Real Analysis

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Lecture - 39

Integrability

Well yesterday we saw some basic definitions of Riemann integrals as well as Riemann stieltjes integrals timely upper sums lower sums. What is meant by lower Riemann integral or lower Riemann stieltjes integral, similarly upper Riemann integral as well as Riemann stieltjes integral. And we have seen that we say that the function is Riemann integrable, if the upper Riemann integral coincides with lower Riemann integral, similarly in case of Riemann stieltjes integral and then we have seen that using this definition we can decide about the integrability or otherwise of very few functions. And so we need some good criteria to decide whether a function is integrable or not. And in order to come up with some useful results of that type.

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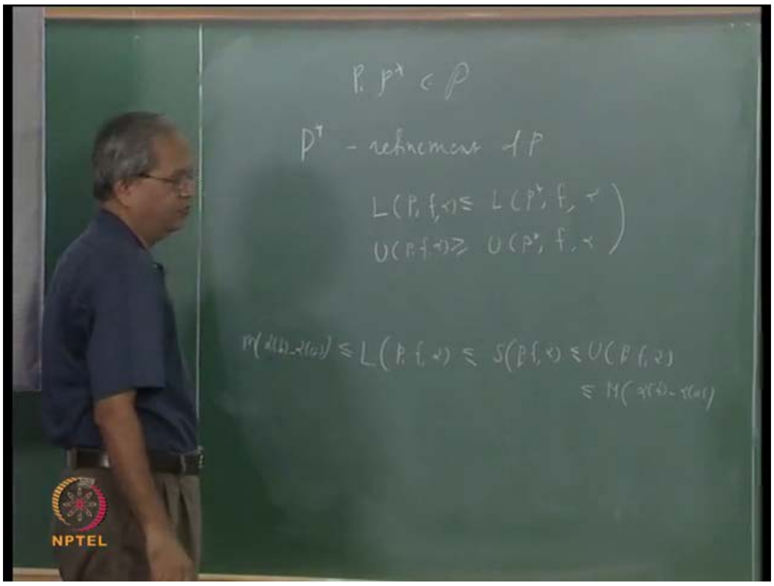

We started discussing what is meant by refinement of a partition and what happens to lower and upper sums, when you consider refinement. So, let us recall that last result.

So, what we have seen was that suppose, so we take P star is a refinement of p. And see P and P star both are partitions, so both P and P star both are partitions. That means P star contains all the points of P plus possibly a few more. And we are saying that in this case what happens is that lower sum of lower sum will increase and upper sum will decrease right. So, what we know that L P f is less than or equal to L P star f and U P f is bigger than or equal to U P star f right. Now, this has a very important consequence namely using this we can prove that every lower sum is less than or equal to every upper sum. See till now what we know not only this in fact we have proved something which is for Riemann sums.

But, we also putted for Riemann stieltjes sums right, see what we know for any partition. Let me again recall that we have we have shown that for any partition this L P f alpha is less than or equal to, let us just say this is stieltjes sum S P f alpha this is less than or equal to U P f alpha. And let me again recall that also this whole less than or equal to big M X alpha b minus alpha a and this will be bigger than or equal to small m times alpha b minus alpha a. But, using this what we can show is that if you take any two partitions then lower sum with respect to any partition is less than or equal to upper sum with respect to every partition. And that will enable us to prove that the lower integral is always less than or equal to the upper integral that is our idea. So, let us let us just take in the same way suppose let P and Q be two partitions let P and Q be two partitions.

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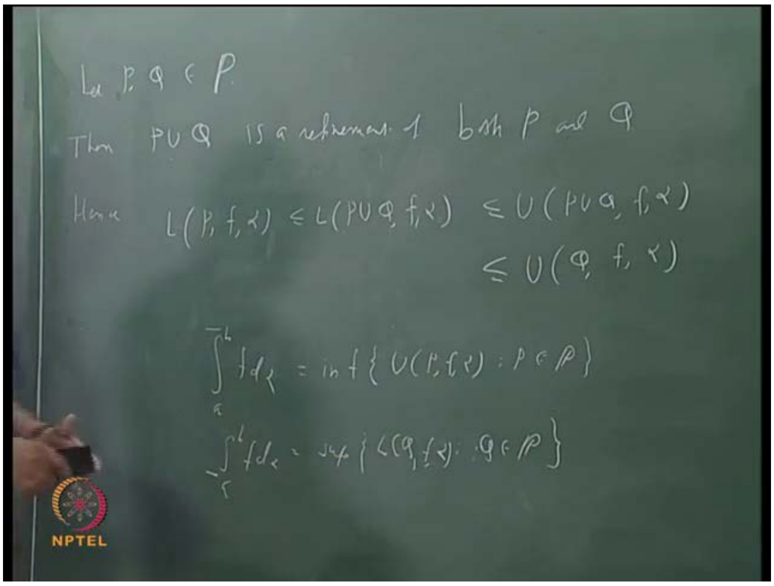

Then look at P union Q P union Q is refinement of both P union Q is refinement of P as well as refinement of Q right. So, P union Q is refinement of both P and Q, so we can apply these results. So, lower sum of P union Q will be bigger than or equal to lower sum of P. So, we can say that hence lower sum of P union Q sorry I want it the other way. That is lower sum of P is less than or equal to lower sum of P union Q right. Because P union Q is a refinement and lower sum of P union Q will obviously be less than or equal to upper sum of P union Q let us use this notation.

Of course, actually when f and alpha are fixed for the discussion we can simply say L P and U P etcetera. But, anyway let us be more precise, so then L P f alpha is less than an equal to L P union Q f alpha. And we know that for any partition lower sum is less than or equal to upper sum of that partition. So, this is something we have already seen there, so less than or equal to U of P union Q f alpha. And now we use the fact that P union Q is also a refinement of Q. So, the upper sum of P union Q should be less than or equal to upper sum of Q. So, this means this is less than or equal to Q f upper sum of Q right.

So, what did we prove that lower sum with respect to if you take any two partitions P and Q lower sum with respect to any partition is less than to upper sum of with respect to any other partition. That means that all lower sums are not only that lower sum of a particular partition is less than or equal to upper sum of that particular partition. But, all lower sums are less than or equal to all upper sums.

Now, let us recall the definition of upper and lower integrals what was what was this upper integral integral a to b f d alpha with one line here this was infimum of U P f alpha infimum taken over r right. And similarly lower integral was supremum of all lower sums. So, let just to avoid confusion let me use some other letter here, let us say this is supremum of L Q f for Q in the partition P L Q f alpha. Now, what we can say is that let us say you fix a partition Q.

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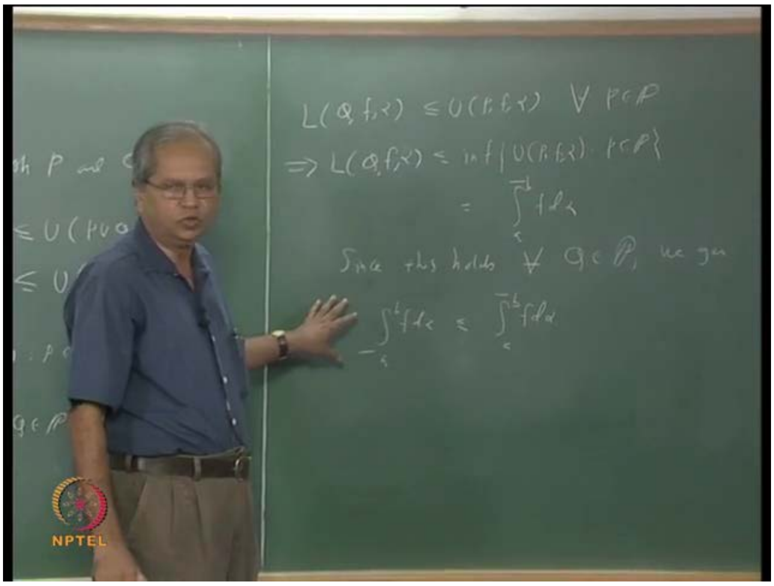

Here take some partition Q from here then you can say that L Q f alpha is less than or equal to U P f alpha for every P right. Suppose you take a particular partition Q here then the lower sum with respect to that partition is less than or equal to every upper sum. Which means this is a lower bound and if it is a lower bound this is a greatest lower bound infimum of is greatest lower bound. So, any your own must be less than or equal to infimum, so that means this implies that L Q f alpha is less than or equal to this infimum of U P f alpha for P in. But, this is nothing but the upper integral right. So, what did we prove if you take any partition Q then it is lower sum is less than or equal to the upper integral it is lower sum is less than or equal to upper integral.

Now, what we can say is that this is true for every Q this is true because there is no particular property of Q. That we have taken any arbitrary partition and fix every lower sum is less than or equal to this number. That means that this number is upper bound of this set. So, this is number upper bound of this set and this upper integral is the least upper bound. So, that least upper bound must be less than or equal to any upper bound right. So, that shows that since this since this holds for every Q in since this holds for all partitions Q. This holds for every Q in this set of all partition P we get supremum of this must be less than or equal to this number or which is same this saying that. The lower integral a to b f d alpha is less than or equal to upper integral a to b f d alpha.

So, remember to take the stock of whatever we have done so far is we approve that once a bounded function is given the upper integrals. And lower integrals will always exist and we say that the function is integrable. If these two numbers coincide and what is the extra thing that we proved. Now, is that the lower integral is always less than or equal to the upper integral. So, to show that your function is integrable what is required to show, now we must show that upper integral is less than or equal to lower integral if because this inequality always holds this always holds. So, to show that your function is integrable we must show that this upper integral is less than or equal to lower integral. And to do this also we use a very standard technique in analysis which is used very often. For example, when you want to show that one number is less than or equal to other.

And suppose you cannot do it directly easily then you for example, if you wanted to show that alpha is less than or equal to beta. And you cannot do it easily directly then you show that alpha is less than or equal to beta plus epsilon for every epsilon. Then that will follow for every positive epsilon and then from that we will follow that alpha is less than or equal to beta.

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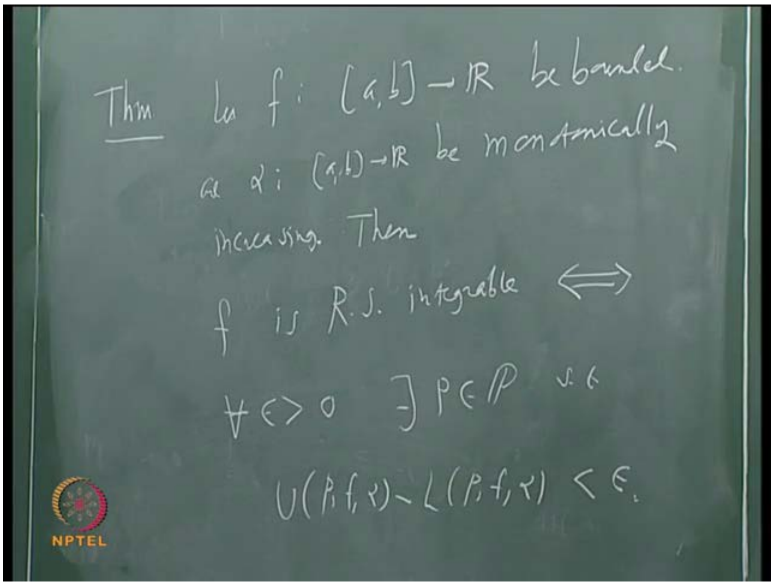

This is a very standard technique used in proving inequalities of this type you should you would have come across this kind of proofs fairly often. So, that is what we do and taking this into account, now we will give you relatively a easy criteria to decide.

Whether a function is integrable or not and then we shall use that criteria to show that several functions that we come across in practice are integrable, so what is that criteria.

Let me let us take that as a theorem, so that is a theorem. So, as usual let f from a b to R bounded then we say that f is integrable. Let us we can prove stieltjes integrable directly instead of going for reminder. Let f be bounded alpha from a b to R b monotonically increasing then f is integrable. Let us use this notation f is let us say or fine f is Riemann stieltjes integrable.

If and only if and only if what should happen is that for every epsilon bigger than 0 suppose you have given some epsilon for every epsilon bigger than 0. You should be able to find a partition such that the difference between the upper and lower sum of that partition should be less than epsilon. If that can be done then function is Riemann stieltjes integrable for every epsilon bigger than 0. There exist a partition P such that of course upper sum is going to be always bigger or equal to lower sum. So, we can simply say such that U P f alpha minus L P f alpha is less than epsilon.

And that means all that we need to do that given any epsilon we should find a partition P. Such that for that partition P the difference between the upper and lower sum is less than epsilon. Suppose you do that then that is enough that is enough to conclude that the function is Riemann integrable or Riemann stieltjes integrable. In this depending on which kind of sums you take it will Riemann integrable or Riemann stieltjes integrable.

And in proving this we shall make use of the observation which I have made just now namely that. Whenever you want to show that alpha is less than or equal to beta you try to show that alpha is less than or equal to beta plus epsilon foe every epsilon. That is what that is what we shall use for that, we note the following. See remember we have already shown that every lower sum is less than or equal to the in fact every upper sum is less than or equal to.

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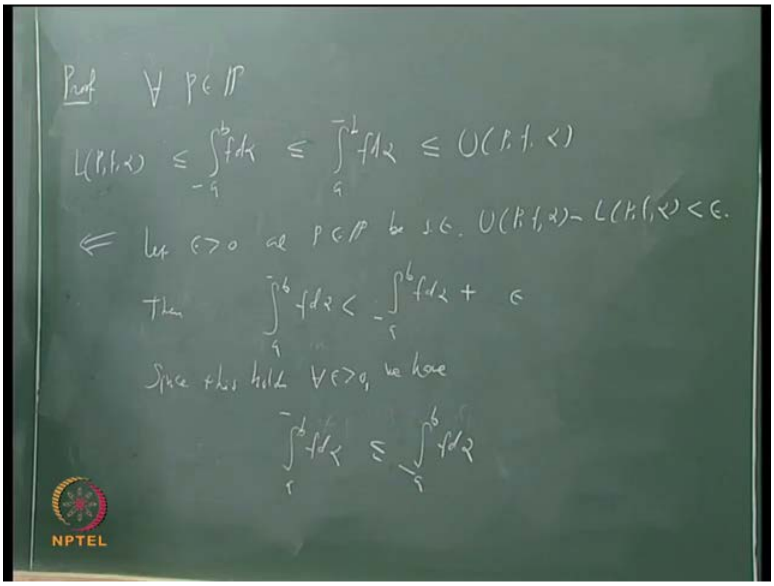

Let me just again recall the observation because upper integral is the infimum of upper sum infimum of the upper sums. So, this will always be true, so for every partition P this is something that we know always L P f alpha. In fact let me write it like this you can always say this that integral a to b f d alpha. There is upper integral this is less than or equal to U P f alpha right that is always true right. Because this is infimum of sums of these types, so particularly this is lower bound. And now we have proved that upper integral is always bigger than and equal to lower integral right. And it is also clear that lower integral will always be bigger than or equal to any lower sum because lower integral is the supremum of all the lower sums alright.

Now, after this observation does the proof of this part becomes obvious suppose we know that given any partition given any epsilon there exists a partition P. Such that the difference between upper and lower sum is less than epsilon, if the difference between the upper that is see if the difference between these two is less than epsilon. Then these two numbers are lying in between those two, so the difference between those two also must be less than or equal to epsilon right. So, what so what is the proof here, so this true for every P. Now, lets us look at we are assuming this that for every epsilon bigger than 0.

There exists a partition P etcetera and using that we are showing this, so that means we are proving this part we are proving this part. So, let epsilon be bigger than 0 and P belong to P be such that. Such that U P f alpha minus L P f alpha is less than epsilon right. So, does it follow from there that difference between upper integral and lower integral is also less than epsilon that is then integral a to b f d alpha minus the lower integral is less than epsilon.

Alright or the same inequality I can write in a slightly different manner instead of writing this minus this is less than epsilon I shall say this is less than this plus epsilon, that is the same thing right. And that is what we wanted to get and for every epsilon this number upper integral is less than lower integral plus epsilon.

So, upper integral must be less than or equal to, so since this holds for every epsilon for every epsilon bigger than 0. We have we have this integral upper integral is less than or equal to the lower integral. That completes the proof of that completes the proof of this part. Now, we will for this part this part means what suppose f is Riemann stieltjes integrable then given epsilon there should exist a partition P such that this happens.

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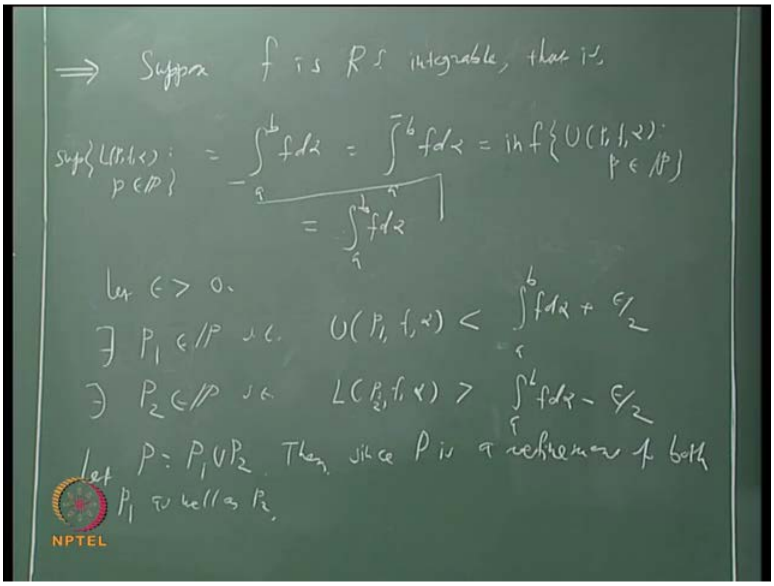

So, suppose f is Riemann stieltjes integrable what does this mean it by definition, this means the lower integral is same as the upper integral. That is this lower integral a to b f d alpha this is same as the upper integral a to b f d alpha alright. And what we have to show is that this if this the case then every epsilon we can find a partition P with that property. So, let us start with this let epsilon be bigger than 0 let epsilon be bigger than 0. Now, here it will also help to recall the definitions of this what is this again, let us recall this is nothing but the infimum of the upper sum right. P belongs to script P and this P is the thing but, this supremum of the lower sums.

Again the way in which infimum and supremum are used in mostly it proves is as follows this is also, something you would have come across very often. What is the meaning of infimum, infimum is the greatest lower bound right. Infimum is the greatest lower bound if it is greatest lower bound if you take any number bigger than that than that then it is not a lower bound if you take any number. Now, if something is not lower bound means what there should exist some number in that set, which is strictly less than that number. So, for example now we are given this epsilon bigger than 0 suppose I take something like this plus epsilon or this plus epsilon by 2, then that is not a lower bound.

So, there should exist a partition here such that the corresponding sum of that partition should be less than this plus epsilon by 2 right. So, one can say that there exists a partition suppose I call it partition P 1 that there exists a partition P 1 in script P. Such that what should happen U P 1 f alpha should be strictly less than this upper integral into a to b f d alpha plus epsilon by 2, do you agree is that clear. Because this if all of this if all of this were bigger than or equal to this would have become lower bound that is not the case right. Use the same thing here what is the supremum is the least upper bound supremum is the least upper bound.

So, that means if you take any number smaller than that that is not an upper bound that means, there should exist something here which is bigger than that number right. So, we consider it similarly there exist P 1 sorry P 2 in script P, such that what should happen L P f alpha should be bigger than. Because this integral lower integral sorry this should have been upper integral here because we have used this part. So, this lower sum is less than the lower integral minus epsilon by 2 this is P 2, so if you had. So, what is our aim actually we want to find a partition P such that for that partition P U P f alpha minus L P f alpha is less than epsilon.

Suppose this P 1 and P 2 were the same if P 1 and P 2 were the same then we could have simply said it this is less than epsilon by 2. This is bigger than this minus epsilon by 2 and these two are the same actually that is that is our starting assumption right. So, suppose you call this common number as sub number I that is the integral or in fact that is what we have called integral a to b f d alpha right. This is the common value is what we have called integral a to b f d alpha right. So, I can forget about this, so this upper sum is less than this number epsilon by 2 this lower sum is bigger than this number minus epsilon by 2.

So suppose we subtract you will get, so because when you subtract this the inequality will be reverse this will be cancelled. And you will get U if P 1 and P 2 were same you would have got U P f alpha minus L P f alpha less than epsilon. Which is what we require, but that is not the case in general because we have obtained P 1 and P 2 from these definitions. So, in general they will be different in general they will be different but again. Now, it is obvious what is to be done in such a case yeah take the take the refinement take the union and that will satisfy both theses inequalities. So, let P be equal to P 1 union P 2 then P is a refinement of both then since P is a refinement both P 1 as well as P two.

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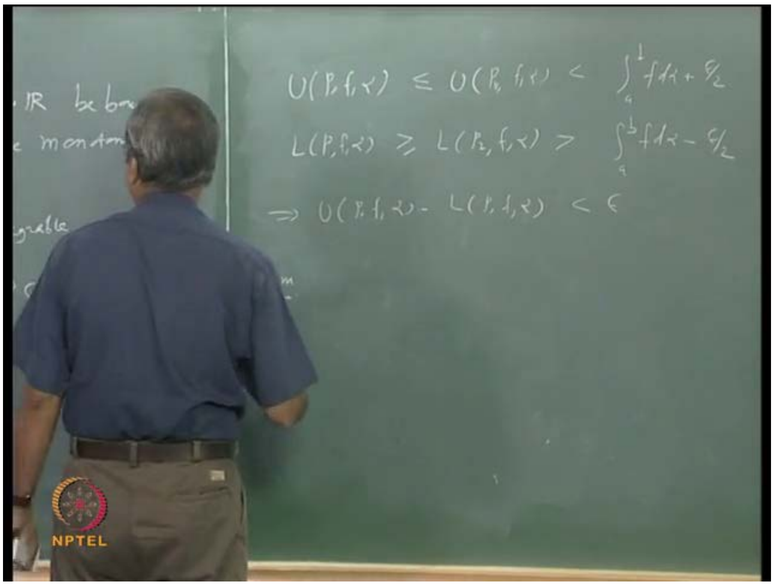

What we will get is that I think I will continue there what will get is U P f alpha this is less than or equal to u since P is a refinement of P 1. Upper sum with respect to P is less than or equal to this, so u P 1 U P 1 f alpha and this is less than integral a to b f d alpha plus epsilon by 2. And L P f alpha is bigger than or equal to L P 2 f alpha because lower sums increase P comma f comma alpha. So, this is bigger than integral a to b f d alpha minus epsilon by 2 and this implies that this means U P f alpha minus L P f alpha is less than epsilon, this completes the proof. So, what we have shown is that if f is Riemann integrable given any epsilon we can find a partition P, such that for that partition U P f alpha minus L P f alpha is less than epsilon. And other way we have shown earlier if this property is true.

Then the function is Riemann integrable Riemann stieltjes integrable alright. Now, as I said in the beginning this is a very useful criteria and this decides this helps us in deciding that about fairly large classes of functions are integrable. So, the such first important class is that of continuous function this will help us show to show that every continuous function is Riemann stieltjes integrable. And so that that takes care of fairly big class, so that takes care of fairly big class.

So, once you know that a function is continuous then you can immediately say that it is integrable. So, if I think I will write it here itself here because these assumptions are standard for whatever we want are going to. So, what is the additional thing we require that, if f is also continuous. Then it is Riemann integrable Riemann stieltjes integrable.

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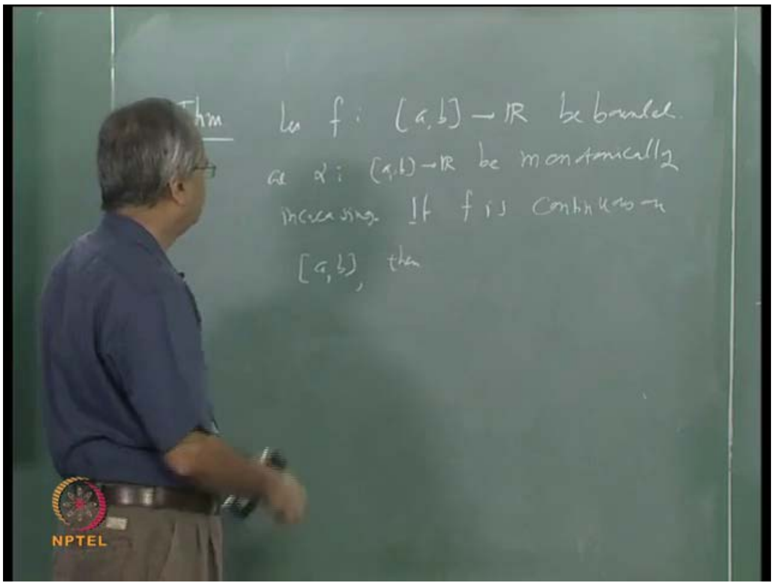

If f is continuous on a b then I think we shall first this for this theorem it is better to prove it first for Riemann integrals rather than Riemann stieltjes integrals. So, I will remove this part in fact once we say it is continuous we need not to say it is bounded also. Because it is a closed end bounded interval I think it is better to write it as a separate statement, so in fact we can write it in a short form and that you know this class C a b right.

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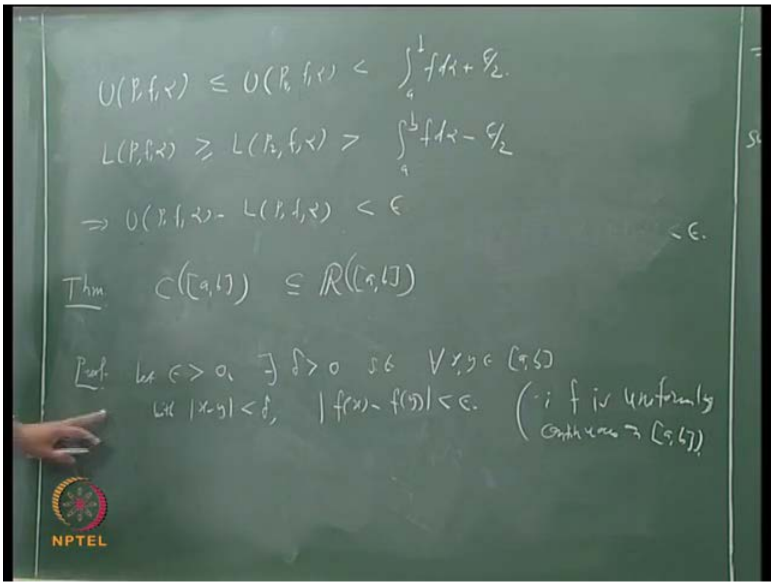

It is the set of all continuous functions defined on a b continuous, let us say it is continuous real value. And then all that we want to say is that every continuous function is Riemann integral. So, C a b is contained in this R a b or in words this means that every continuous function is Riemann integrable every continuous function is Riemann integrable and for the proof for the proof. We are going to use the previous theorem, so the way in which we will prove this that given any epsilon we shall produce a partition. Such that for that partition the difference between the upper and lower sum is less than epsilon and since we are talking about Riemann integral this alpha will not it will just U P f minus L P f.

So, let epsilon be bigger than 0 then what we can do is that we can think of some number for which. Let me come to that little later basically we are going to use the fact that this a b is a compact set. So, every continuous function on a compact set is uniformly continuous of course, it is bounded etcetera that is all there, but it is also uniformly continuous it is also uniformly continuous. So, we can say that there exists there exists delta bigger than 0 such that such that.

If you take any two points in a b whose difference is less than delta, then the difference between f x and f y is less than epsilon is less than epsilon there exists a delta bigger than 0. Such that for all x y in a b with mod x minus y less than delta mod of f x minus f y is less than epsilon. This is because f is uniformly continuous on a b this is because f is uniformly continuous.

So, here where you are using the fact that continuous function on a compact set is uniformly continuous, something that I suppose you have already done continuous function on a compact continuous function. On a compact matrix space is uniformly continuous only thing is that see ultimately what we want is that U P f alpha minus L P f alpha that should be less than epsilon. So, here for the time being instead of taking this epsilon I shall take this epsilon 1. Let us say we shall choose this epsilon 1, depending on this epsilon later. Because ultimately what we want is the difference between the upper and lower sum should be less than epsilon.

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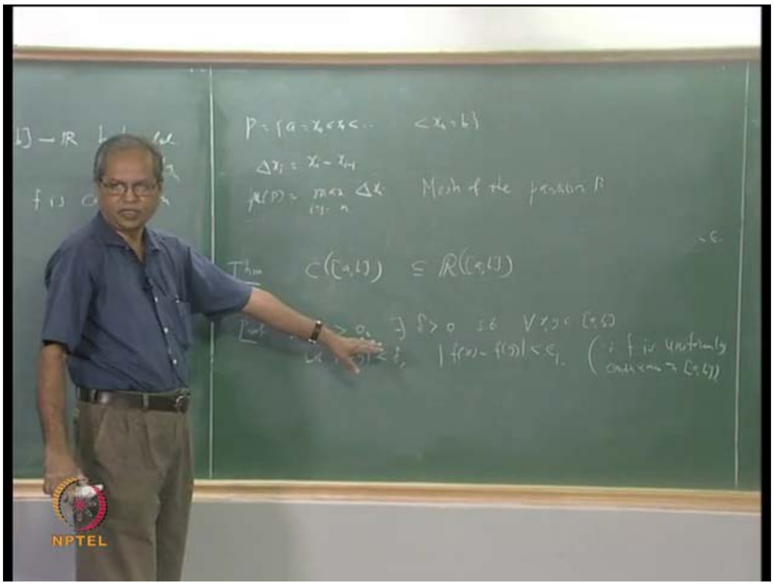

So, we shall epsilon we shall choose this epsilon 1 afterwards, now what we do is that we take a partition in such a way that the length of all the partitions is less than this delta length of all the partitions is less than this delta. So, for this I think it is better to introduce a notation here which may be useful subsequently also let me write here. Suppose, P is a partition let us say P is a partition which is of this type a is equal to x naught less than x 1 etcetera less than x n equal to b. Then we have said that we will call this delta x i to be x i minus x i minus 1 x i minus x i minus 1. And that is the length of the ith sum interval length of the ith sum interval, there n such intervals you take the maximum of it you take the maximum of that.

That is called mesh of the partition and denoted by mu of P maximum of delta x i going from 1 to n, so this is called mesh of P alright. Now, coming back to this proof given an epsilon we have found a delta such that whenever mod x minus y is less than delta mod f x minus f y is less than some number depending on epsilon. That choice we will do later, now what we do is choose a find a partition find a partition. Such that the mesh of that partition is less than delta.

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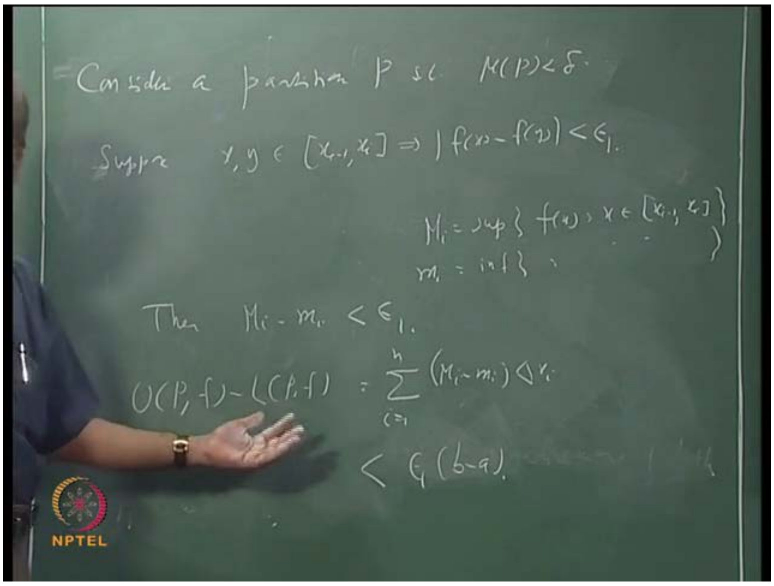

So, consider a partition consider a partition P, such that mesh of the partition P is less than delta is it clear that such a partition can be chosen very easily. For example, one possibility is that I can I can take all of these intervals are of the same length I can take all of these intervals of the same length. If that happens then in each of this delta x i will be b minus a by n there are n such things right. That if you choose x not equal to a x 1 is equal to a plus b minus a by n then x 2 is equal to 2 times a plus b minus a by n etcetera. Then the length of each partition each sub interval will be b minus a by n.

Then can we always choose n such that this b minus a by n is less than delta right. Whatever the number delta is given b minus a tends to 0. We always choose n big enough, such that b minus a by n is less than delta fine. So, this is no problem we can given a delta we can always choose a partition we can always choose a partition. Such that the mesh of that partition is less than delta, that is clear. That is why we need not describe how it is to be chosen. Now, what is going to happen is that suppose you take any x and y in the sub interval then, then for this partition suppose x and y belong to this sub interval x i minus 1 to x i.

Then since this length is less than delta this length is less than delta the difference between function values here difference between f x and f y must be less than this epsilon. So, this implies mod f x minus f y must be less than epsilon 1 fine, now what can we say about this quantity big M i minus small m i is it clear to you that big M i minus small m i is the biggest of all the difference between f x and f y. See what is big M i, let us recall big M i is supremum of big M i is supremum of f x for x belonging to this x i minus one to minus x i. And small m i is infimum of the same there is small m i is the smallest possible value that f can take and big M i is the biggest remember in this case.

This supremum I can say is the maximum also because f is a continuous function because f is a continuous function there will exist some point at which this maximum is occur. Similarly, there exists some point at which this will occur and for any other any x and y the difference between f x and x y must be less than or equal to this big m i minus small m i. But, what we know is that for any x and y that difference is less than epsilon 1 suppose I choose say x for which f x becomes this big M i choose y such that f y becomes this small m i then the difference between these two is less than epsilon right.

So, what I can say let epsilon 1 then I can say then big M i minus small m i is less than epsilon 1 alright. Now, you look at U P f minus L P f is it clear that this is same as sigma i going from 1 to n sigma i going from one to n big M i minus small m i multiplied by delta x i right.

Because U P f alpha is big M i into delta x i L P f alpha is small m i into delta x i. So, the difference will be given by this, but what we do is this number is less than epsilon 1 this number is less than epsilon 1 for each i. So, that we can say that is less than sigma this epsilon 1 into delta x i, but we know sigma delta x i is b minus a. So, this will be less than epsilon 1 into b minus a, but now come back to our idea we wanted to find a partition. Such that this difference is less than epsilon this difference is less than epsilon, so if I take this epsilon 1 as epsilon divided by b minus a. Then this final thing will less than epsilon, so we can say that here let epsilon be bigger than 0.

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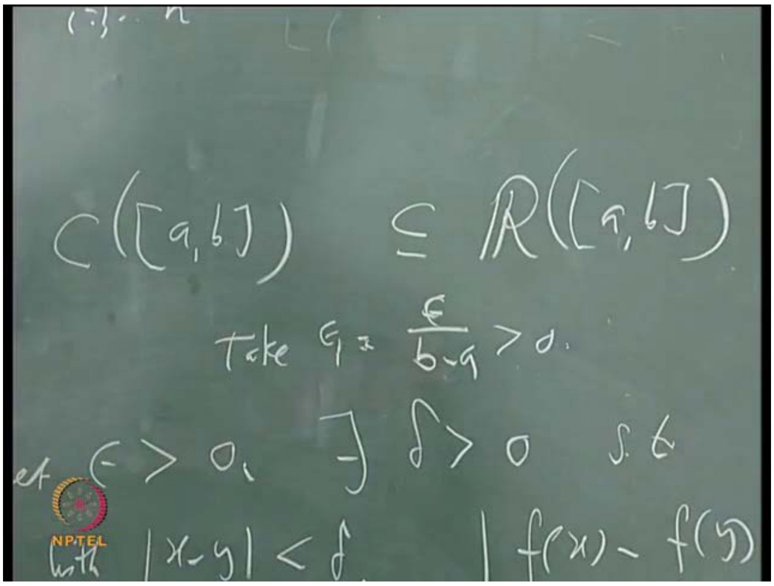

And take epsilon 1 is equal to epsilon 1 is equal to what epsilon divided by b minus a, epsilon divided by b minus a. So, that will also be bigger than 0 is epsilon is bigger than 0 etcetera, so suppose we take epsilon 1 as epsilon 1 by b minus a.

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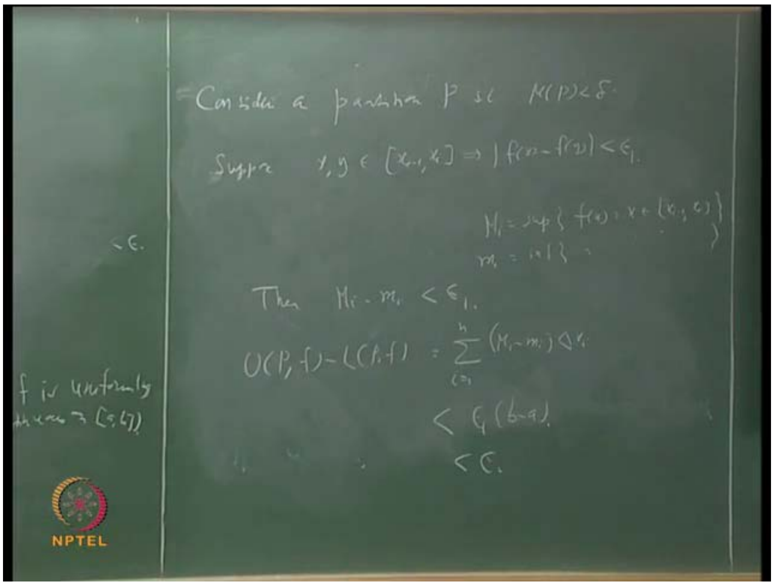

This is less than epsilon is it clear, so again let us recall what did we do we showed that given any epsilon we can find a partition P. We can find a partition P such that for that partition U P f minus L P f is less than epsilon. And we already said that if that happens the function must be integrable Riemann integrable right. Function must be we need not show for any arbitrary alpha we showed it for this.

If we could show this for any arbitrary alpha we could have said that the function is Riemann stieltjes integrable. Now, let me come to that is it clear first of all, now let me come to that problem when can w say that the. Suppose a function is continuous when can we say that it is Riemann stieltjes integrable what will be the what will be the changes required here.

See if you look at this difference instead of U P f you will get U P f alpha minus L P f alpha you will get U P f alpha minus L P f alpha. Then that will be then that will be M i minus m i delta alpha i delta alpha i is alpha x i minus alpha i. This will be less than epsilon 1 into alpha b minus alpha a alpha b minus alpha a, now what happens here is that. We will need to choose a partition in such a way that this big M i. So, that is the only change instead of instead of instead of instead of taking epsilon 1 as a epsilon divided by b minus a, you will need to take epsilon divided by alpha b minus alpha a.

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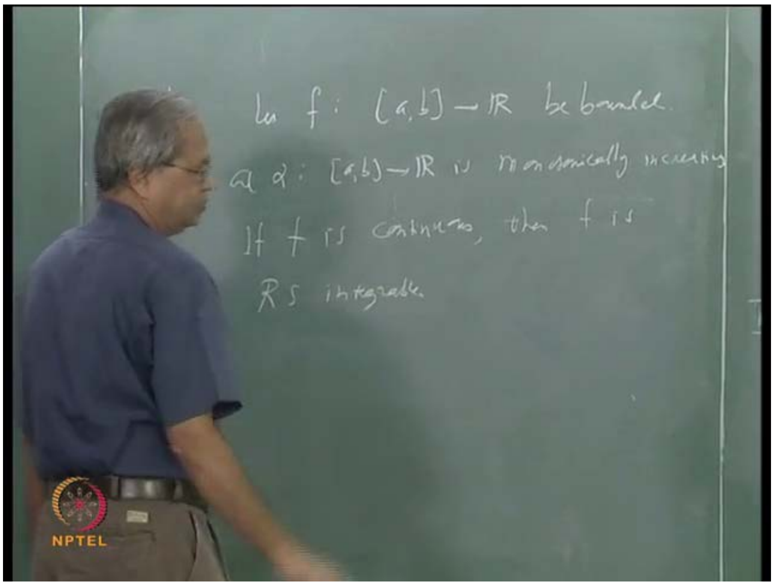

That is all, so let me come back to this what I have started. So, suppose f from a b to R is bounded and alpha from a b to R is monotonically increasing monotonically increasing. Then if f is continuous if f is continuous then f is Riemann stieltjes integrable. We did not discussed this proof separately because there is only one change here namely that instead of taking epsilon 1 as epsilon divided by b minus a. You just take epsilon divided by alpha b minus alpha a, that is the only change rest of the proof will remain as. It is now this tells us that let us come back to this theorem again this tells us that every continuous function is Riemann integrable or Riemann stieltjes integrable depending on what you want.

Now, what happens if function is not continuous at let us say one point what happens if function is not continuous at one point. Then what we can say is that you choose suppose you look at a point at which the function is discontinuous suppose that is the point x naught.

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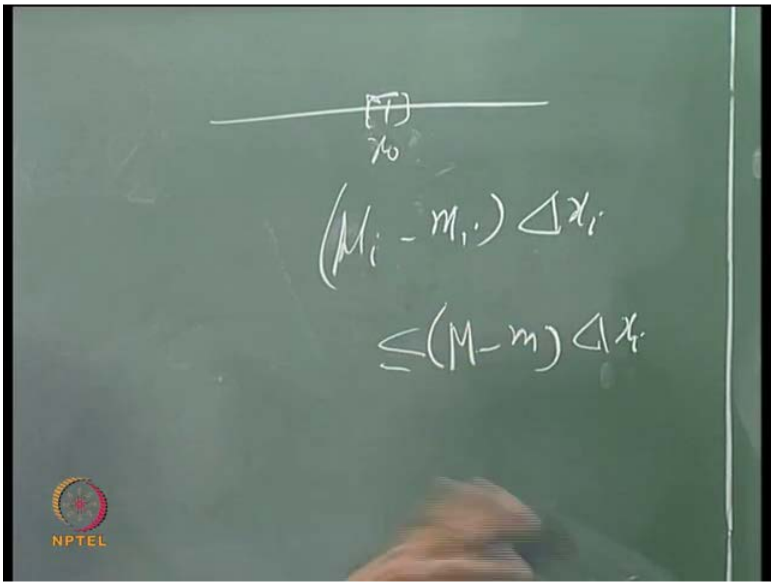

What you do is that you choose this particular interval, this particular interval containing x naught to be very small. We choose this particular interval containing x naught to be very small. So, what will be the contribution due to this interval to let us say basically what we want is that we want to make this U P f minus l P f to be very small. That is the idea now what will be the contribution to this U P f minus l P f it will be this particular suppose. This is delta x i suppose this is suppose this is ith sub interval then in contribution to that from this will be M i minus m i divided by delta x i. Now, what we can do is that for all other intervals we can make this M i minus small m i to be to be less than this epsilon b minus a.

For every other sub interval we cannot do it for this particular sub interval, so what we do is for this particular sub interval we will make this delta x i to be very small over this we do not have control. But, of course still we do not have control means we still know that it is see that we can always say that. This will always be less than or equal to big M i minus small m i this will always and this into delta x i. So, suppose I choose this delta x i so small this delta x i so small means what I choose this x i minus x i very close to x naught. So, that this product is less than epsilon by 2, let us say this product is less than by epsilon by 2. And then choose the other delta in such a way that the remaining part is less than epsilon by 2.

Remaining part is also less than epsilon by 2 then again we can do the same thing. So, what does it mean that even if your function has one discontinuity suppose function is bounded continuous everywhere except at one point then still it is Riemann integral right, but then one can ask go further and ask. So, what is so particular about one point suppose it is discontinuous at two points again we can do the same thing. We reduce for example we can make each there will be two such intervals. And you can make the contribution by each interval less than say here you have taken epsilon by 2. Now, you take epsilon by 4, so it does not matter right. So, what it also means is that there is nothing particular about one or two or anything if a function is continuous everywhere.

Except at a finite number of points then it will be Riemann integrable, but then the natural question is how far we can go in this. That is what if it is discontinuous infinitely many points or countable points uncountable points that is what is the maximum number of discontinuities. That one can allow for a function to be Riemann integrable that is an that is an interesting question only thing is that that question we cannot answer here in this course.

Because that requires some something which is this is something you will perhaps learn when you learn measure an integration the answer to that is that function is Riemann integrable if and only it is continuous. What is called almost everywhere it is continuous almost everywhere what it means is that if you look at the set of all those points where the function is discontinuous.

Then that set is what is called set of measure 0 that set is what is called set of measure 0 to understand what is meant by this you will have to learn what is measure and what is layback measure etcetera. But, roughly speaking what we can say is that set of measure 0 means that you can enclose that set by a countable number of open intervals whose total length can be made arbitrarily small even any epsilon you can. You can enclose that set into union of open intervals such that the sum of all those sum of the lengths of all those open intervals is less than epsilon. So, that is what is called a set of measure 0, but that result since I said that result requires several other things from is measure theory we shall not discuss in this course please stop.