Real Analysis
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Lecture - 35
Mean Value Theorems (Continued)

We discussed Lagrange's mean value theorem yesterday and also saw some of the consequences. Towards the end, I said that we shall be discussing one more mean value theorem and namely Cauchy's mean value theorem, so let us see the statement of that Cauchy's mean value theorem.

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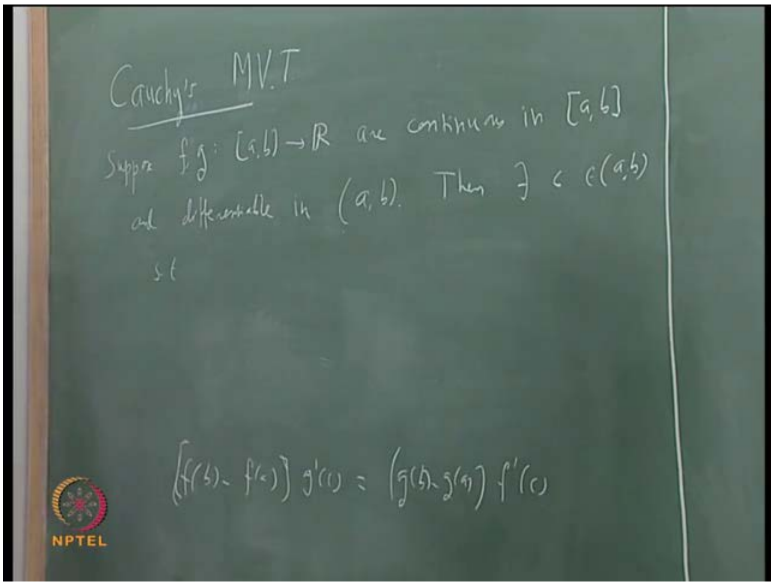

I shall use this standards short form for mean value theorem, main difference is in case of Lagrange's mean value theorem you deal with only one function whereas, in Cauchy's mean value theorem you deal with two functions, so we shall begin with two functions. Suppose, f and g these are two functions from a, b to r are continuous again the assumption are similar. Continuous in this closed interval a b and differentiable in open interval a b then there exist a number c in a, b such that such that f b minus f a divided by g b minus g a that is equal to f prime c divided by g prime c.

This is how it is given in most of the books, but you can see that in this formulation there is a slight problem here, what is the problem, what if this g b is equal to g a what this and similarly, what will happen, if this g prime c is equal to 0. So, in order to avoid that difficulty one can I mean one, there are two ways. One can write it in the beginning assume that g prime x s not 0 in anywhere, if g prime x is not 0, is it clear that this g b minus g a also cannot be 0. That follows from Rolle's theorem because if g b where equal to g a then at some point see g prime x would become 0, but another way to deal with that is we can just avoid this division.

I can write this same formula in a slightly different form, I can write this as instead of writing it like this, I shall write is like this f b minus f a into this g prime c that is equal to g b minus g a into f prime c. If I write it like this then there is no division involve and we do not need to make any extra assumptions. So, we shall we shall write it in this form right now, we do not have to say anything about g prime x not equal to 0 etcetera, now proceeding further for the proof.

Let us make one or two observations is it clear to everybody that Lagrange's mean value theorem is a special case of Cauchy's mean value theorem, that is if we take g x is equal to x. If we take g x is equal to x you will get this Cauchy's we will Lagrange's mean value theorem alright because g x equal to x then g prime will be 1. So, this will be 1 and that will be prime c this will be just b minus a, one more thing it is quite tempting to attempt the proof of Cauchy's mean value theorem by using Lagrange's mean value theorem. For example, one might say that apply Lagrange's mean value theorem to both the functions.

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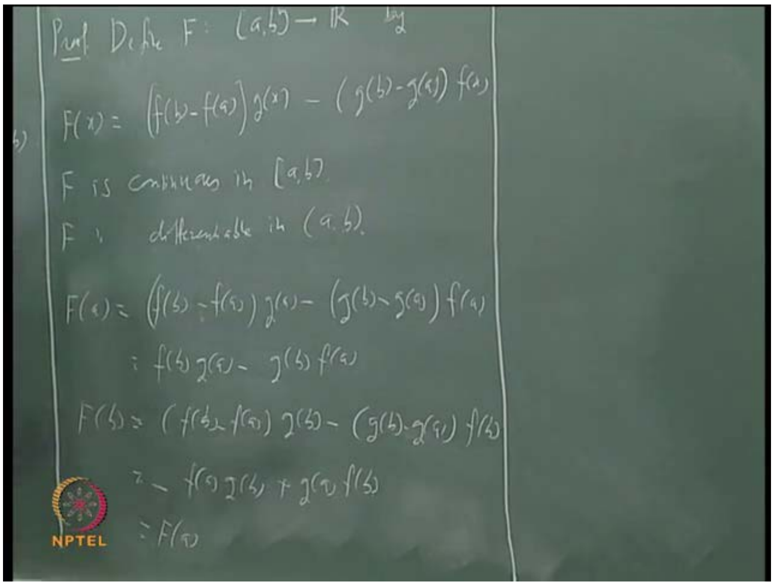

So, using that you will get you will get f b minus f a is equal to f prime c into b minus a. Similarly, g b minus g a is equal to g prime c into b minus a and then just substitute here, but is this correct, what is wrong with this right that is when you are applying Lagrange's mean value theorem to different functions. The point c that you may get will be in general different that c, so this c and this c need not be same. So, correct this there will be some c 1 here then will be some c 2 here and then you will not get this conclusion.

So, Cauchy's mean value theorem has to be approached in our different manner, but once that is clear what we will do is that we shall basically imitate the proof of Lagrange's mean value theorem. To proof Cauchy's mean value theorem though we do not use Lagrange's mean value theorem, but we use the ideas in the proof and what was the idea. We construct some x auxiliary function big F and showed that function satisfied all the properties of all the hypothesis of Rolle's theorem, and using Rolle's theorem we got it.

So, we do the same thing here, so define auxiliary function f, define f from m e to r by big F at x is equal to in fact this expression gives us an idea of what that function big F should be. We shall define it as follows f b minus f a multiplied by g x minus g b minus g a multiplied by f x. Suppose, we show that this function satisfy all the hypothesis of Rolle's theorem that may Rolle's theorem will say that there exist some c such that this big F prime at c is equal to 0. It is clear that will immediately give this because big F prime at c is nothing but f b minus f a into g prime c minus g b minus g a into f prime c.

So, the only thing left now is to show that this big F satisfies all the hypothesis of Rolle's theorem what are those this big F must be continuous in a b. Now, what is this f b minus f a is anyway a constant, so this g is continuous big F should be continuous. Similarly, wherever small f is continuous big F also should be continuous and we are given that small g and small f are continuous in the closed interval a b. So, we can observe that big F is continuous. Next is differentiability again the same argument because whenever small f because only see this big F is some kind of a combination of small f and small g all other things coming into picture are just constant.

So, wherever small f and small g are differentiable big F will also be differentiable right and that is the case in the open interval f is also differentiable in open interval. So, what said big F is differentiable in open interval a b, so the final step is hence by Rolle's theorem, we can say that hence by Rolle’s theorem. Again, this is something that I have forgotten yesterday also we need to check the values of f at a and b. So, what is big F at a so it f b minus f a into g a minus g b minus g a into f of a, so what is that this f a into g a will cancel with this g a into f a, so what will remain is f b, g a minus g b, f a.

Similarly, look at big F at b so big F at b again that is f b minus f a into g b minus g b minus g a into f b, so this time what happens. This f b g b will cancel with this g b f b, so what remains minus f a g b from here minus f a g b from here and minus g, sorry this becomes plus g a f b from there. So, it is same as this, so what we have is f b is equal to f a that is big F at b is same as big F at a, so we have verified that assumption also.

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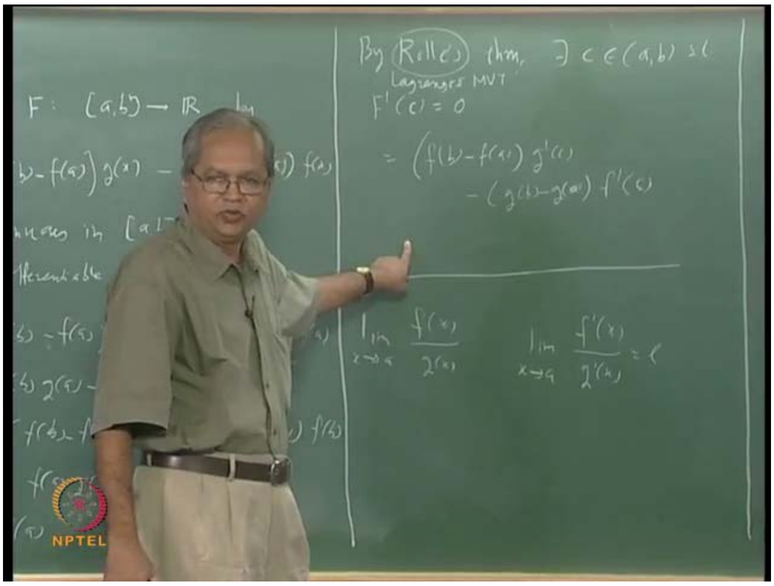

So, we can say that by Rolle’s theorem there exist c in the open interval a, b such that big F prime at c is equal to 0. You say by Lagrange’s theorem, but see when we discuss Rolle’s theorem that time itself we also made a comment that f a equal to f b equal to 0. See for example, suppose f a equal to f b where some other value suppose f a equal to f b is equal to k, then you can sub you can consider a function f minus k right and then that will that will be 0 at both the end points.

So, all that really matters is that the values at a and b are the same it is there is some convention in taking them to be 0, but that is not very crucial because adding and subtracting constant does not change the values of the derivatives the function f. Then, the function f minus k and the derivative of x minus k will be same as the derivative of f.

So, you can, but it is I mean instead the Rolle's theorem if you want, you can say Lagrange's mean value theorem that does not matter. So, what remains now what is this f prime of c, so look at this. So, at any point big F big prime at x is nothing but this into g prime x minus this constant into f prime x.

So, at in particular at the point c this will be same as f b minus f a into g prime c minus g b minus g a into f prime c and saying that this is equal to 0 is same as this equation here that is the that is the proof of Cauchy's mean value theorem. Before, going to the other problem which I mentioned yesterday namely about the types of continuities of the derivatives. Let me also discuss one more very frequent application of this Cauchy's mean value theorem. Especially in the undergraduate calculus you would have come across what is call indeterminate form and forms like 0 by 0 or infinity by infinity etcetera, what are the issues involved.

Suppose, you have functions like let us say you have function like f x f x and g x. Suppose, you want to look at limit of this as x tends to a and suppose it so happens that f x also goes to 0 as x tends to a and g x also goes to 0, if x tends to a then what I say about f x by g x. See, usually this depends on even though both these functions go to the same go to 0 where the limit of f x by g x exist or not which goes to 0 faster. For example, x also goes to 0 as x goes to 0 x square also goes to 0, but whether the limit will exist x by x square or x divided by x that depends on which one goes to 0 faster.

So, in order to discuss that one might want to discuss the rate of convergence and things like that, but we will not get into that these things are decided basically by the derivatives how fast a function goes to 0. So, instead of what is done in the in this so called in determine form is that you look at this f prime x by a g prime x f prime x by g prime x. Suppose, the limit of these exist as a finite real number then the limit of these also exist then the limit of these also exist and that limit is same as whatever is the limit of this.

Suppose, this limit is limit as x tends to a is equal to l, then this limit also exist and that is also same as l these something you may have used quite often, but without perhaps realizing that this really depends on Cauchy's mean value theorem. So, let us just say few things about that connection and then I will come back to the other problem.

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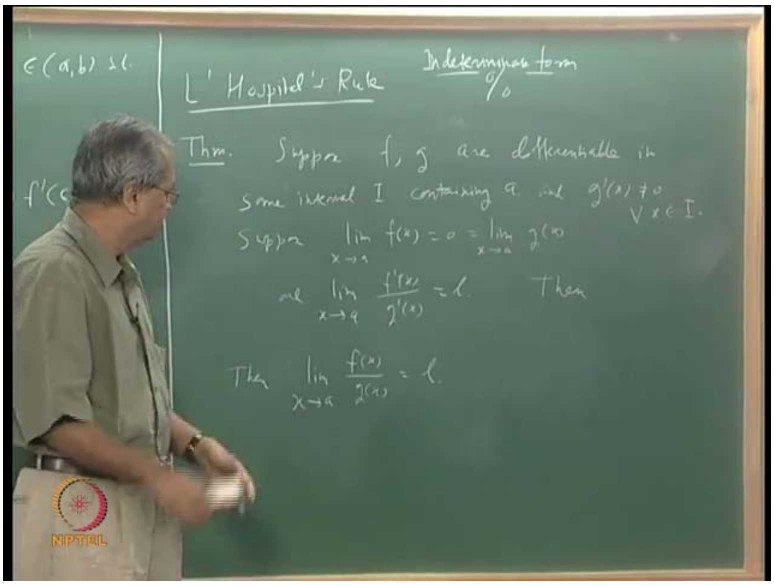

So, this is the this is usually called L'hopitals rule will write this as theorem of course, since now we are looking at the point limit at the point a we will prefer that a is not the end point of the interval. So, let us say that let us say suppose f and g are differentiable in some interval. Actually, as you know strictly speaking when we talk of limits the function need not be define at that point a still we can talk of limit of f x as x goes to a. So, one could have said that it is differentiable in some interval i containing a except perhaps at the point i, but we will not go into that kind of final points.

Now, those refinements one can make later, now my objective is simply this to illustrate that the Cauchy's mean value theorem is crucially involve in this. In, this rule suppose further that limit of f x as x tends to a is 0 and limit of g x as x tends to a is also 0 this is so called indeterminate form 0 by 0. This is how it is usually described indeterminate form 0 by 0 and limit of f prime x divided by g prime x as x tends to a is equal to l then then the conclusion is that limit of f x by g x also exist and that limit is same as l.

Then, limit of f x by g x as x tends to a is equal to a, remember this symbol means all the things that the limit exist and its value is l. Now, let us see how one proves this and where exactly where this Cauchy's mean value theorem comes into picture. Of course, we have to assume here one more extra thing here, we avoided assuming that g prime x is equal to 0 by rewriting the theorem in this form, but there we need to discuss this ratio f prime x by g prime x. So, here we cannot avoid assuming that g prime x is not equal to

0 in i. So, there is one more assumption here, if g are differentiable at some interval i containing a and g prime x is not 0 for all x in i.

Let us first see what is the meaning of this statement, let us use our usual epsilon delta definition, what is the meaning of saying that limit of f prime x by g prime x is equal to l. It is given epsilon bigger than 0, there exist delta bigger than 0 that something happens for those epsilon and delta, what is that something we shall write. So, let us just rewrite that statement, let us say that epsilon bigger than 0 is given.

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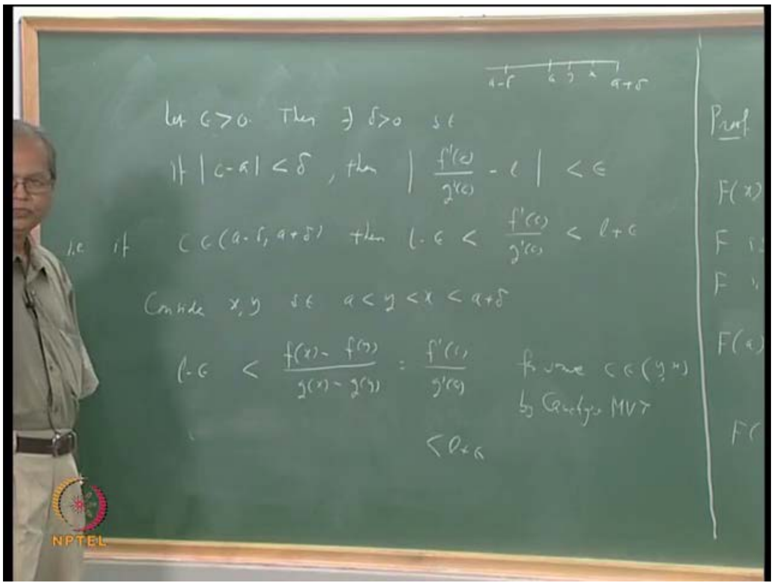

Let epsilon be bigger than 0, then there exist delta bigger than 0 such that whenever mod x minus a is less than delta, the difference between this should be less than epsilon difference between this should be less than epsilon. If mod x minus a is less than delta, then mod f prime x by g prime x minus l is less than epsilon. Only thing is now I will make a slight change in the notation, here instead of using this letter x, I will change it to c because I want to use the mid value theorem and use the property of c.

So, I will say that if mod c minus a is less than delta, then mod f prime c divided by g prime c minus l that is less than epsilon. Now, let us rewrite this statement saying that mod c minus a is less than delta, its same as saying that c belongs to a minus delta to a plus delta. Similarly, this means that if that is we can say that is if c belongs to a minus a plus delta, then this f prime c by g prime c lies l minus epsilon to l plus epsilon.

Then, l minus epsilon less than f prime c by g prime c, this is less then l plus epsilon and this should happen for all c in a minus delta to a plus delta. Now, let us consider some point x and y in this interval a minus delta to a plus delta considered x y such that first let me see a less than y less than x less than a plus delta and look at this ratio f x minus f y, f y divided by g x minus g y. We can say that f x minus f y divided g x minus g y for this since if f and g satisfy, if f and g are differentiable in the interval i containing a, so in particular those are differentiable even in this interval y to x.

So, we can say that by Cauchy's mean value theorem, there exist some c lying between y and x such that this is same as f prime c by g prime c, so this is equal to f prime c by g prime c for some c for some c in the interval. For the time being, let me simply say the interval y to x and this is by Cauchy's mean value theorem. Now, does this interval y to x contained in this in this interval y to x, this is the y interval y to x is a because a is less than y. If you just draw the diagram here, suppose this is a, let us say this is a minus delta, this is a plus delta.

We have taken y here and x bigger than that, so interval y to x is a part of this interval a minus delta to a plus delta and we have said is that for every c in this interval f prime c by g prime c should be less than l plus epsilon and bigger than l minus epsilon. So, it applies in particular for this c also, we can say that this is less than l plus epsilon and bigger than l minus epsilon, we shall just take this in equality, further then see what happens.

So, what we have here l minus epsilon less than f x minus f y divided by g x minus g y this is less than l plus epsilon. This is true for which y and which x it is true for any y and x which satisfy this, so what I would now say is that you fix this x and let this y approach a because this is inequalities true for all y this inequalities true for all y. So, we can say that fix x, let y tend to a if y tends to a, what will happen to f y, look at what we assumed here, we have assumed that limit of f x tends to a e f x is 0.

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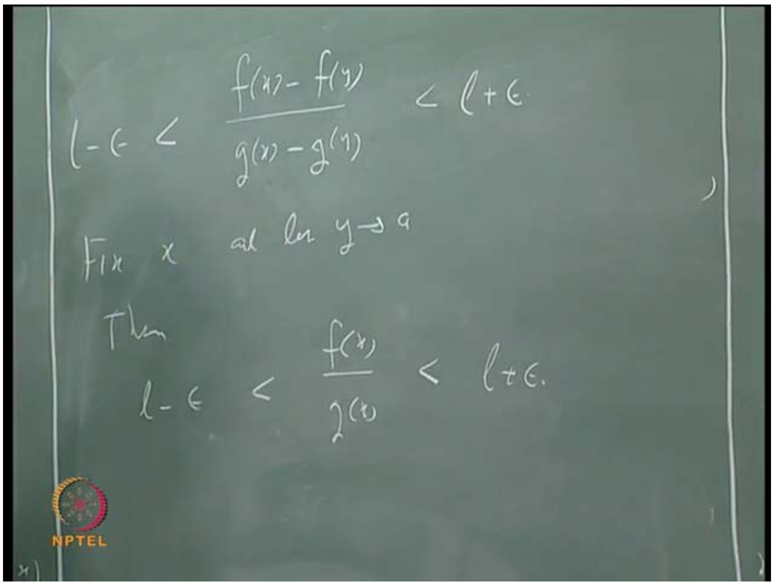

So, whether you take x or y, it does not matter here, so f y will go to 0 as y tends to a and similarly, g y will go to 0 as y tends to a. So, what it will mean is that in the limit f x by g x will between l minus epsilon to l plus epsilon, so then what will I say is that minus epsilon less than f x by g x less than l plus epsilon because these two numbers will become 0. These two numbers will become 0 and that is that is what this says, that is what does this mean that given epsilon you should be able to find a delta.

Whenever x lies between a minus delta to a plus delta f x by g x should lie between l minus epsilon to l plus epsilon that is what we have shown, so this last statement means this that limit of f x by g x as it tends to a is equal to l. So, we have seen here that this in case of this 0 by 0 basically uses Cauchy's mean value theorem. Now, I will give you an exercise if you have followed this proof, then do the same thing for the infinity by infinity form right. Just try to imitate whatever we have done for that what is that, there will be only one change, here this will be replaced by infinity and I believe you know what is meant by saying that limit of f x is as x goes to a is infinity.

Use the definition, what is the definition that given any positive number m there exist a delta so that f x be bigger than m for all x in a minus delta to a plus delta that is that is a meaning of saying that limit of f x is x tend to a is infinity. Similarly, for x so do the same thing what we have done here for 0 by 0 form to infinity by infinity form and try to try to do it yourself.

Now, let us come to the consideration of the theorem which we have mentioned earlier namely that we want to show that derivative of a function cannot have jumped this continuity to do that. Let us also observe one more thing, let me remind you one we had yesterday we had defined what is meant by local minimum and local maximum. We have said that if has a local minimum at the point x naught, then if it is differentiable at that point.

Then, that derivative must be 0, now we can we say the same thing in a slightly different language, because that is how we are going to use in this theorem. Suppose, we know that the function is differentiable and at a point x naught and the derivative is not 0 at that point can we say that there can be no local maximum or local minimum.

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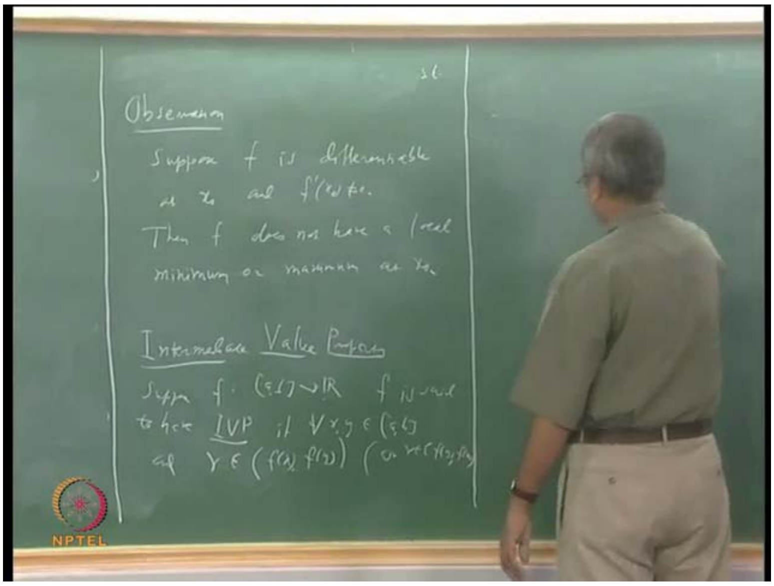

Let me just write this is the this is the observation suppose f is differentiable at x naught and f prime at x naught is not 0, f prime at x naught is not 0, then f does not have local minimum or maximum at x naught. It is basically written in the same statement in a different language, if f in fact did have a local minimum, then since it is differentiable the value of derivative should be 0, which is not the case. So, it cannot have a local minimum similarly, it cannot have a local maximum, then there is one more property that we shall recall perhaps you would have heard of this what is called intermediate value property.

This is usually discussed in connection with discussing connectedness that one can prove that a defined continuous function or image of a connected set. Other continuous function is connected and in the real line only connected sets are intervals, if you define a real valued function on a connected set, then its image must be in interval. Once you say it is an interval means what if the function takes any two values, it must take all the values in between those two values.

This is precisely what is called intermediate value property and this property is there for the real valued function defined on a connected set, but it can be there for other functions also and derivatives or of the functions are one such class. We will prove now before doing that let us also write up the definition what is meant by saying that intermediate value property what is the property what is meant by intermediate value property.

Let us say, suppose let us take usual f is defined from a b to R, now we will say that said to have this I will write this as IVP intermediate value property. If suppose you take any two points in x y if for every x y in a b and let us say I will any point say at the point gamma in the interval. Now, it depends on whether f x is smaller or f y is smaller, suppose f x i smaller gamma is in the interval f x to f y suppose gamma in f x to f y or gamma in f y to f x, that is one of the possibilities, what does this mean?

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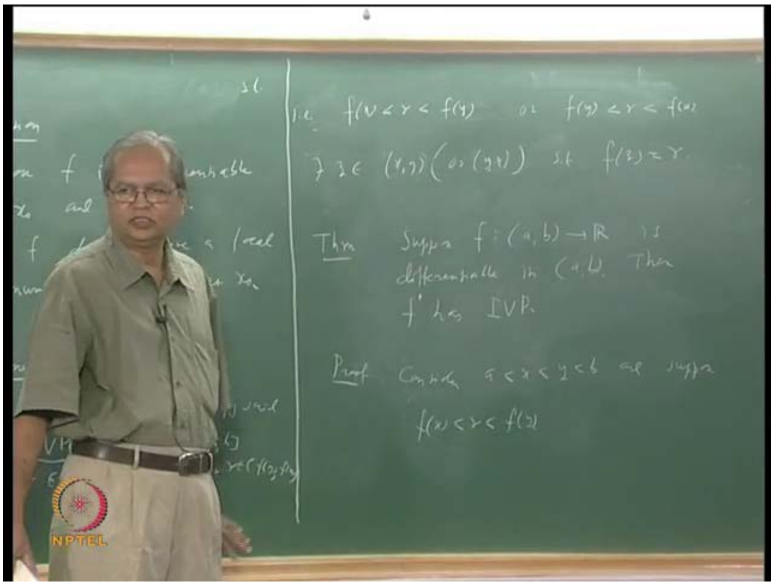

This means is this that f x less than gamma less than f y or let us let us just explain it that is or f y less than gamma less than f x. This is what you mean by saying that x take some point lying between f x and f y if f x is less than f y this first thing will happen if f y is less than f x that you know. So, what does this intermediate property says that there should exist some point in the interval going from x to y again it may be from x to y or y to x depending on which ever is smaller.

So, for every such point x y and a b, there exist let us say that point i z in in x y or y x depending on which ever is smaller, depending on such that f of z is equal to this point gamma f of z is equal to this value gamma. If you take any two values if, sorry to if you take any value lying between f x and f y there exist a point z that point z also lies between x and y. In fact it cannot be x x or y so in fact one should write here open intervals because z cannot be x or z cannot be y, so open interval here also y x such that f of z is equal to gamma.

Now, what do we want to say that if a function is differentiable, if a function is differentiable then it has intermediate value property, so that is the theorem, of course we are talking about differentiability, we can just take open interval suppose f from a b to R. I will take now open interval a b to R is differentiable in this open interval in a b, then f has intermediate value property, so not f f prime has intermediate value property f is differentiable a b means f prime is different as a function at all these points. That function f prime has intermediate value property, let me again remind you that how did we start this whole thing?

We started with a question whether there exists any function whose derivative is integral part of x. After proving this theorem, we can get negative because no function can have integral part of x as its derivative because all derivatives must satisfy intermediate value property and the function integral part of that does not satisfy that. Now, let us come to the proof, we will just take one of this cases because here we will define it is to consider all such possibilities, we will just assume that x is less than y and f x is less than f y. So, consider a less than x less than y less than b and suppose f x is less than gamma less than f y.

If we show in this particular case, the proofs for all the other cases will be similar x is less than y f x is less than f y, then we can similarly, one can prove that f y is less than f x or y is less than x and things like that. So, what we will do is, we shall also define new function which depends on the given function f and used some these things for that function.

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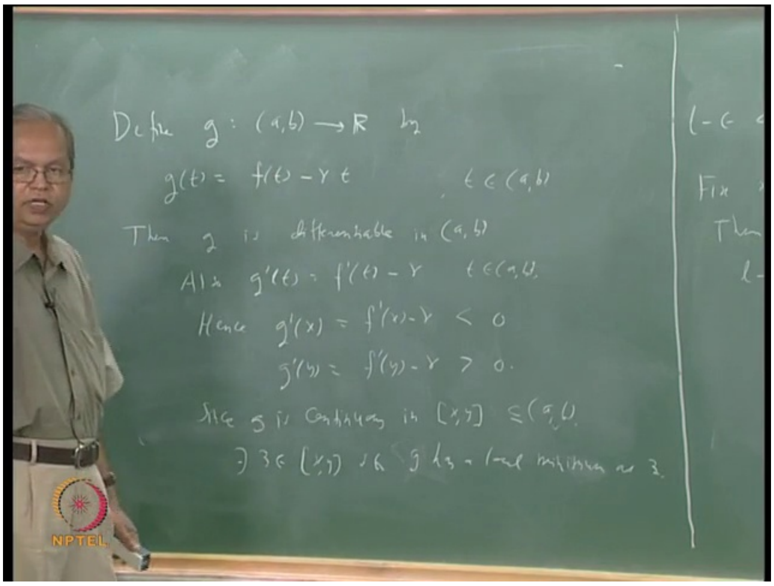

So, let us say define g from a b to R by g of x or both we have already used the letter x, let us use some other letter here g of t as f of t minus lambda times t, let f of t minus lambda times t for for t belonging to a b. Now, what can we say about g, first of all is it clear that g is all also differentiable in a b because what is g f is differentiable and lambda times t is also differentiable in fact its derivative is lambda. So, we can say that that g is differentiable in a b what is g prime at any point g prime at t is nothing but f prime at t minus lambda. We can also say that we can also record it also g prime at t is equal at f prime at t minus lambda for t in it.

So, what is g prime at x g prime at x is x prime at x minus lambda, I think here it should have been f prime at x f prime at y because we remember we are we are trying to show that f prime has intermediate value property not f. So, we should have taken f prime x less than gamma less the f prime y and I think here also I want to take this not lambda, but that constant gamma so we will make that change.

So, g prime of t is f prime t minus gamma and this is also g prime x is f prime x minus gamma, so let us come back to the question g prime x is f prime x minus gamma so what do we know about this is less than 0, f prime x minus gamma must be less than 0. So, g prime x is less than 0 what about g prime y g prime y is f prime y minus gamma what about this gamma is less than f prime y, so f prime y minus gamma must be bigger than 0. So, f prime y gamma is bigger than 0, now our argument is as follows, you look at this function g in the interval x to y look at this function g in the interval x to y right g is differentiable in a b.

Hence, it is also continuous in a b and hence continuous in the interval x to y that is a compact z, so every continuous function on that compact z must have a minimum. Let us say that is that since g is continuous in x y which is an interval contained in a and this interval is this z is not compact, but this is this side is not compact, but this is g. When you look at g defined on this interval it has a minimum it has a minimum, so within one can say that suppose I call that minimum as z there exist such that f has I will simply say f has a local minimum at z.

Now, there is only one question whether this z is an interior point of x y or one of the end points, suppose z is equal to x it will be in that this g has a, I am sorry not g has a it will mean that this g has a local minimum at x g has a local minimum at x. It is possible look what we know about g prime, x g prime, x is less than 0 and that is how we start the observation is a function is differentiable and its derivative is not 0.

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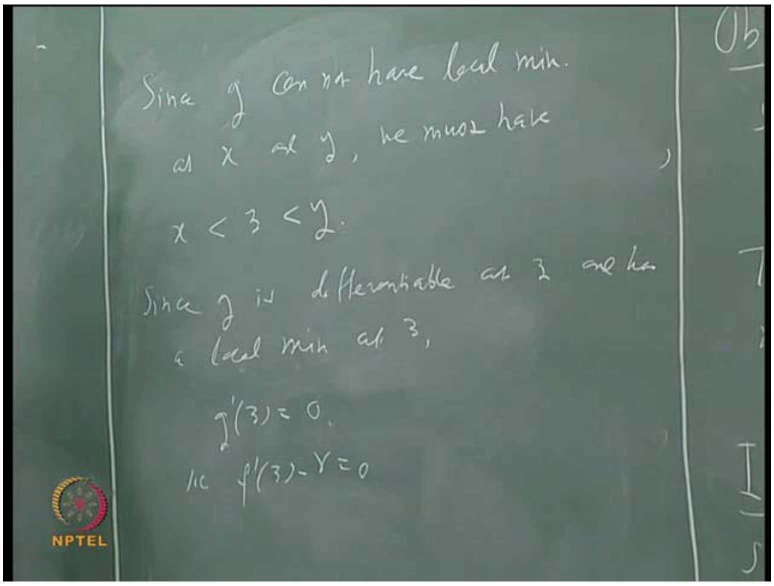

Then, it cannot have a local maximum or local minimum so that means since we know g prime is strictly less than 0, so it is not 0 so g cannot have a local minimum at x.

Similarly, it cannot have a local minimum at y because g prime y is strictly bigger than 0 fine so what does this mean it means this point z must be point.

So, let us say since what is our argument since x and y since g cannot have local minimum at x and y, we must have x strictly less then z and z strictly less then y, x strictly less than z. So, z is an integral point what is what is more, what we know that g is differentiable throughout a b, so in particular at the point z also and it has a local minimum at z.

So, what is the meaning of this g prime at z must be 0 since g is differentiable at z at z and has a local minimum at z g prime at z must be equal to 0, but what is g prime of z g? Prime of z is nothing but f prime of z minus gamma g, f prime of z is equal to that is f prime of z minus gamma equal to 0 and that is what we wanted to show that is given any number gamma which lies between f prime x and f prime y.

There exist z between x and y such that f prime at z is equal to gamma is clear. So, we have shown that derivative of a function must satisfy this intermediate value property, is this clear and as a corollary of this we will show that the derivative cannot have jumped discontinuities. It can have discontinuities of the other type, but it cannot have jumped discontinuities, but since the time is up, that will take up tomorrow.