b integers, so what we can say is that if there exist let us say a naught a 1 a n belonging to Z of courses all of them should not be 0. So, let us just say a n is 0 that a n minus 0 and should happens is that this Z should be root of the polynomial.

That means a naught a 1 plus z plus etc that means a naught a 1 plus Z plus a n Z to for end this should 0 is the root of a, of course it should take all them to be 0 then it will be in that every number. So, we do not want it that is why we take non zero polynomial and why do not we ensure that the polynomial is non zero at least one coefficient is non zero, we shall ensure that a n is not 0. So, this covers everything, now how do we, how do we know that certain number is algebraic or not obliviously all integers are algebraic number, we can just, we can just take the polynomial.

For example, if you number 2 i can other polynomial Z minus 2 it is a root of that if i take root 2 i can Z square minus 2 that it is root of that, it is a root of that, so root two algebraic number. Now, to decide whether a particular number is algebraic or not is not very easy, for example root of in case of root 2 is fine.

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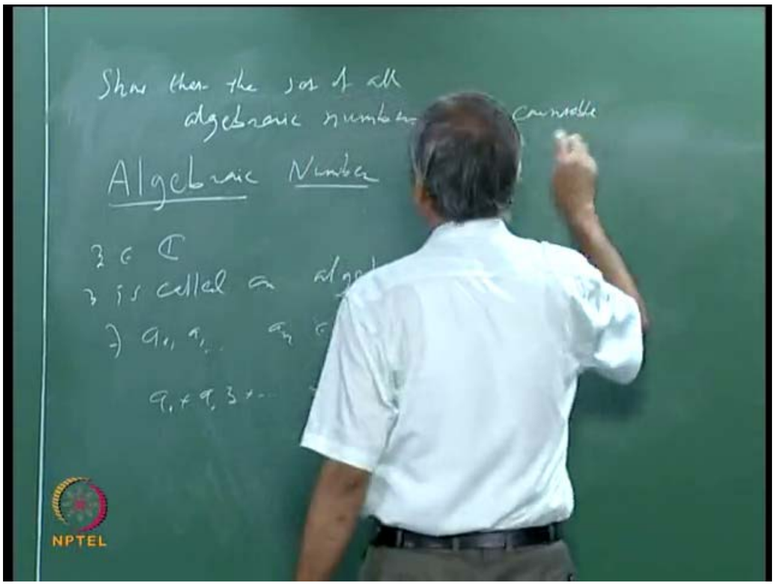

But, suppose I give a number like e or pie you cannot easily whether it is algebraic or not in fact it is not that they are not algebraic numbers, but that is very difficult to prove. Now, what is problem the problem is this show that the set of all algebraic numbers is countable that is the problem show that, show that the set of all algebraic numbers is countable. I give you a hint, I will not give you the full solution, I will just give a hint what do you what are the ways of showing that the set is countable.

The most recent way that we saw is you can show that is a union of countable sets or union of finite sets whatever, so I will say that you try to show that this is a countable union of finite sets, you can show that this is the countable union of finite sets. Now, you have see, so many examples of the sets which are countable, which are countable that is finite or countable infinite. Now, this may raise a question whether there exists uncountable sets or not, whether they do exists uncountable set or not and that is something that we shall answer by now.

How does one show that certain set is not countable by definition you have to show that there is no 1 1 correspondence between that and the set of all natural numbers that is one way of showing. Then once you show that some set is not countable then any set which is numerically equivalent to that is also not countable. How to go about showing that we shall, we shall discuss in next class, we shall stop this for today.