Real Analysis
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Lecture - 38
Integration

We begin the discussion of Integration today. As you know in the undergraduate calculus courses usually differentiation is taught first. Then that is followed by integration, but logically speaking or even historically speaking integration was discovered much earlier, compared to differentiation. If you look at the geometric interpretations, as you all know that derivatives represent the slope of the tangent of a curve. Whereas, integrals represent the areas under the curve. Prevail know how to find areas under the curve much before they knew anything about the tangents.

Also, the process of differentiation and integrations are in some sense inverse of each other that is also something it is known but, from this geometric interpretation it is not immediately clear, how these two are inverses of each other. for that we need to go into the details of theory of integration. So, we will begin with what is normally called Riemann integrals.

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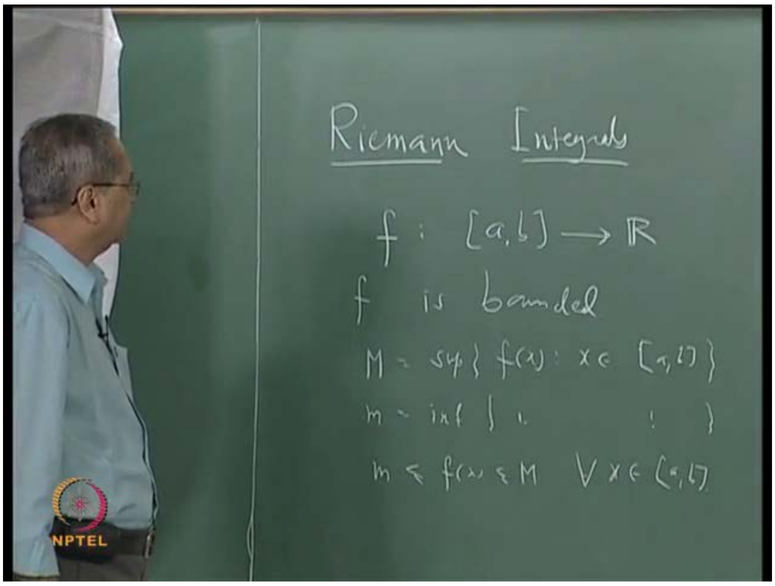

So, here the idea is as follows f is a function that is defined on some interval a to b real valued function, a to b to R. We also assume that f is bounded since f is bounded. We can take some numbers which are the least upper bound and the greatest lower bound of this f. So, let us say suppose if we denote this numbers by m. So, let us say that m is a supremum of f x for x in a b. Let us say small m is infimum of f x for x in a b. So, in particular this means that for every x in a b small m less than or equal to fx less than or equal to big m, for every x in a b. Now, the first thing to be done for this Riemann integral is what is called partitioning of this interval.

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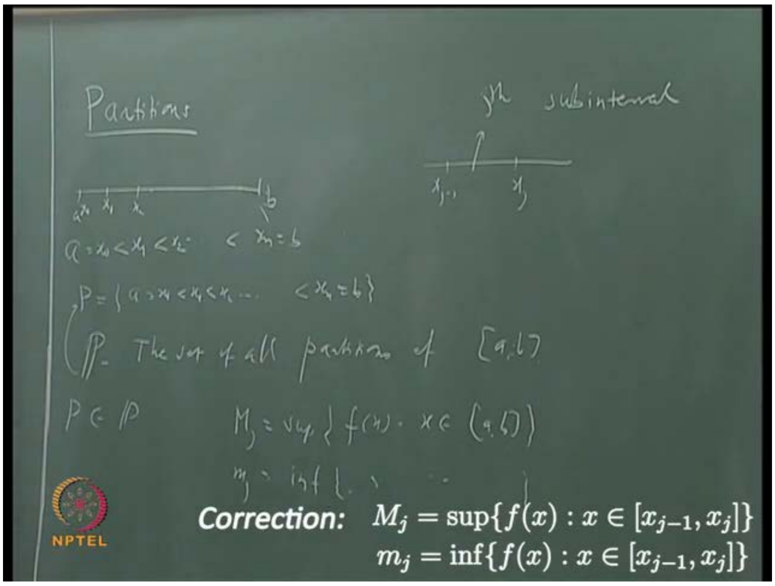

So that is the next thing that we want to see discuss what is meant by partition or partitions a. Partition of a b is nothing but suppose this is the interval a b we sub divide this interval into some sub intervals. So, and it is usually labeled like this first point a is called x naught. Next, let us say next point x 1 next is x 2 etcetera and going like this the last point b is called x n. What we want is x naught should be strictly less than x 1, x 1 should be strictly less than x 2 etcetera. So, finally x n minus 1 will be strictly less than x n.

So, such a set of point is called a partition of a b. So, this P usually we will talk like this a equal to x naught less than x 1 less than x 2 etcetera less than x n and x n is equal to b. Since, we may also want to refer to various partitions. We also like to give some notation for the set of all such partitions. So, that we basically put P the set of all partitions of P partitions of a b.

Then let us now consider one such partition some P in script P suppose, it is this partition. Then for such a partition we look at let us say this x j minus 1 to x j. That is called j th sub interval, this is called j th sub interval. So, there are n such sub intervals. So, whatever we have done for this full interval we also do the same thing for each of those sub intervals. That is we look at supremum and infimum this time not over the whole interval a b, but just for this sub interval x j minus 1. It is obvious that if a function is bounded on a whole interval a b. It will be also bounded on the all such sub intervals.

So, we denote those numbers by big M suffix j that is supremum of f x, for x in a b and small m suffix j, as infimum of this x in a b. Now, having defined these numbers big M suffix j and small m suffix j. Then we define what are known as lower sums and upper sums corresponding to this function f. So, we will call it upper sum.

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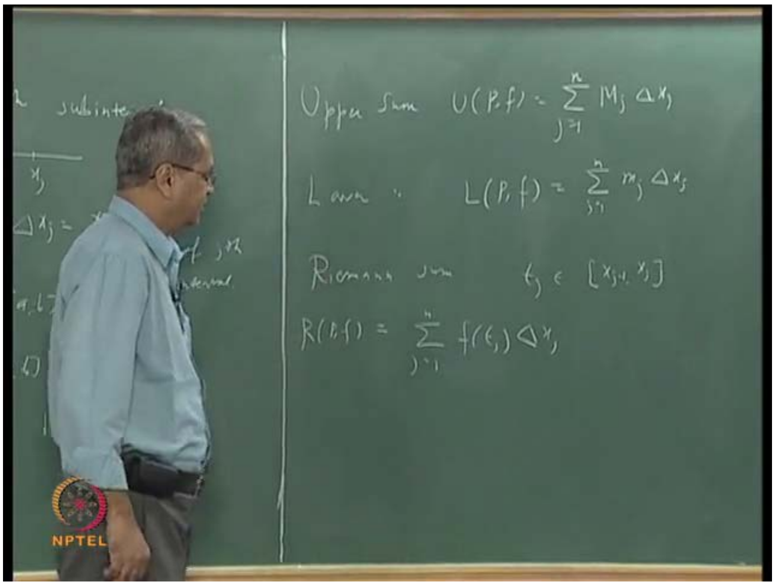

Since, this upper sum will depend on the function f and also on the partition P. So, we denote that by U P f, this is nothing but sigma m j. That is multiplied by one more notation for this length of this j th sub interval. We denote that by delta x j delta x j is nothing but x j minus x j minus 1 that is nothing but the length of j th sub interval. So, that is you take the this number big M suffix j multiplied by that by the length of the j th sub interval. Take the sum for over all such sub intervals j going from one to n that is that is called upper sum.

Then similarly one can define what is meant by lower sum. That is denoted by L P f for lower and instead of this big M suffix j. We take that small m suffix j sigma j going from 1 to n small m suffix j delta x j. Now, in addition to this upper and lower sums we also have what is called Riemann sum. Now, here there is one more extra thing is required what we do in this case is that, we choose some point. Suppose, we call that point t j in the j th sum interval x j minus 1 to x j. So, we make that choice t j in the sub interval x j minus 1 to x j. Instead of taking this number either big M suffix j or small m suffix j. We take the value of the function at this point f of t j. Then multiply that by delta x j and then take the sum for j going from 1 to n, this is called Riemann sum.

Now, the Riemann also be depend on P and f, but in addition it will also depend on the choice of this points t j in the sub intervals x j minus 1 to x j. So, strictly speaking notation should be something like R P f. Also, t j here, but since that will make the notation very complicated. We do not usually include it here unless we have to make some specific reference to the way in which the choice is made.

We will not bother about this, we will say that it is understood that t j is an arbitrary point in the sub interval, x j minus point to x j. Now, before proceeding further it may be useful to have a look at what this sums represent geometrically. For example, suppose we have this interval a to b and suppose this is a. Suppose, this is a graph of this function.

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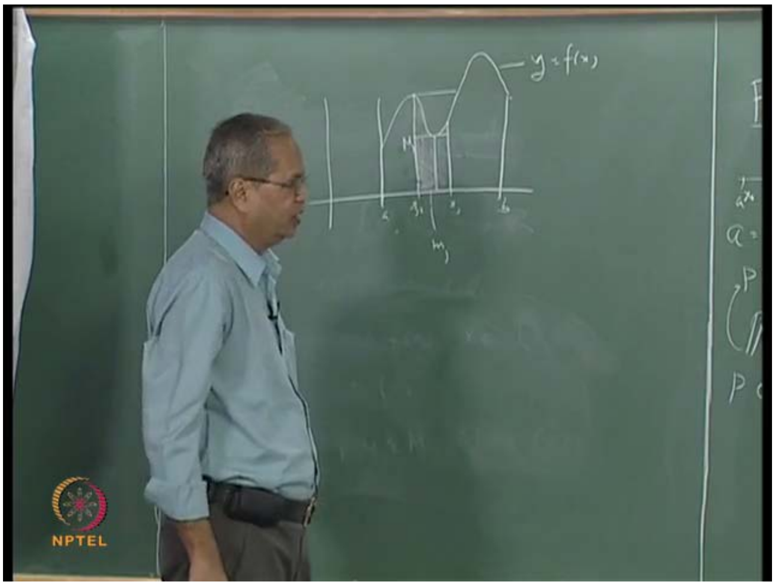

Let us say this is the graph y equal to f x, then when we sub divide this interval into sub intervals. Suppose, this is x j minus 1 and that is x j, then it will be it is this sub intervals. So, this is f of x j this is f of x j minus 1. Now, what is small m suffix j small m suffix j is the minimum value of f in the interval x j minus 1 to x j. So, let us say that will occupy here. So, that is small m suffix j it is small m suffix j and what is big M suffix j? That is the biggest possible value of f in the interval x j minus 1 to x j. So, for example in this case it will occur here. So, this will be big M suffix j and this will be small m suffix j. So, when you multiply this big M suffix j or small m suffix j by this x j minus x j.

Let us take this case first, which will correspond to the lower sum m j into x j minus x j minus 1. That will represent the area of this rectangle and when you do it for all such sub intervals. That is what you will get in the lower sum. Similarly, if you do it for m j this big M j it will represent this area, but what you see is that each of this lower sum. The upper sum approximates the area under the curve lower sum approximates from below. The upper sum approximates from above and a Riemann sum is lying between these two. So, from the definition of this m j and this big M j and small m j. It is clear that this f of t j, that is we small m j is less than or equal to f of t j, this is less than or equal to big M suffix j.

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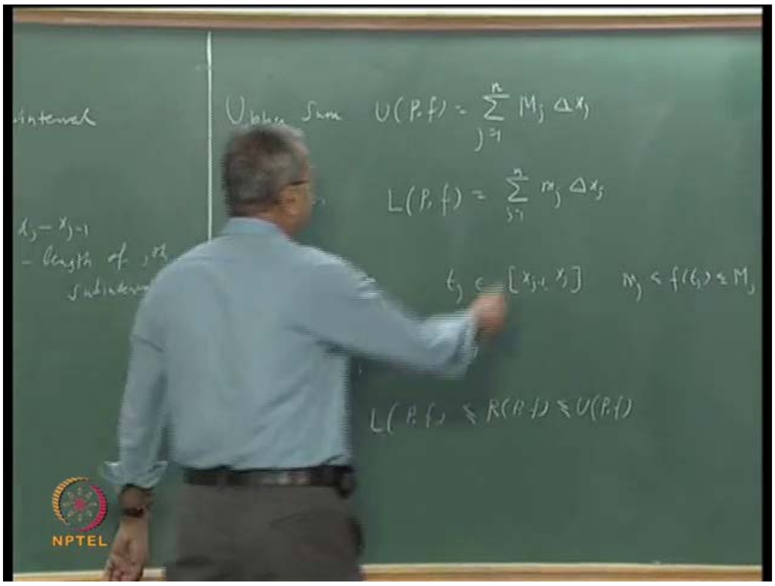

Lower sum is nothing but you multiply each of this number by delta x j. Similarly, do the same thing for this that you will get Riemann sum and do the same thing for that you will get the upper sum. So, what you obviously have is the following that is we get L P f is less than or equal to R P f, that is less than or equal to U P f this is obvious.

One more thing we can see is that this big m j is a supremum over this sub interval x j minus 1 to x j. Whereas, this big M is supremum over the whole interval a to b. So, this big M suffix j will be less than or equal to m. So, this will be less than or equal to m into sigma j going from 1 to n delta x j. This sigma j going from one to n delta x j is nothing but the length of the whole interval, which is nothing but b minus a.

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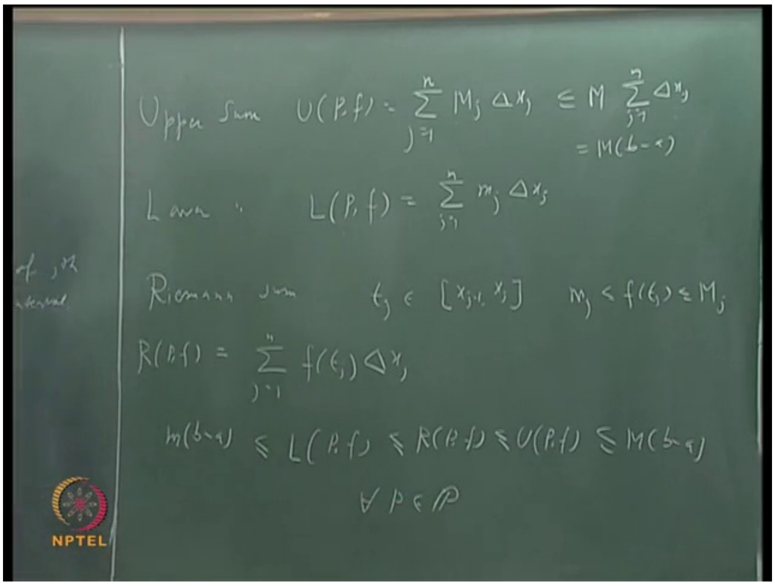

So, this is m big M into b minus a that is all the upper sums or less than or equal to big M into b minus a. Coming back to this picture big M is the maximum over this. For example, in this case something like this big M. So, big M into b minus a is nothing but the area of this rectangle. So, all the upper sums will be less than or equal to area of this rectangle and similarly, but can see that lower sum P f L P f. That will be bigger than or equal to small m into b minus a this inequality is true for every partition, this is true. Whatever, be the partition this we can say always what is the inequality.

That is small m into b minus a less than or equal to lower sum less than or equal to Riemann sum less than or equal to upper sum, less than or equal to big M into b minus a, for every P in script P. Next, what we do is we look at all these upper sums. So, look at this set that is set of all U P f where P belong to script P.

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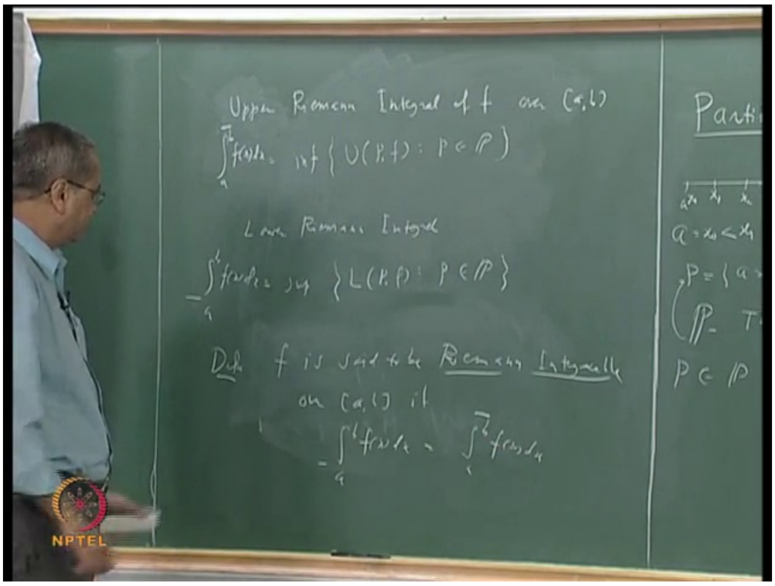

That is take all possible partitions and take upper sums corresponding to those partitions. Then we know that every upper sum is bounded below by this number small m into b minus a. So, we can take the infimum of all these upper see these are all numbers for each partition you get different number here, this set is bounded below. So, we can think of it is infimum, this infimum is a real number. So, that is called lower Riemann integral it is called lower Riemann integral of f over the interval a b and denoted by this. This is called upper Riemann integral and is denoted by this integral a to b f x, d x. This bar over the above the integral sign denotes the upper Riemann integral.

Similarly, if you look at all the lower sums L P f for P in script P that set is bounded above, because every lower sum is less than or equal to this number. So, we can take its supremum that is the least upper bound and that is called lower Riemann integral. That is denoted by integral a to b f x d x it is lower Riemann. So, we put one bar below this integral sign. So, that is lower Riemann integral, so once the function is bounded on the interval a to b this upper and lower Riemann integrals. Those will always exist and we can show subsequently, that this will be always less than or equal to that. That is lower Riemann integral will be always less than or equal to upper Riemann integral.

Now, we come to the main definition when these two coincide that is whenever upper and lower Riemann integrals coincide. We say that the function is Riemann integrable over the interval a b. So, that is the definition, f is said to be Riemann inetgrable over a b if is integral a to b f x d x, that is lower integral is same as integral a to b f x d x. When this happens that is when the upper and lower Riemann integrals coincide whatever is the common value. That value is called the integral of f over the interval a to b.

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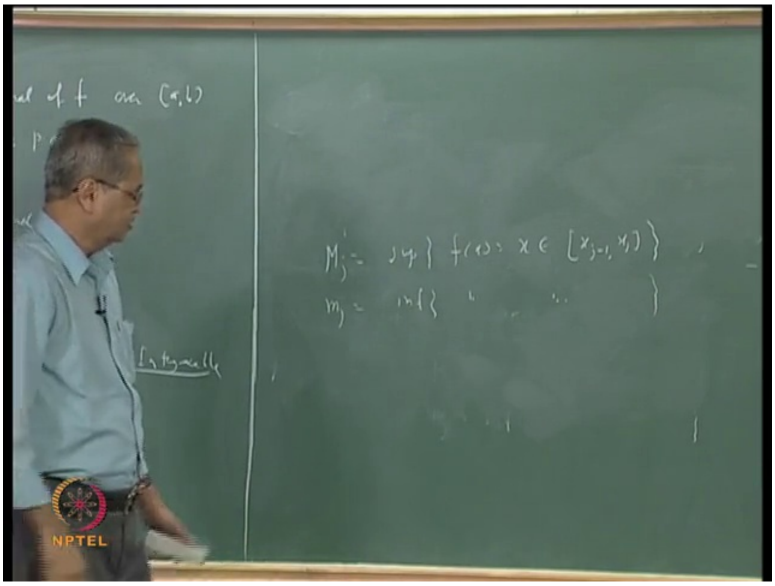

Repeat, that once again big M suffix j that is supremum of f x for x in the sub interval x belongs to x j minus 1 to x j. Similarly, small m suffix j is the infimum of f x, where x belongs to did i write a b here in the earlier? So, that is wrong this is correct. So, coming back to this when the lower and upper Riemann integrals coincide. We say that the function is Riemann inetgrable and whatever is the common value, that is called the integral of f. So, in this case that is in this case means, when the function is Riemann integrable the common value is called the Riemann integral of f integral of f on the interval a b.

This is denoted by this symbol just the integral sign denoted by integral a to b, f x d x. This time no bar either below or above it is just the Riemann integral of the function f. In fact another notation for that is just this integral a to b f. This is also convenient in many cases, because you can see that this x really does not mean much. Because, what is integral a to b f x d x is same as integral a to b f y d y or f t d t. So, that is why this is simply called dummy variable, because you are ultimately summing or integrating over that variable. Whereas, the value depends only on the function f and on the interval a b, that is why this notation is little more convenient.

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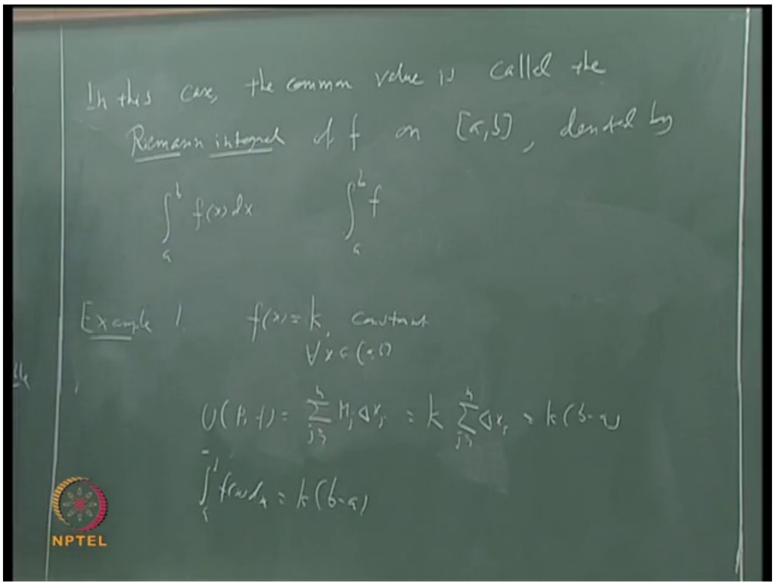

Now, before proceeding further let us think some example of the function, which is Riemann inetgrable and also a function, which is not Riemann inetgrable. So, let us just see example suppose we take the simplest example. Let us say f x is equal to x or even simpler let us just take this. This also will involves little more work just. Suppose, we take f x is equal to just a constant k constant. Suppose, f x is equal to k for all x in a b. Now, if f x is equal to constant then small m j as well as big M j over any sub interval will have the same value K.

So, that is what we take this U P f whatever be the partition U P f that is sigma j going from one to n m j delta xj. Since, in that sub interval x j minus 1 to xj the function is going to be a constant its maximum value is same as k. So, m j is equal to k for each j. So, it is nothing but sigma k times delta x j. So, that k will come outside the summation sign. So, this is nothing but k times sigma delta x j j going from 1 to n. We have seen that this sum is nothing but b minus a.

So, this is nothing but k times b minus a and this k times b minus a is regardless of whatever was the partition P. So, it is infimum over see this number is nothing but k times b minus a for every partition P. So, it is infimum also will be the same. So, in other words integral a to b f x d x. That is upper integral this is nothing but k times b minus a. In a similar way we can also observe that the lower integral or the lower sums will also take the same value. That is L P f, this is nothing but sigma j going from 1 to n m j small m j into delta x j.

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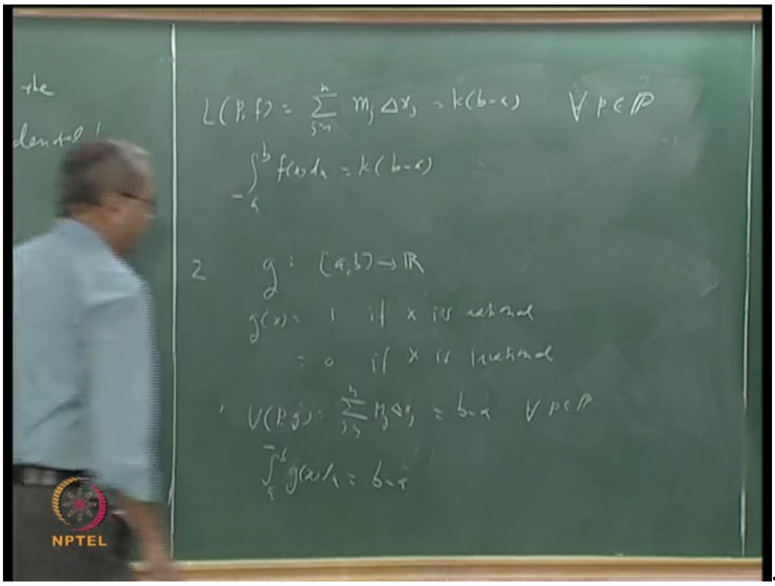

Since, this small m j is also equal to k, because the function is constant throughout the interval and hence on every sub interval. So, this will also become k times b minus a. So, the lower integral integral a to b f x d x. This is again also for every partition for every P in script P. So, this will also be k into b minus a. Since, this coincide we can see that the function this function is Riemann integrable. It is integral is nothing but k times b minus a. This is of course a trivial example, but our idea here is to just explain the definition.

Let us take one more example, so suppose I take the example. Suppose, g is defined as follows g from a b to R. Suppose, we define g as follows g x equal to let us say 1. If x is rational and zero if x is irrational. Now, this time we will see that whatever sub interval you take. Since, it is going to take the each sub interval is going to contain rationals as well as irrationals. So, the minimum over each sub interval will be 0 and a maximum over each sub interval will be 1. So, if you look at the upper sums that is U P f sigma j going from 1 to n big M j into delta x j this big M j is 1 for each j. Because, that is the maximum over the j th sub interval.

So, this value is nothing but b minus a, hence see this is this is true for every partition this is true. For every partition what we have is that integral upper integral a to b. This should be g, because the function we are thinking of here is g. So, upper integral g x d x that is b minus a. On the other hand if you look at the lower integral or first look at the lower sum. So, L P g this will be sigma j going from 1 to n small m suffix j into delta x j, where this small m suffix j will be 0 for each j, because every sub interval is going to contain at least one irrational number.

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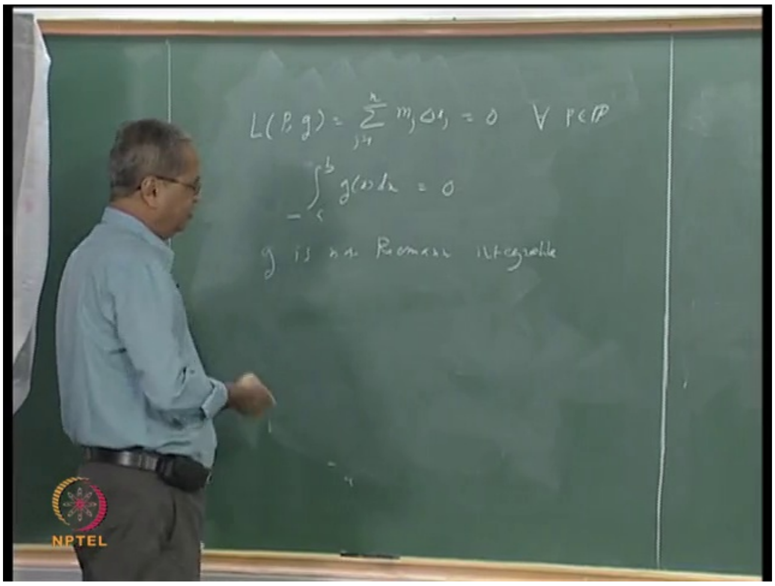

There the function is taking the value 0. So, this sum will become 0 again this is true for every partition. Hence, the lower integral a to b g x d x this lower integral this is 0. So, upper integral is b minus a and the lower integral is 0. So, these two are different, so this function g is not Riemann integrable g. Now, our next objective will be to in general. Now, it is not possible to decide whether a function is integrable Riemann integrable or not. By using the definition like this and calculating upper integrals and lower integrals except in some very special cases like this. For an arbitrary function computing this upper sum. It is supremum or lower sum and it is infimum etcetra that will be quite difficult.

So, we need some verifiable criteria to decide whether a function is Riemann integrable or not, but before developing that we will just one small extension of this idea. So, that that little bit of more work we extend this whole concept to a little more general type of integral. That is called Riemann’s Stieltjes integral.

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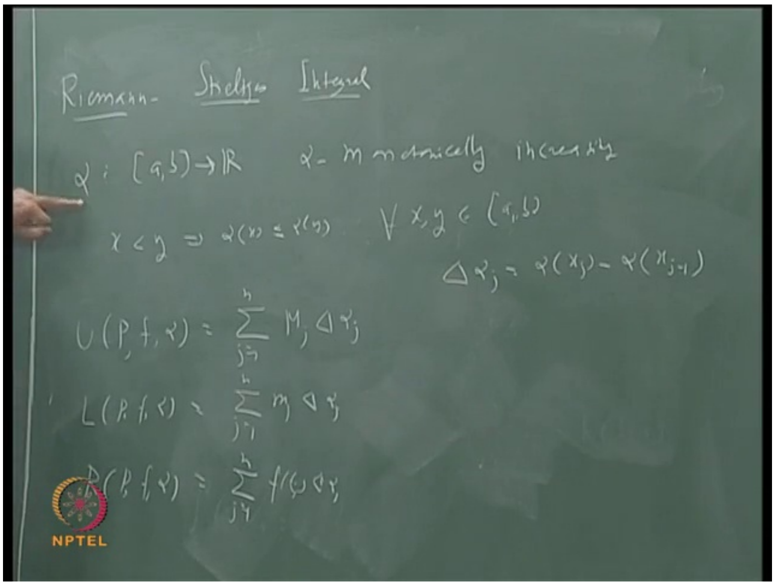

This Riemann integral is a special case of this Stieltjes integral. Now, what is happening here is that as usual f is a bounded function on the interval a to b. We take one more additional function let be the function alpha, which goes from a b to R. We take this alpha to be a monotonically increasing function alpha monotonically increasing monotonically. Let us recall that this means that x less than y this means implies alpha x less than or equal to alpha y, that is monotonically increasing. If we want to say strictly monotonically increasing, it will be in that x less than y implies alpha x strictly less than alpha y for all x y in a b.

We will make a small change in the definition of this upper sums and lower sums with, where it comes to talking about Riemann Stieltjes integral. That small difference is as follows this time will it will also depend on this function alpha. So, it is denoted by U P f alpha that is sigma j going from 1 to n big M suffix j into, what I called delta alpha j. Delta alpha j is nothing but alpha at x j minus alpha at x j minus 1. So, this upper sum is nothing but big M suffix j multiplied by delta alpha j.

Similarly, lower sum nothing but sigma j going from 1 to n small m suffix j multiplied by delta alpha j and also Riemann's sum R P f alpha. That is sigma j going from 1 to n f at t j multiplied by delta alpha j. You can see that why do we say that Riemann sums or Riemann integrals is a special case of this. It is because if you take the function alpha x equal to x. Then delta alpha j is nothing but x j minus x j minus 1, which is nothing but delta x j in which case this will simply become upper sum U P f. This will simply become lower sum L P f etcetera.

In other words Riemann integral is a special case of Riemann Stieltjes integral. If you take the function alpha x equal to x that is a monotonically increasing function. Now, once having defined the upper and lower sums in this fashion, we also note the following. Namely, that each lower sum that is Riemann sum will lie between lower sum and upper sum.

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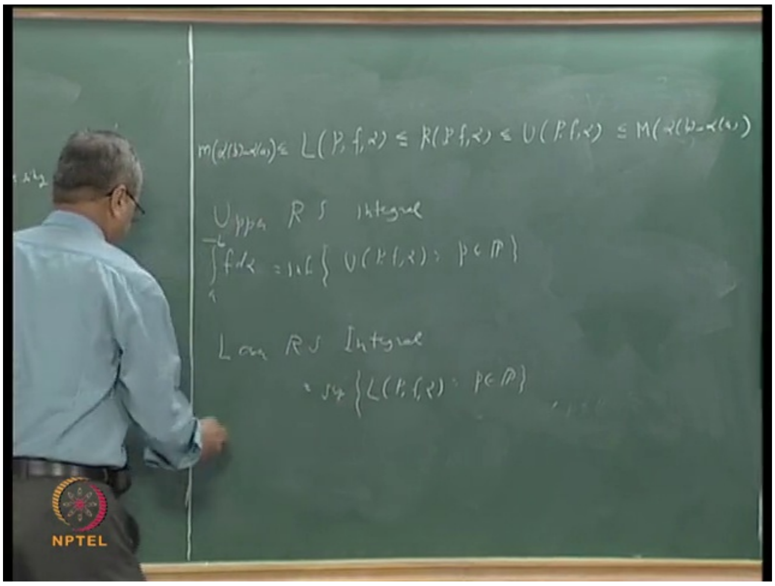

That is L P f alpha will be less than or equal to R P f alpha. This is less than or equal to U P f alpha and as usual we can say that this m j is less than or equal to m. So, this is less than or equal to m into sigma j going from one to n delta alpha j. Delta alpha j will be nothing but sigma delta alpha j is nothing but alpha b minus alpha a. So, this is less than or equal to big M into alpha b minus alpha a. This is bigger than or equal to small m into alpha b minus alpha a.

So, similar in equality if you take alpha x equal to x this becomes m into b minus a and that becomes big M into b minus a. So, that is why we say it is a small extension of the ideas. As usual we define this upper Riemann Stieltjes integral upper Riemann Stieltjes integral as follows. That is again you take this set of all U P f alpha for P belonging to script P.

Take it is infimum that is upper Riemann Stieltjes integral and denoted by integral a to b f d alpha with this bar on top on the integral side. Similarly, we define lower Riemann's Stieltjes integral this time we take the lower sums. So, L P f alpha P belonging to script P and supremum of this, this is lower Riemann Stieltjes integrals that is integral a to b f d alpha with a bar below this integral sign. As in the case of Riemann integral we say that the f is Riemann Stieltjes integrable. If these two integrals coincide and whatever is the common value that common value. We will call the Riemann Stieltjes integral of f denoted by integral a to b f d alpha that is without any bar.

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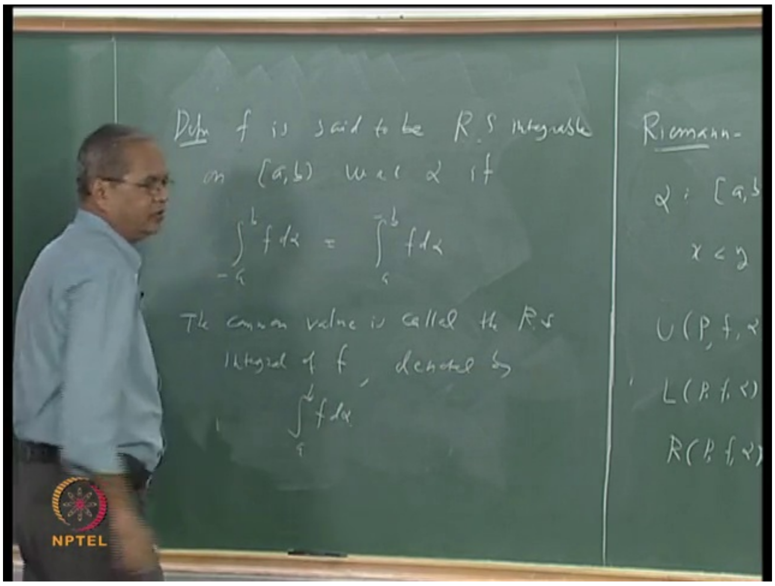

So, we just recorded that is this is the definition f is said to be Riemann's Stieltjes integrable on a b, with respect to this monotonically increasing function alpha, with respect to alpha. If these two integrals coincide if integral lower integral a to b f d alpha is equal to the upper integral integral a to b f d alpha. This common value is called a Riemann's Stieltjes integral of the function f with respect to this function alpha common value. That is whatever is the common value of the lower and upper integral the common value is called the Riemann's Stieltjes integral of f denoted by integral a to b f d alpha.

So, now coming back to the problem of deciding which function is Riemann integrable or which function is Riemann Stieltjes integrable. Again, as I said earlier that requires some criteria or some tests and that requires some more work. So, just by using this definitions either of Riemann integrals or of Riemann Stieltjes integral. It is possible to decide the integrability in case of very few functions, but after going through some of these criteria. In particular some of the theorems we shall be able to decide this question in case of a fairly large number of classes. To discuss that we need some more properties of this partitions. So, let us say that we take two partitions P and Q.

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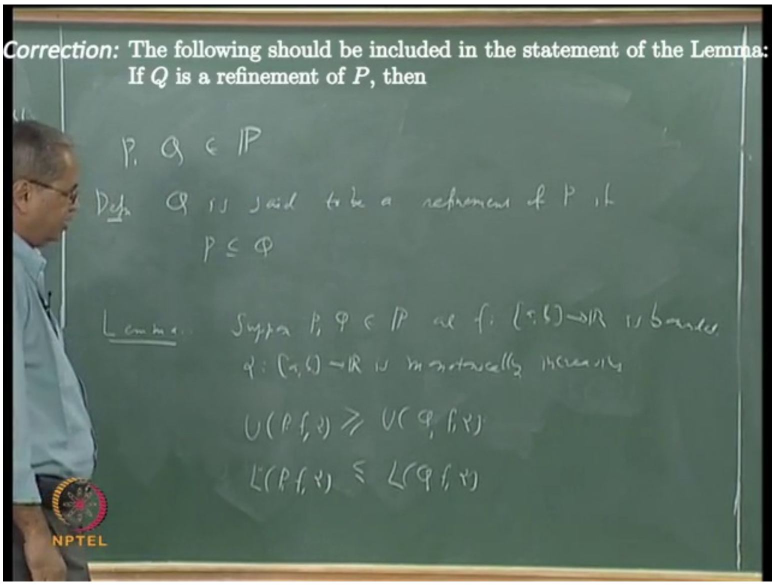

Suppose, these are the two partitions then we say that one partition is a refinement of the other. If as a set Q contains P we will say that is a definition Q is said to be to be a refinement of P. If P is a subset of Q what is it mean that Q contains all the points of P. Possibly some more remember all the partitions must contain the first and the last point a is x naught and b is x n. Those two points are there in all the partitions in between, whatever points are there in P. Those are also there in Q and perhaps Q may contain some more some more points.

So, what we want to know now is that, suppose we take a refinement of a partition. Then how do the upper and lower sums change, that is the first thing. So, let us see to that let us call that as a lemma. Suppose, P and Q are the partitions suppose P Q are partitions of the interval a b and f is f from a b to R is bounded. Then what we want to say is that we want to compare these two. That is the alpha from a b to R is monotonically increasing. We want to compare the corresponding upper and lower sums of these two partitions. That is we want to compare these numbers U P f with this u opposing U P f alpha and U Q f alpha.

Similarly, the lower sums L P f alpha and L Q f alpha. So, what we want to say is that when we take a refinement the upper sums decrease. That is U P f alpha should be bigger than or equal to U Q f alpha. The lower sums increase that is L P f alpha is less than or equal to L Q f alpha. To repeat again whenever we take a refinement of a partition the upper sums decrease and the lower sums increase, that is what is lemmas is. Let's now see how we can prove this.

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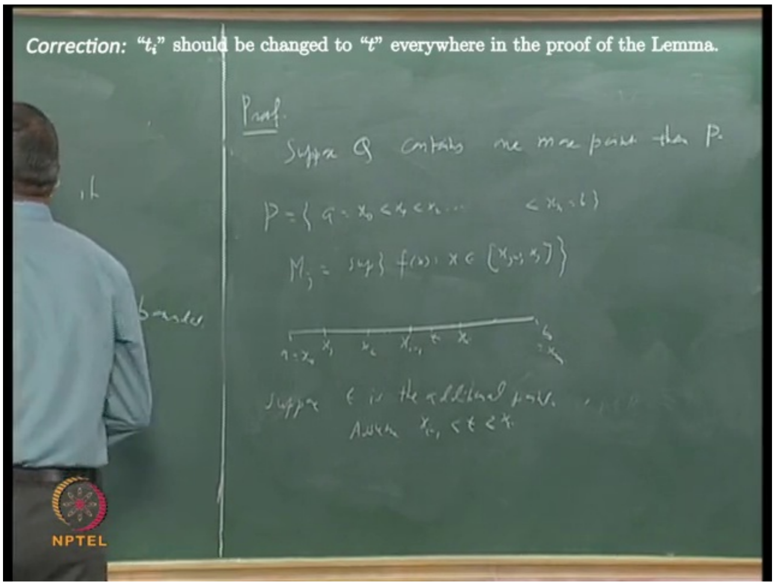

Since, if P is equal to Q, there is nothing to be proved, then both these will be simply equal. So, let us say that Q contains some more points of P. To begin with let us first assume that Q contains just one extra point. Suppose, Q contains that is one more point than Q than P. Suppose, Q contains one more point than P. Let us say P is a equal to x naught less than x 1 less than x 2 etcetera. Last point is x n and that is equal to b and let us look at the upper sums first. Let us say big M suffix j is a supremum of f x in x j minus 1 to x j. That is supremum is taken over this sub interval.

Let us just draw this picture suppose this is a, a is x naught. Then we have x 1 somewhere x 2 etcetera. This last point b is x n there are some point x 2 etcetera. All these points are there in Q also all these points are there in Q also, but in addition there is one more point. Now, that point has to be obviously different from all these points x naught x 1 x 2 x n. So, since it has to be different from all these points it cannot be it. It cannot be none of these x j x j minus 1. So, it has to lie properly in some sub interval it has to lie properly in some sub interval. So, suppose that point is let us say that point is t.

Suppose, t is the additional point now that point has to lie in some sub interval. So, suppose that sub interval is let us say x i minus 1 to xi x i minus 1 to xi. Of course, the point t cannot be xi minus 1 or x i it has to be properly between that interval. So, suppose that point is t. So, suppose t is the additional point, so let us say assume x i minus 1 less than t less than x i. Now, look at the upper sums corresponding to the partitions P and Q. Let me write it here sigma j going from one to let me write it U P f. That is sigma j going from 1 to n big M suffix j delta x j delta alpha j.

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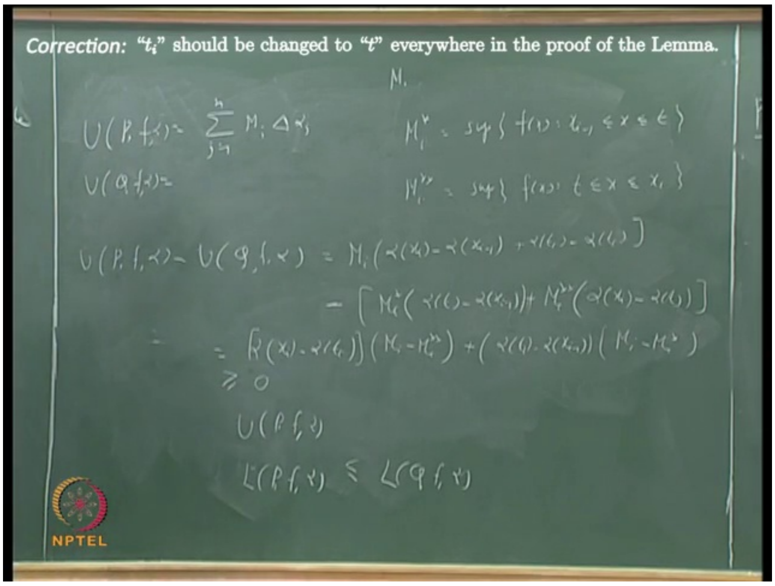

U Q f that is here we have to write somewhat carefully, U Q f What will be happen to U Q f? For writing that U Q f for the interval x naught to x 1 to x 2 etcetera xi minus 1. Similarly, x i to x i plus 1 for all those other intervals the terms corresponded to this. Other only difference will occur when you take this interval x i minus 1 to x i x i minus. So, for that what we will have to do is we will have to look at supremum over this sub interval xi minus 1 to t and t to x i.

So, suppose I give some notations for this so mi let us say m i star is supremum of f x for xi minus 1 less than or equal to x, less than or equal to t. So, that is the supremum over this interval that is m i star. Suppose, this I call mi double star that is supremum over f x this time. It will be t less than or equal to x less than or equal to x i. So, if you look at this

U Q f for all j s not equal to i the term is the same. For the corresponding interval xi minus 1 to xi, it will be mi star multiplied by this t minus x i minus 1. Plus, m i double star multiplied by x i minus t.

So, suppose we look at the difference between the two. Basically, we want to show this that U Q f alpha is less than or equal to U P f alpha. So, we look at the difference between the two. So, we look at U P f alpha minus U Q f alpha what will be the difference? The terms the difference will be, because of this sub interval x i minus 1 to x I, because all the terms coming from the other sub intervals are the same. So, when you subtract all the other terms are going to cancel. what is going to remain from here is this m i into delta alpha i.

Let us write delta alpha i in the full form for delta alpha i. It is alpha x i minus alpha x i minus 1. That is because of this and what comes from this it is minus it is m i star into alpha t minus alpha x i minus 1. Plus, mi double star alpha x i minus alpha t alpha x i minus alpha t. Here, also here also yes it should be U P f alpha and here also U Q f alpha that is right.

Now, what we can do here that we can write this same thing by adding and subtracting. This term alpha t can write this as I can it is alpha x i minus alpha x i minus 1. I can say plus alpha t minus alpha t. So, if I do that you can see that it will be alpha I will just take this first alpha x i minus alpha alpha x i minus alpha t. This multiplied by mi minus mi double star and what remains plus alpha t minus alpha xi minus 1, that into here also m i minus m i double star m i minus m i star here.

Now, the thing is to compare mi and mi double star m i is the supremum over this whole interval x i minus 1 to x i. Where, is mi star is supremum over this sub interval, let us also recall, what is m i anyway it is. It follows from there m i you just take j is equal to i m i is supremum of f x for x in the interval x i minus 1 to x i. So, the supremum over this bigger sub interval is obviously bigger than or equal to supremum, over this smaller sub interval. So, that means m i is bigger than or equal to both m i star as well as mi double star. So, alpha x i minus alpha t i those are anyway non negative numbers, because alpha is monotonically increasing. So, if you look at this alpha x i minus alpha t is bigger than or equal to 0 m i minus m i star is bigger than or equal to 0.

Similarly, here alpha t minus alpha xi minus 1 is bigger than or equal to 0 and m i minus m i star is bigger than or equal to 0. So, this whole thing is sum of two non negative numbers. So, this is bigger than or equal to 0. That proves that U P f alpha is bigger than or equal to U Q f. That is upper sum corresponding to P is bigger than or equal to upper sum corresponding to Q. That means if you take a refinement the upper sums reduce in a similar way one can show that lower sums increase. Because, what will be the change here you will take small m i star and small m i double star. In that case since we are taking infimum over a smaller interval. That will be bigger than or equal to infimum over the bigger interval. That is what we will have to use.

Now, there is only one thing, which we have assumed here, we have assumed that Q contains one more point than P, but suppose you take an arbitrary case Q contains. Let us say finitely m points more than P. Then you can take the you can consider partition Q one Q two etcetera. At each stage Q contain one more extra point. Then use the same inequality. So, by induction we prove that if Q contains a finitely many more points than P, then by the repeating this same argument. That many number of times we will be able to prove that upper sums reduce and lower sums increase by taking the refinement. I think we stop it there.