Real Analysis
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Lecture - 37
Differentiation of Vector Valued Functions

Well, so far we have discussed the differentiability of functions which are defined on some open interval of the real line, but also take values in the real line.

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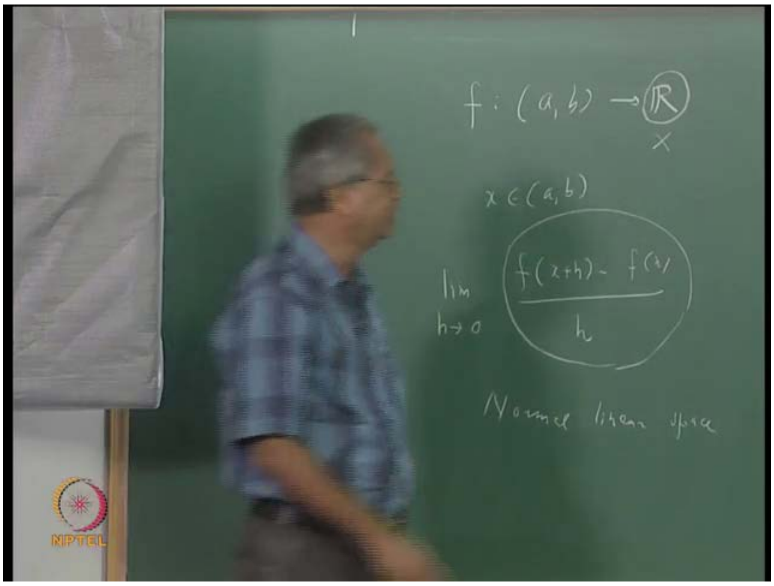

That is to put it in symbols we have taken functions of this type a b to R, now our idea is today to discuss a few things about what happens when we change this codomain, change this codomain to some x. So, first of all, we would like to see what are the properties that are required for this codomain to satisfy, so as to make this whole concept of differentiability meaningful in that structure.

So, if you recall let us suppose you take some x in a b how do we or when do we say that the function is differentiable at the point x. If you recall the definition let us recall that we take this quotient f of x plus h minus f of x and then divided by h and take the limit of this as h tends to 0. Now, when f was going from a b to R all it was fine this makes sense, but suppose, now this objects are coming from some other, some other set x then f of x plus h is a member of x f of x is also member of x. Then what is by subtracting those two in an arbitrary set x, so obviously you cannot, this cannot be an arbitrary set x there has to be some meaning of subtracting two elements.

In other words there has to be some algebraic structure in x, so addition or subtraction of two elements in that first makes sense not only that this 1 by h is a real number, so you are multiplying by that. So, multiplying by real number to an element of that also should have some meaning and I think by, now you would understand what I am coming to since you are also doing the course on linear algebra. Simultaneously, what is required is that this must be a real vector space that is for, given two elements in this there should be some meaning of saying when the two elements.

What is the meaning by addition or subtraction of two elements and what is meaning by multiplication of a real number and that? So, the minimum thing that is required is that x must be a real vector space that is clear, but, that is not enough suppose x is a real vector space this quotient or this fraction makes sense. But, we want something more we want to take the limit of that as h goes to 0 limit of that means the whole process of limit taking limit also must be meaningful in that vector space. So, you cannot take vectors totally arbitrary vector space on the vector and you have seen till now that you can talk of limits when it is a metric space.

So, what we need is that this should be a vector space totally vector space it should also be a metric space and, of course there should be some relationship between that metric and the algebraic operations. The most natural way to define these things is what is called a normed linear space and, so what is a normed linear space it is a real vector space and on that a function called norm is defined. That function norm leads to a metric that, so this norm is something like the absolute value in the real number see in the case of real numbers.

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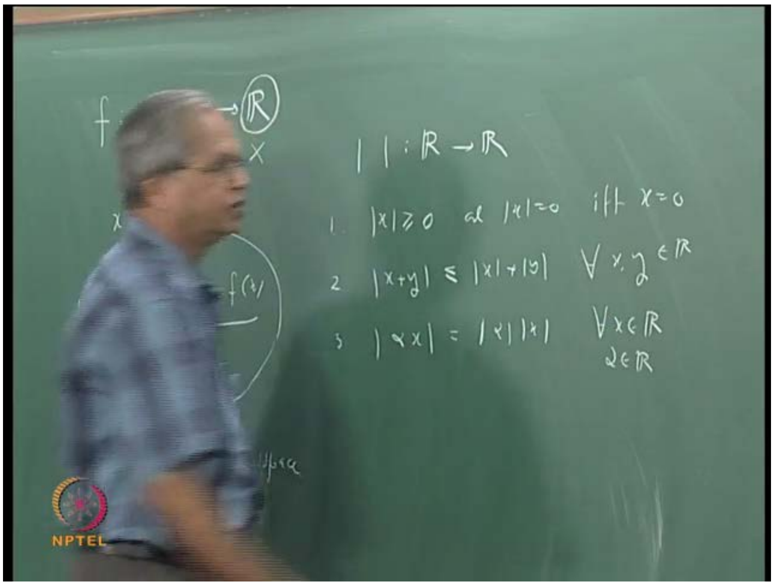

Suppose we look at the, suppose you take this function R to R then what are the properties of this function we know that absolute value of x is always bigger than or equal to 0 and it is equal to 0 if and only x is equal to 0. So, that is one thing and second thing that you know is that absolute value of x plus y is less than or equal to absolute value of x plus absolute value of y for all x y in R. Third thing that you know that if you take it alpha times x, if you take alpha times x then this is absolute value alpha times x is same as absolute value of alpha into absolute value of x.

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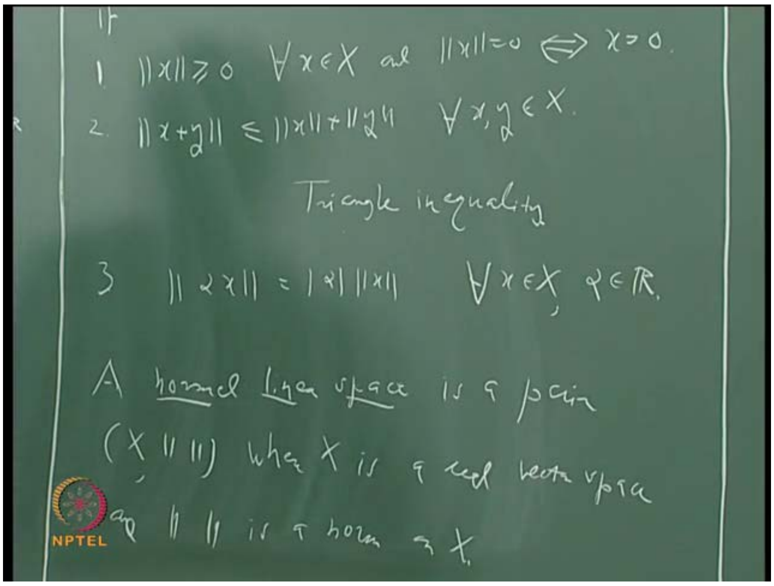

This is true for every x in R and also alpha in R, now instead of R suppose I take, replace this by some other vector space. Here, then we take a function which goes from that x to R and satisfies all this properties, if that happens that function is called a norm that function is called a norm.

So, let us, let us just look at that, so let us say x is a real vector space of course we can define something similar in the complex vector space also. But, we shall not go into that much of generating x is a real vector space and suppose we denote this function by norm, norm is a function which goes from x to R. So, we will say that a function from x to R is called a norm, is called a norm x on x if it satisfies the property which are similar to this wherever I have written, here absolute value you replace that by that by norm.

So, first property is that norm of x is bigger than or equal to 0 for every x in x and it is equal to 0 if and only x is equal to 0. Second property is this norm of x plus y is less than or equal to norm of x plus norm of y for every x y in x this is a property which is usually described as triangle inequality. Third property is norm of alpha times x is same as absolute value of alpha into norm of x, this happens for every, for every x in x and every alpha in R you might have wondered. While writing the properties of the substitute value why I took alpha, here because I could organ this as simply y into x, but what are the conventions follows is that if x is a vector space over the some field R or c R.

Whatever it is you normally denote the elements of the vector space and underlying field by different sets of symbols. So, for example when can use the convention that will denote the elements in x which are vectors as by the usual lower case English letters x, y, z etcetera and elements in R which are scar, scalars by this let us alpha Greek letters alpha, beta etcetera. So, a normed linear space is nothing, but a vector space on which a norm is defined, you can say that normed linear spaces are paired. So, a normed linear space normed linear space is a pair x norm where x is a real vector space and norm is a norm x you can compare this definition.

With the definition of a metric space we define first what is spend by a metric given any set x we define, of course it is not a function from x to R it will be function from x cross to R satisfying some properties. Similarly, that is called a metric on x and a metric space is a pair x comma d if d is a metric one x it is similar. Now, what is the relationship between a normed linear space and a metric space, it is that a large number of metric spaces are basically normed linear spaces. Every normed linear space is a metric space of course converse is not true there are, so except for example you have discrete metric space. That is not a normed linear space because you can define discrete metric for any set, whereas normed you can talk of only on a vector space.

But, most useful examples of metric spaces are normed linear spaces by the way there can be different norms on the same vector space just there can be different metrics on the same set. Similarly, there can be different norms on the same vector space, so as vector space is those two will be the same but, as normed linear spaces those two will be different objects suppose I take two different norms then there will be two different objects.

Now, what is the relationship between vector space and A, this normed linear space and a metric space as I said just now every normed linear space is a metric space. How does that happen for that you have to do a function which goes from x cross x to R starting from, starting from this norm. Again, for example what is this, here you have the properties of absolute value on the higher line how is the metric on x, so called usual metric on x is defined distance between x and y norm of x minus y is a norm of x minus y. You do the same thing in arbitrary normed linear space you define distance between x and y as norm of x minus y distance between x and y as norm of x minus y that is a metric.

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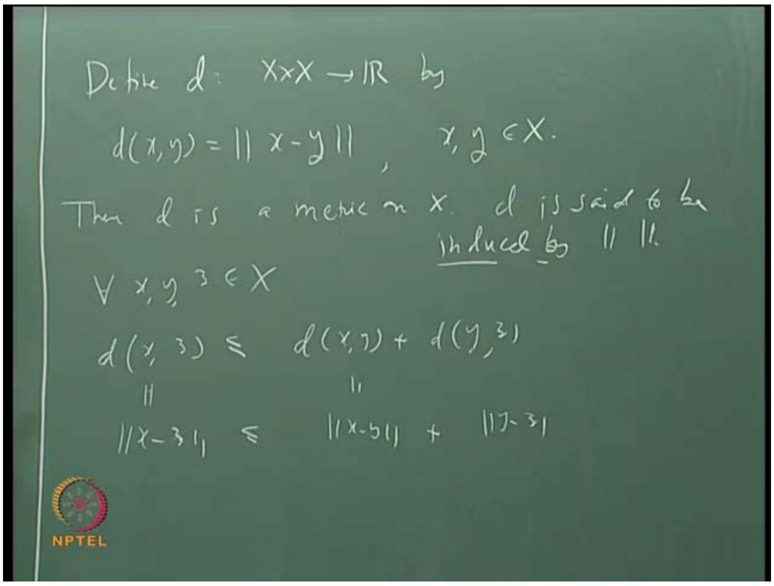

So, you can define d from x cross x to r by d of x y as norm x minus y then d is a metric 1 x d is a metric 1 x, now how does one check that d is a metric 1 x. Let us infinitely fairly easy we can quickly verify this for example what is required for showing that d is a metric. First of all d x y should be bigger than or equal to 0 for all x and y is it true that follows from this first property. Here, what is the next thing that you require that is d x y is equal to 0 if and only x is equal to 0, does it follow again suppose this is 0 then you must have x minus y equal to 0 which is same as x equal to again it follow.

So, that first property follows from, here what is the other thing that you require, you require of d of x y should be same as d of y x what is what will be d of y x from this. Norm of y minus x is it same as norm of x minus y and from what does it follow, it follows from here suppose you take alpha is equal to minus 1 then norm of minus x is same as norm of x. So, that will give norm of x minus y is same as norm of y minus x and what is other thing required.

You require that if you take for all x y inside in x you require the d of x z should be less than or equal to d of x y plus d of y z and this is what you called a triangle inequality in metric spaces. Now, what is d of x z this is nothing, but norm of x minus z what about this d of x y norm of x minus y and this is norm of y minus z. So, can you say that norm of x minus z will be always less than or equal to norm of x minus y plus norm of y minus z and again from what does it follow this property. Here, for example you take A as x a as let us say x minus y d x y minus z then x minus z will be a plus b and that will give this property. Here, that is why this is also called triangle inequality that is why this is also called triangle inequality alright.

So, every norm every norm leads to a metric every norm leads to a metric, so this is A, d is a metric on x and this metrics is called induced by this norm it is called induced by this term it is metric on x, d is said to be induced by this term of course. Here, one may ask, now this question, it is suppose you are given a vector space and a metric on d is every metric on vector space induced by some norm again. The answer is no, there are metrics which do not come from any other norms because norms will have some more properties compared to the metric. It is possible to say when exactly a metric is induced by a norm it should draw, but, I think that discussion will take us too far away from what we basically want to discuss.

So, instead of getting into various properties of this norms and the relationship of metrics and properties of normed linear spaces that is basically a subject of functional analysis. In functional analysis, you will learn normed linear spaces and operators on normed linear spaces.

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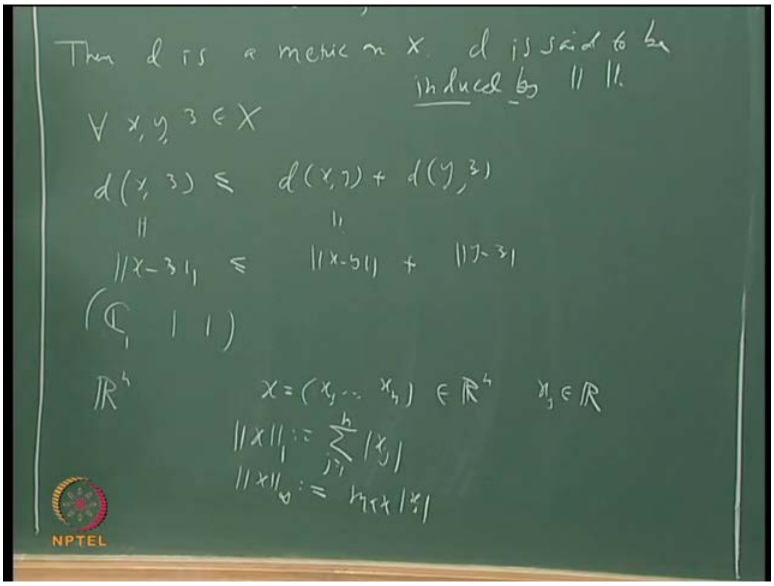

So, coming back to what we want to discuss we just have see some well known or usual examples with which we deal with. So, one example is this R with d usual absolute value that is a norm R with a usual absolute value that is a norm, similarly you know that absolute value can be defined on C also C, if C is a complex number you can talk of what is meant by its absolute value. That also satisfies all this properties, so that is also a norm, so C and you can regard C also the real vector space, you can regard C also as a real vector space.

By the way what you have defined is what is called a real normed linear space because if you take you are starting with a real vector space. So, it is called a real normed linear space if we instead of real vector space, if we are started with a complex vector space it would have been a complex normed linear space. But, let us confine our attention to real normed linear spaces then what are the other obvious examples that you would have already seen. These spaces are n, space is R n with again with various norms on that we can define we can define various norm on this spaces R n.

So, let us just take one or two well known examples, for example this is, so any element x suppose I take x in R n then that is of the form x 1, x 2, x 3, it is n double where x 1, x 2 x n are real numbers. So, each x this belongs to R n that means each x is in R, so how does one define norm there are various ways as I said on the same vector space you can define various norms. So, for example 1 is called normed suffix 1, so normed suffix 1 that is nothing, but you take mod x 1 plus mod x 2 plus just take the sum of all the absolute values of all the n coordinates.

So, this is sigma mod x j, j going from 1 to n in fact that is called verb norm or sometimes called element norm and, similarly there is also what is called norm suffix infinity. Instead of taking the sum of the absolute values, you can take the maximum of absolute values that is called norm suffix infinity that is maximum. Anyways there are n numbers mod x 1, mod x 2, mod x n take the maximum of this, so that is maximum of mod x j.

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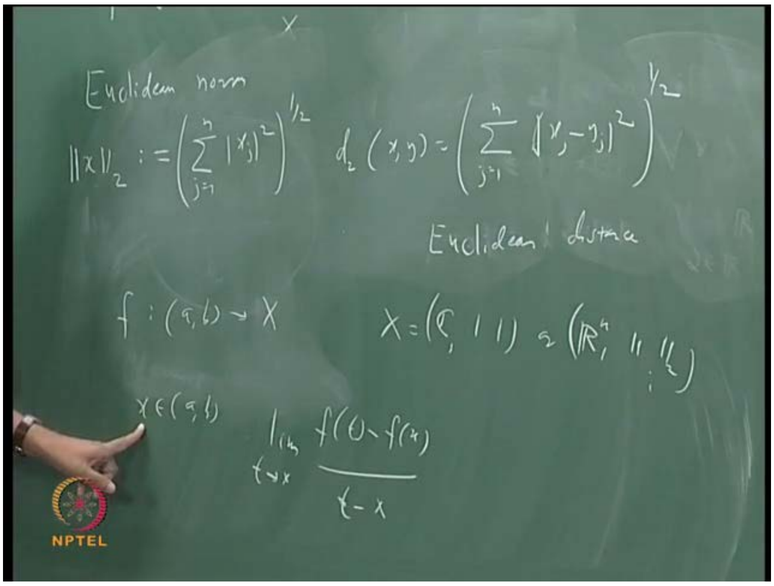

Another example of norm on R n which is most popularly used it is called Euclidean norm and that is denoted by this norm suffix 2, norm suffix 2 that is norm of x suffix 2 this is defined as follows. By the way, I suppose you are familiar with this symbol this means whatever is there on the left side its definition is on the right of this symbol. So, this is sigma mod x j square, j going from 1 to n and then takes its square root, take its square root, so similarly instead of taking this power 2. Here, one can also take some other power between 1 and infinities, mod x j to the power p and then whole root of power 1 by p that is called norm suffix p.

So, one can define any number of norms on a given vector space and each of this term will lead to a metric each of this term. For example, this norm will lead to a metric this term is also lead to A, for example if you look at the definition if you look at d x y nothing. But, norm of x minus y suppose the corresponding metric if I call d 1 then done x y will be sigma j going from 1 to n mod x j minus y j and, similarly this suffix infinity. Here, if I take say d suffix 2 of x y that will be sigma j going from 1 to n mod x j minus y j whole square and then this is to 1, 2 and this is what is called the Euclidean distance, Euclidean distance or Euclidean metric.

You can see the obvious reason for that because when you take n is equal to 2 if you are looking at R 2 or R 3 this is the usual distance between the two points. That is why the corresponding norm is called Euclidean norm and corresponding distance is called Euclidean distance. So, for our discussion of this differentiation of the functions let us confine our self to these two examples either C with this absolute value or R n with one of this norms, R n with of course. I have not proved that these functions defined a norm, defined a norm that requires some it is fairly easy to check in these cases, for example suppose it is obvious that each of them is less than or equal to 0.

In fact this something that you should know even while discussing the metric, the real thing that deals verification is a triangle inequality. Other properties are usually straight forward where is the definition of A, distance is given to verify that it is a metric other. Similarly, in the case of norm Whilst definition of a norm is given to verify this and this is usually trivial, for example if you look at, here norm of alpha times x that is nothing. But, sigma mod alpha x j that mod alpha will come outside this summation sign and that is nothing, but mod alpha times norm x.

Similarly, in case of the other norms real verification where some work is required is for this triangle inequality that is also fairly easy in case of this two because this will mod x j plus y i. You can say that mod x j plus y j is less than or equal to mod x j plus mod y j and then separate in the sum, but for something like this for you have something like this to prove that the triangle inequality is satisfied. You have to go through some standard inequalities the proof like this satisfied this term satisfies this triangle inequality requires what is called Cauchy's norm inequality and also followed by Minkowski inequality.

They are very well known inequalities this will not go into the proof but, using those inequalities one can show that that triangle inequality is satisfied by this function. So, now let us go back to our this thing I take this f from a b to R not a b to R a b to x where what is x is either x can be, so see x can be in general any normed linear space x can be in general any normed linear space for definition. But, for most of things mostly practical purpose we can take x as either the C with this absolute value or R n with any of those norms. Let us say norms of x 2, 2 or 1 or you take any of those norms, but most frequently we deal with this norm because that is the one which coincides with the usual concept of distance in 2 and 3 dimensions.

Now, let us take some x in a b then, now we can talk of f of instead of talking x plus h etcetera let me just take any other number t which is also there in a b. So, say I can say f of t minus f of x you think of this t as x plus h, so saying that h goes to 0 is basically same as saying t goes to x, so I will take f t minus f x divided by t minus x. Now, all this makes sense f even though f t and f x are in this x, since x are one of this f t minus f x is well defined that divided by t minus x is also well defined. Now, we think of limit of this as t tends to x and, now limit is also of course limit may or may not exist.

But, the concept of limit is well defined because this is a metric space, we can define in a usual way that suppose this limit is, of course this limit will also be some element in x, this limit will also be some element in x. So, wherever it exists and if it exists we can define it in usual way that is given epsilon, they should exist some delta bigger than 0 etcetera that is if this limit exists we shall call, we shall say that the function is differentiable at this point x.

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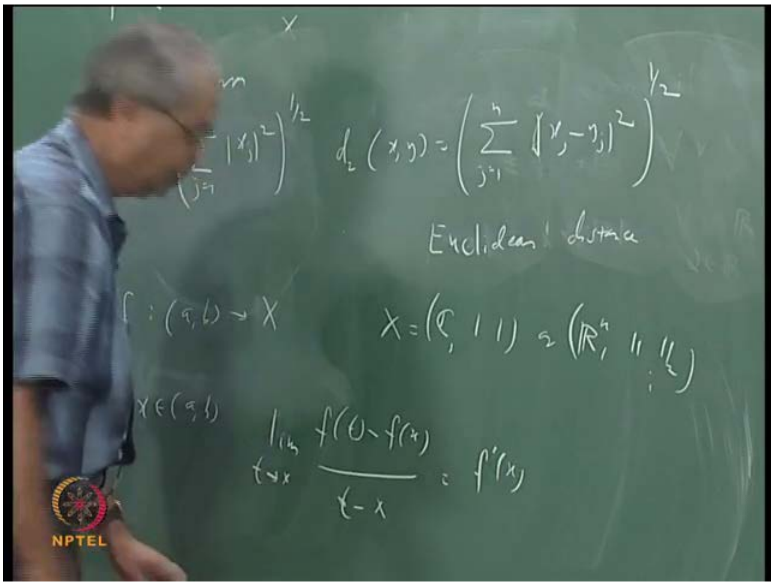

Whatever is that limit that will be called derivative of f at that point x and this will denote that by as usual f prime x, so this whole topic which I have discussed just now is called differentiation of vector valued functions.

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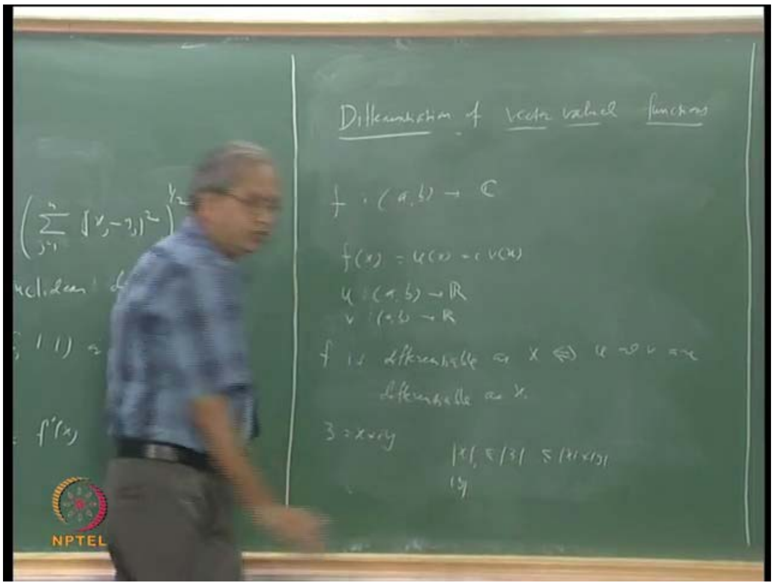

This is called differentiation of vector valued functions, so this prime x will also be an element in this space x either in C or R n etcetera. Now, the question is fine you can define what is meant by differentiation of this vector valued functions, but can you do something further above that. Of course, one can also ask one more question why just replace this codomain by some others thing why not think of replacing this, also yes that can be done. But, it will lead to some more complications, so we will not go into that, that is something you may perhaps learn in your course in advance calculus.

That is where you talk of differentiation of functions which are also defined in R 2, R 3 and things like that anyway this is little simpler and we shall we shall discuss here. So, the question is what other things go through obviously one can define what is meant by derivative then whatever are there very routine things about the derivatives those will go through.

Even for this vector valued functions and what are those, for example the theorems about the derivatives of this sums, for example if you take two functions f and g and suppose f and g both are differentiable at point x then f plus g is also differentiable. The value of derivative is same as f prime x plus g prime x product we cannot talk about in general because there may be no product in this x. It is just a vector space, it is just a vector space, so given two elements in x the product may not be defined. But, if you have some more additional structure on x you may talk about the product also, but let us not go into certainly.

There is no meaning of talking about of course you can talk of product, here product in C because you have there is a definition of product of two complex numbers. But, suppose you take two elements in R n you can talk of dot product etcetera, but not the usual product because that will not be in R n. Similarly, you can talk of multiplying, but you can talk of multiplying by a real number to a function f. For example, you can talk of the function alpha times f when alpha is a real number and its derivative also will be same as alpha times f prime x.

Then one can also see the following things we can, for example let us first take this case c, so when for example suppose f is a function from a b to C. Now, what is this mean it means that f x is a complex number, it means that f x is a complex number and each complex number has a real and imaginary part. So, I can, suppose I call the real part as say u x and imaginary part as v x then for each x in a b you have defined these two real numbers. In other words every such complex goes to a function leads to two real valued functions u and v both from a b to R, similarly v from a b to R.

Now, the obvious question, here is what is the obvious question suppose we know that f is differentiable at a point x can we say that this u and v are also differentiable. Similarly, what about the converse suppose we knew that u and v both are differentiable at a point x can we say that f is differentiable. Now, again the answer to this question both of this question is yes that is f is differentiable at x if and only if u and v both are differentiable at x. Let me just write that, here this f is differentiable at x if and only if u and v are differentiable at x.

How does one prove this in fact I would say that this follows from the property of the absolute value function on the complex numbers and what is that property that is suppose we are given any complex number z. Say z is equal to x plus i y then how are the absolute values of z and x written in this case one can say exactly that norm z square is x square plus square root of x square plus y square. But, that is not something I am looking at, what I want is the following that is can we always say that mod x is less than or equal to mod z.

This is something we can always say can we also say that mod z is less than or equal to mod x plus mod y this also something we can say and similarly, here you take mod x or mod y both are less than or equal to mod z. Now, we just apply that to this f x is equal to u x plus i v x, so suppose you write here f x is equal to u x plus i v x then and similarly, f of t will be u t plus i times v t what is going to happen is this.

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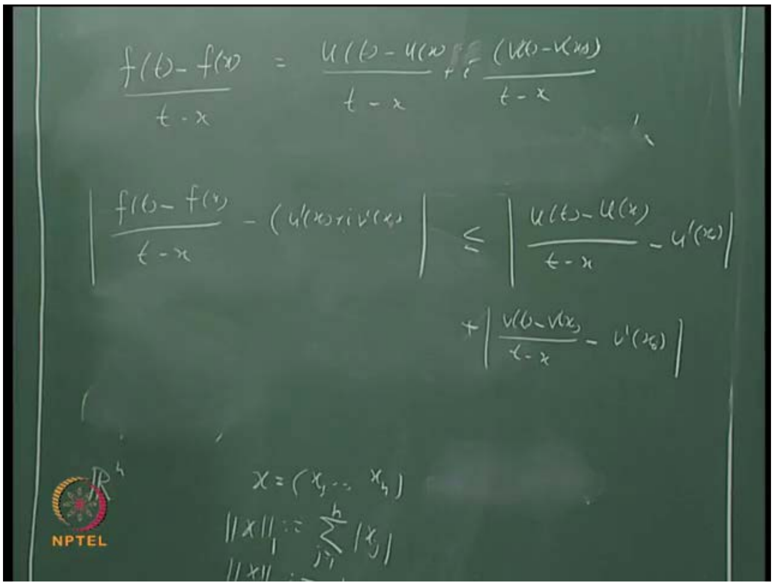

So, f t minus f x, this will be say suppose I write f t as u t plus i times v t and f x as u x plus i times v x, so this will be simply u t minus u x plus i times v t minus v x. Suppose you divide this by t minus x and this I will write this i here i this also divide by t minus x then let us say that suppose the limit of this exists u t minus u x divided by t minus x suppose this exists. Then that we would call u prime at x and, similarly if the limit exists that we will call v prime at x. But, then the limit of this plus i times this should also exists and that is where we, that is where we use this property what we can say is that suppose you take this fraction f t minus f x divided by t minus x.

Let us say minus this u prime at x naught plus i times v prime at x naught and suppose you take the absolute value of this then use this mod z is less than or equal to mod x plus mod y. So, remember f t minus that is u t minus u x divided by t minus x minus u prime x naught that is a real part of this whole thing, remember t minus x is the real number t and x both are coming from this interval a b t and x, so t minus x is the real number. So, what one can say is this is less than or equal to u t minus u x divided by t minus x minus u prime x naught plus v t minus v x divided by t minus x minus v prime x naught.

Suppose we assume that this has a limit and it is limit is u prime x naught which is same as saying that this goes to 0 as t goes to x and similarly this also goes to 0, as t goes to x. Which is same as saying that because this left handed is less than or equal to that this will also go to 0 as t goes to x, which is same as saying that the limit of this exists as t goes to x and that limit is same as u prime x plus i times v prime x. In a similar way, you can prove about the converse also what did we prove just now that if u and v both are differentiable at a point x then f is differentiable at that point.

Its value is nothing but, u prime x plus i times v prime x conversely suppose if f is differentiable at a point x and suppose its derivative x prime x then real part of f prime x will be the derivative of u and imaginary part will be derivative of v. In a similar way for that you will use this part of inequality for that you will use, this part of inequality because you can say that if you take only this much that will be less than or equal to this that will be less than or equal to this. But, this is about the complex number, now with small modifications, with small modifications we can we can extend it to this R n also. Here, what is involved see in C there are only two things real part and imaginary part, but what we can say is let me now take this situation.

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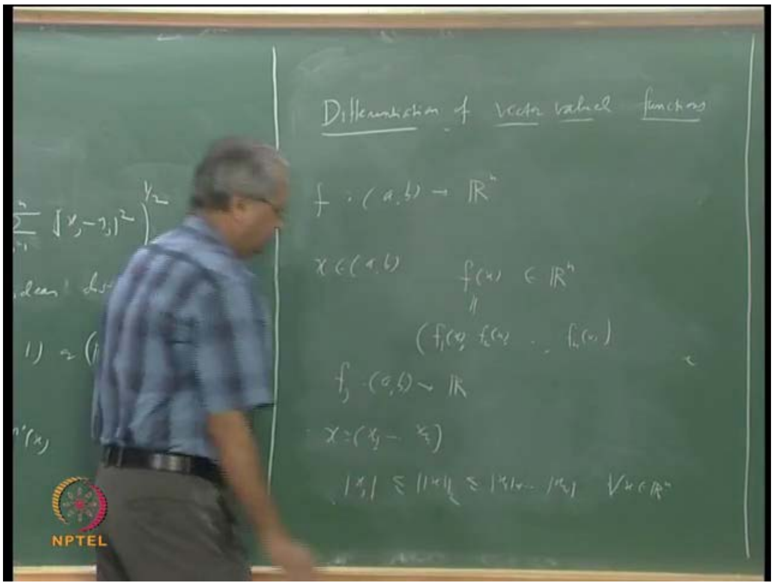

Suppose this is slightly more in what is the sense there are more components that is all suppose you, now take f from a b to R n, now you take let us say x in a b what will be f x, f x will be a member of R n, f x will be a member of R n. Now, if it is a member of R n what is R n, R n is a R n consists of a n types of real numbers that means this f x will have n components, f x will have n components. So, I can call suppose I call the first component as f 1 x, so I can suppose I can call this f x as follows, I will call the first component as f 1 x, second component as f 2 x and nth component as f n x.

So, what is, what is similar to this with the complex number what is the similarity, there you had only two real and, here you have n components. So, what this mean that if you have one R n valued functions that corresponds to n real valued functions is it clear. Suppose you have one function f which takes value from a b to R n corresponding to that you will have n such functions f 1, f 2 and each of those functions will go from a b to R. So, suppose I call each j-th function, here is f j, f j will be a function from a b to R, f j will be a function from a b to R.

Now, coming back to differentiability what will be the question and the answer, here what you would ask all those things suppose I knew that f is differentiable at a point x what can you say about each of this n functions. So, what we must expect is that each of this n functions are differentiable at that same point x and what will be the value of derivative. It will be nothing, but see f prime x naught that will also have n components its first component will be derivative of this, its second component will be a derivative of f 2 etcetera.

That is what we should expect and that also can be proved again by essentially the same idea here that is just as we have. See in that case of if the complex number z is x plus i y, we wrote it mod x plus mod is equal to mod z and mod z less than or equal to mod x plus mod y etcetera. Similarly, for each suppose you take x is equal to x 1, x 2, x n and you take any norm points any of those norms, then each of those norm has a following property that if you look at mod x j, if you look at mod x j then that is less than or equal to norm x.

Whatever norm you take whatever norm you take and that is less than or equal to mod x one plus mod x 2 plus mod x n, this is true for every x in R n since I said that most often we will be discussing this n 2 norm. Let me write it for this and this is what we shall, you will use to show that show the following I will remove this.

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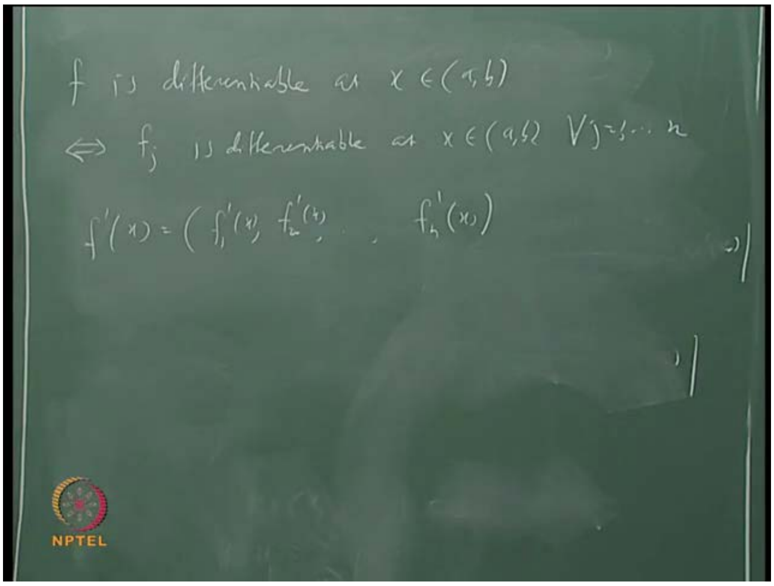

That is f is differentiable at x in a b if and only each f j is differentiable at x in a b. That is whenever f is differentiable for all j, for all j equal to 1 to n that is if f is differentiable f 1, f 2, f n are differentiable and if f 1, f 2, f n are differentiable then f is differentiable. Not only that we can say something about the value of derivative, we can say that in this case that is wherever see there are two statements. Here, and we are saying that this happens if and only if this happens and whenever any one of this happens, we also have this and f prime at x is equal to f 1, prime x f 2, prime x etcetera and finally f n prime x.

Similarly, one can say what happens if you take the composites of two functions etcetera, now this is these are the things which go through in a similar way as in the case of functions going from a b to R. But, you should not be under the impression that everything goes through there are few things which fail also. There are few things which are true in the case of functions like this f from a b to R and which are not true in the functions of this type.

One of the most important in such properties is the mean value theorem, but we have been recalling what is, let us take the case of Lagrangians mean value theorem. What did that say that if you take the function from instead of this we said it functions should be continuous in the open interval a b and the differentiable in the open interval a b then what should happen? There should have just some point C in the interval a b, so in that f b minus f a that is equal to f prime C multiplied by b minus a. So, what I want to say is that such a thing does not happen in case of vector valued functions and what is the way of showing such a thing does not happen, this is only one way that is you have to give an example.

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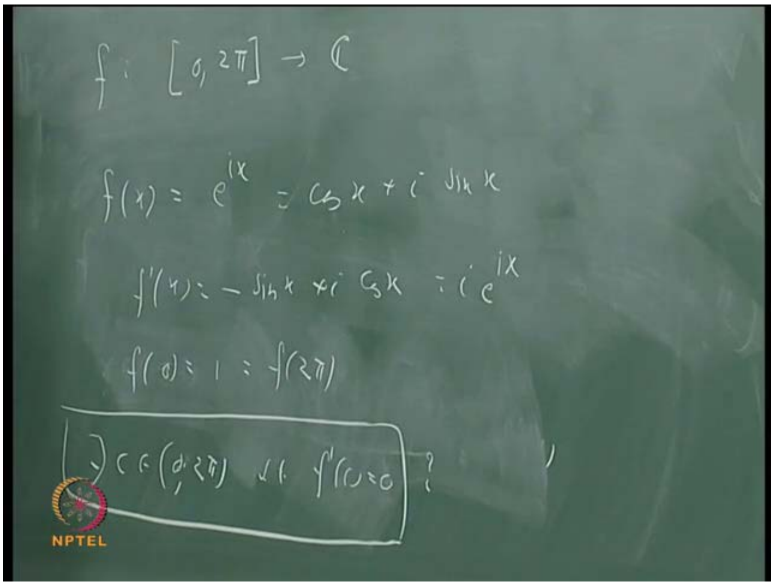

So, let us look at this function I will take the interval as let us say f from at the interval at 0 to 2 pi C and define f of x as i will just say e to the power i x or you wanted in the usual form write as cos x plus i times sine x is this, this function continuous 0 to 2 pi is it also differentiable. We can use what we have shown just, now it is a complex function if it is differentiable if and only if its real and imaginary parts are differentiable.

Real and imaginary part are nothing, but cos x, sin x they are well known differentiable functions and we can also say what are the derivatives we can say that this is f prime x is same as derivative of this with respect to x. So, that is minus sine x plus i times cos x and you can see that that is same as this times e to the power i x, so it is continuous in the closed interval 0 to 2 pi open in fact differentiable also very well. But, we are only looking at this interval. What is f at 0, f at 0 is 1, what is f at 2 pi, f at 2 pi is also 1, now if mean value theorem is true what should happen.

What should happen is there should exists some C in the open interval 0 to 2 pi such that f prime C is equal to 1, of course we are not asserting this we are checking this statement, whether this is true or false. That is the question, sorry f prime C is equal to 0 that is right, you are right f prime C is equal to f 2 pi minus f 0, that is 0.

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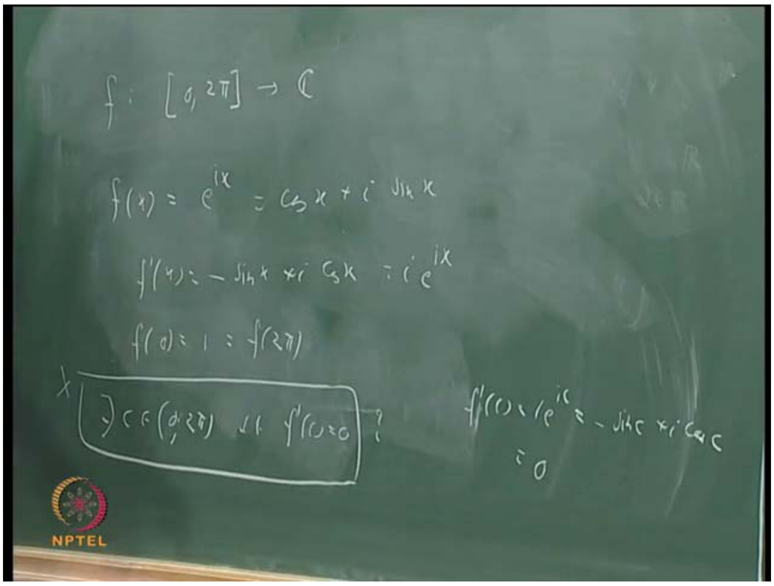

What is f prime C, f prime C is equal to either you write in this form i e to the power i C or minus sine C plus i cos C whichever way and this has to be 0, is that possible. What is the reason why it is not possible, is it such a difficult question, what is the absolute value of f prime C what is mod f prime C, 1. So, apparently 0 that is it right for no C f prime can be 0, for no C f prime c can be 0, so we can say, so what we are saying is that in fact there exists no C for which this is a first step.

If you take any C in fact you can say that if you take any C in a interval 0 to 2 pi absolute value of f prime C is equal to 1, so it is not going to be 0 for anything is that clear. So, mean value theorem is false and hence since mean value theorem is false whatever we have proved using mean value theorem we cannot expect that to be true. For example, something like in fact Lagrangians mean value theorem does not make sense at all because it will involve dividing a function one function by, so dividing one vector by the other.

So, forget about Cauchy's mean value theorem Lagrangians mean value theorem is first Cauchy's mean value theorem does not make any sense, then we proved this L'hopitals rule. Using mean value theorems those also will not be true those also will not be true for complex valued functions. So, whatever we have got for real valued functions using Lagrangians mean value theorem we cannot expect any of that to be true. Any of that to be true for vector valued functions that we see in particular complex valued functions and functions in take the values in R n, I think we will stop with that.