Real Analysis
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Lecture - 33
Differentiation

So, we shall continue our discussion about the real valued functions defined on some subsets of the real line namely the intervals. So, the last class we were discussing the meaning of symbols like this that is…

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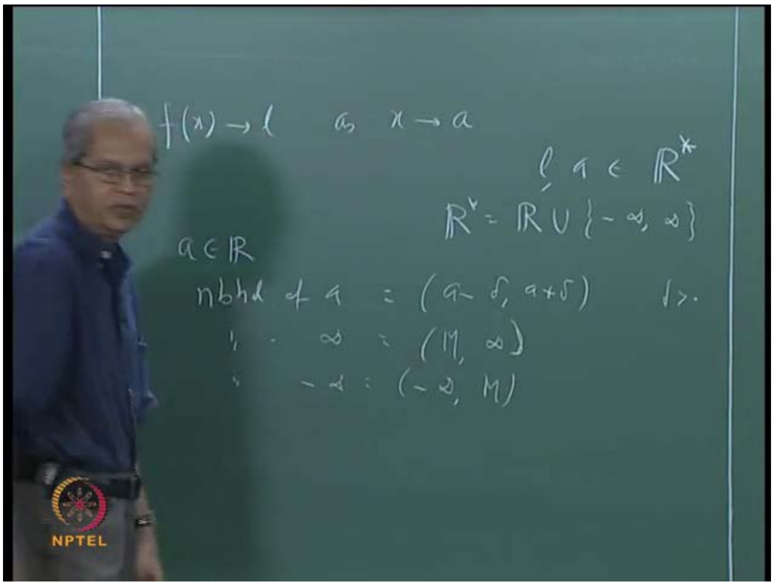

Let us write it in this fashion f x tends to l as x tends to a. And we have already defined what is the meaning of this when l and a both are real numbers. Now what we want to do is that or what rather what we have done in the last class is that. We extend the meaning of this idea of the limit by allowing this l as well as this a to b in the extended real line. So, l and a can be both in the extended real line and remember that extended real line is usual real line in addition with these two symbols minus infinity and plus infinity.

So, what is the additional thing that has happen is that now l can be a real number or plus or minus infinity. And similarly, a can also be a real number or plus or minus infinity. Now let us recall what was the idea how we have define, what we have seen is that. If either a or l is a real number then we take neighborhood of a, that is that we take as something like a minus delta to a plus delta. That is an interval containing a or a minus epsilon to a plus epsilon either one of those and neighborhood of infinity that we take as something like this m to infinity. That is nothing but the set of all x such that x is bigger than m. And similarly, neighborhood of minus infinity that we take as minus infinity to m minus infinity to m, and so what is the meaning of this? f x tends to l as x tends to a, we say that this means let me again write.

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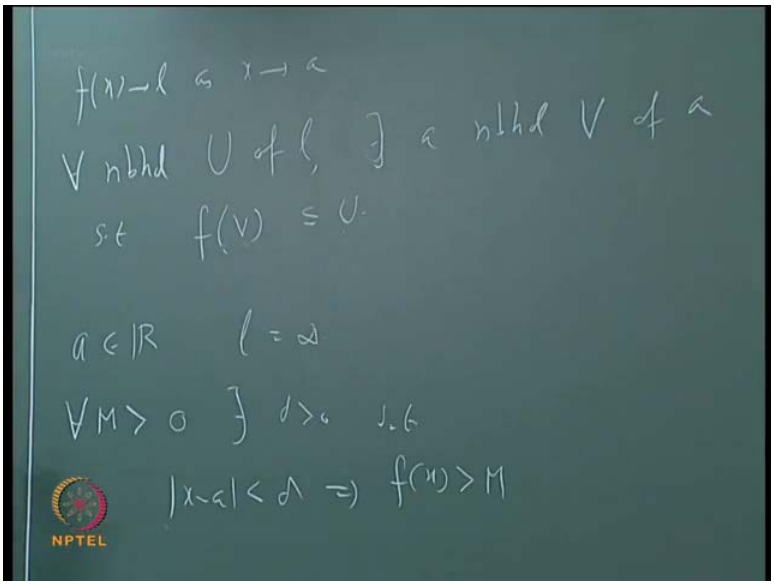

So, f x tends to l as x tends to a. This means that for every neighborhood u of l there exist a neighborhood v of a. Such that whenever, x belongs to v f x belongs to u or which is same as saying that we will say that f of v is contained in u. And when a and l both are real numbers we take this neighborhood u as l minus epsilon to l plus epsilon. And v as a minus delta to a plus delta and then our definition becomes for every epsilon there exist delta such that etcetera right.

So, similarly using these neighborhoods we can translate these definitions of also in the usual manner. If you want to get. Let us just take once special case. So, suppose let us say a, its real number and l is infinity. That means f x tends to infinity as x tends a and a is a real number. So, will be the meaning of that for every neighborhood u of infinity neighborhood of infinity is like this m to infinity. So, it will that for every m bigger than 0 there exist. Now neighborhood of a is… So, v is a minus delta to a plus delta and u is m to infinity, u is m to infinity.

So, there exist delta bigger than 0 such that what should have happen f of v is contain in u that means f of a minus delta to a plus delta should be contain in u. That is mod x minus a less than delta. This means x belongs to v right mod x minus a less than delta that means x belongs to v this should imply f x bigger than m right. f x bigger than m means f x lies in this. f x lies in this m to infinity right. That is meaning and similarly, you can rewrite the definitions by considering various combinations. Where a can be plus or minus infinity or l can be plus or minus infinity or one of those say real numbers and other infinity.

All source of possible combinations are possible. We shall not go into separate discussions of all these cases this one definition covers all those possibilities. By just this understanding that neighborhood of a real number is of the form a minus delta to a plus delta or a minus epsilon to a plus epsilon. And the neighborhood of plus or minus infinity are of this form. Now using this definition we can also show that. Whatever are the usual theorems about this sums and products of limits etcetera that we have discussed. They are also true in this new situation also. So, we shall not go into very detailed proof of that.

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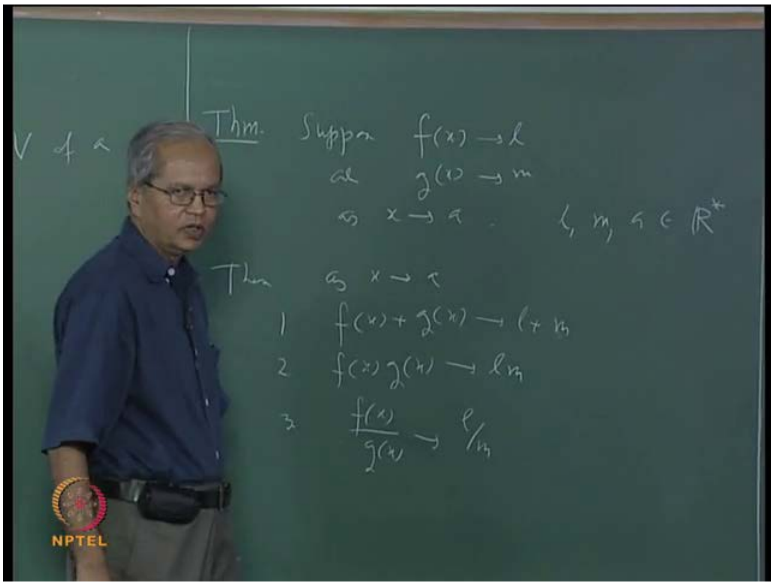

I will just take those theorems because those are somethings which we shall be using in future. So, suppose f x tends to l and g x tends to m as x tends to a. And here l, m, a all can be all are in extended real line. That is l, m, a all are in extended real line. So, any of those can be plus or minus infinity or a real number, then again as x tends to a.

First thing is f x plus g x tends to l plus a, sorry l plus m. And then f x into g x tends to l into m. And last require last thing is f x by g x tends to l by m, and if you remember in the usual way of stating a theorem. We stated these two things as they are only ways stating the last theorem we say that towards if m is not equal to 0. That is because if m is 0 l by m is not defined, but that is the, that is the care now we have to take in all these cases. Because it can happen that l plus m is also not defined because we have seen that… Suppose, you have l is infinity and m is minus infinity then this is infinity minus infinity that is also not define. So here also if it is infinity into infinity into 0 that is also not define. So, what I will say is that all these things are true provided the right hand sides are well defined.

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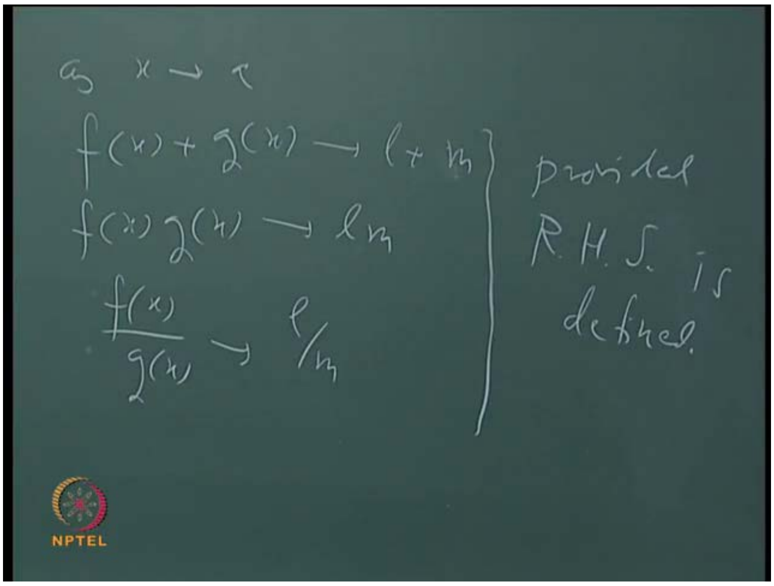

That is all these things are true provided right hand side well I will simply write R, H, S is defined. We shall not discuss the proof of this theorem because there are no new ideas involved, Basically the same idea will hold here. Now let us again let us go to down discussing new concept here. New in the sense we are discussing it for the first time.

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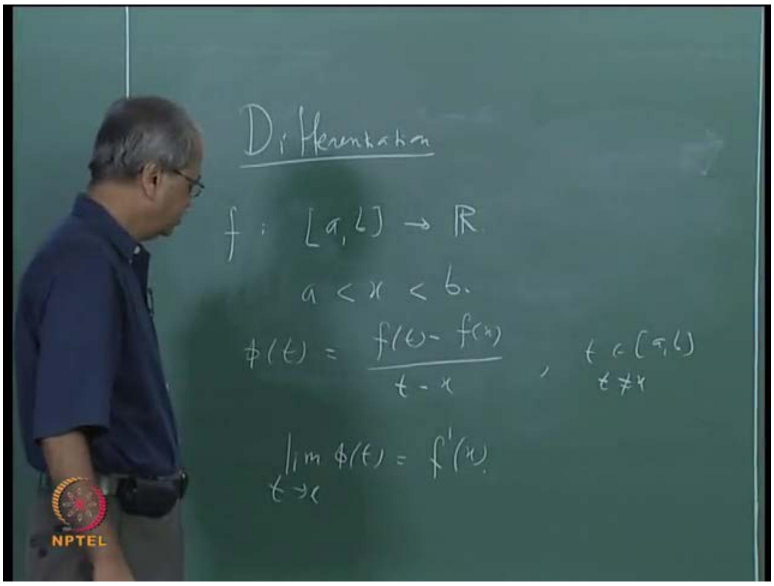

But, you would have already come across this in your undergraduate courses namely differentiation. So, let us just recall a few things. Again here as I said when we are discussing the real valued functions. It is not our aim to just repeat whatever you have already learnt during your undergraduate courses. What we want to do is that we shall just review a few things and try to emphasize where exactly the ideas of real analysis come into picture. Like, things like connectedness compactness. See what could have happen is that why I use those ideas earlier we got realizing that that is the thing that is involved. So, we shall point out whenever such things happens. So, let us begin with the definition. so suppose f is a function and that is define from a, b to R.

Suppose we take some x. Now for the convenience let me say a less than x less than b. Then we will define… Let us say a new function. Let me call it phi, phi of t. We define that is f t minus f x divided by t minus x. Obviously this cannot be define if t equal to x. So we will say that this is define for all t in a, b and t not equal to x. Or you can say t in a, b minus x. So, now we consider limit of this function that is limit of phi t as t tends to x. Of course, phi is not defined at x, but that is not require for considering the limit that is what we have seen. For defining, for talking about a limit the function need not be define at that point.

So we can talk about the limit of phi t x t tends to x. if this limit exist, and now when I, when I thinking of this definition I mean the limit exist as a real number. Now I am not talking about this extended definition of limit right. Now if this limit exist as a real number then we say that f is differentiable at that point x. And whatever is that limit a real number that will be call a derivative of f and will denote that by… let us say denote that by f prime of x. Of course, there are many other notations d, f by d, t at t equal to x etcetera. Let us not bother about that right now. For the time being we will say… we shall… So, let will say that if limit exist.

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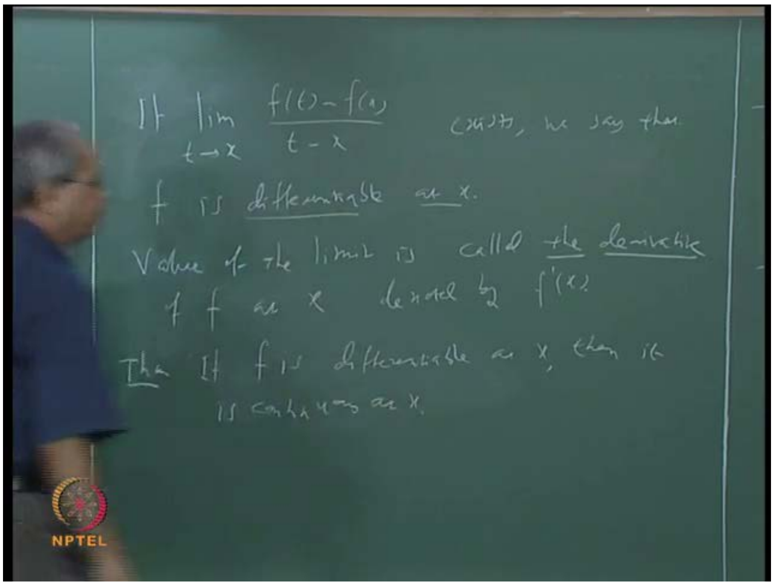

If limit of… Ok let me give this function in full f t minus f x divided by t minus x as t tends to x exist. We say that f is differentiable at x. And the value of the limit, the value of the limit is called derivative of x, value of the limit is called the derivative… Remember we have already prove that whenever limit exist it has to be unique. There cannot be two different values of limit. So with the limit exist there is one unique value that is the value real number, which will be called derivative. That is why the derivative of f at x. And will denoted by f prime x. Now one well known thing which about the derivatives or differentiability that you have already learnt I am sure is that. Whenever a function is differentiable a point it is continuous at that point.

So, let us just recall that. If f is differentiable at x then it is continuous at x. It follows immediately by looking at the definition of the derivative. That he call that since the whole thing is happening inside an interval all points are limit points. So, when we say that the function is continuous what will be in that limit of f, t, s, t tends to x must be f x. That is what we require right, and to see that we just look at the difference f t minus f x.

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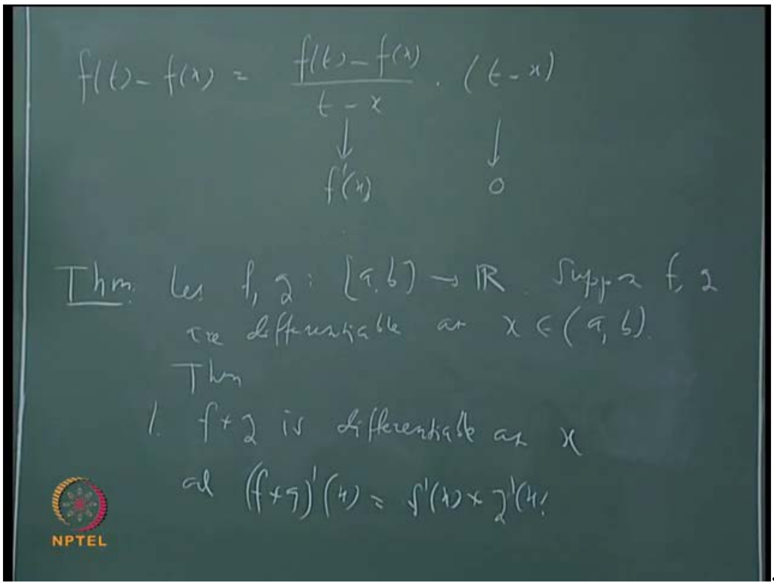

And write that as f t minus f x divided by t minus x and then multiplied by t minus x. And consider the limit of both sides as t goes to x. We already know that since f is differentiable at x this goes to f prime x, this goes to f prime x. And t minus x obviously goes to 0 as t goes to x so this goes to 0, right? So, the product will be 0. So, that means that the limit of that is same as the limit of f t minus f x as t goes to x is 0, which is same as saying that limit of f t as t goes to x is f x. And that is same as say that f is continuous at x. Of course, this something that you already know, but the point is we have made a crucial use of the fact that f is differentiable at x here.

So this proof will also, obviously will not work if f is not differentiable at x. But, of course, that immediately does not say that the converse is false. To show that the converse is false again I just said that there is only one way. We have to think of a counter example that is counter example of a function, which is continuous, but not differentiable at some point. Again I am, I am sure you have come across the function is well known function namely the absolute value of x or mod x, which is, which is continuous in fact at all points, but it is not differentiable at x equal to 0. And one can give several examples like that. There is there is no derif of examples.

All right, then we shall also just quickly review the theorem about the various operations on derivatives. Suppose we take say two functions f and g define on the interval and suppose both are differentiable at a point. Then we want to know what happens for the sum and product etcetera. Now before that let me also say one more thing here. Here we have talked about limit of phi t as t goes to x. And when we say nothing about the limit it is the usual limit. But, we have also considered for the real valued functions what is meant by left hand limit and right hand limit etcetera. It is possible to consider that also, it is possible to consider that also and that sort of thing will be call left derivative and right derivative etcetera.

So, it is quite and again from whatever we have learned about the limits it is well known. That if the limit exist then both left limit as well as the right hand limit will exist. But, the converse is false that is it is possible that the left hand and right hand limit may exist but the limit cannot exist, which is what we have called the discontinuity of the first type namely jumped discontinuity.

So, using those ideas we can define what is meant by left hand derivative of a function. What is call say we can say function is differentiable from left or differentiable from right. And those values we can define as a left hand derivative and right hand derivative etcetera. But, we are not going to do that kind of thing here. Will… I mean it is possible to do that, but since no really new concepts are involve we shall not go into that. There is only one thing that we note you should notice here. I have taken here a strictly less than x less than b. That is because we have considering the usual limit. If you take x is equal to a since there is nothing on the left of a. So, the only limit you can consider is the right hand limit.

So, at the point x equal to a if you want to talk about derivative it will be only right handed derivative. Similarly, at the point x equal to b it will be only left handed, left hand derivative. So sometimes we may need to talk about that kind of thing at the end at the n points of interval. So, at that time we will mention, but otherwise we shall not go into too much details about the left hand and right hand derivative. Coming back to what I have said just now.

Suppose we take two functions like f, n, g from a, b, to R. And suppose f, g are differentiable at x in a b. Again I will take x in thus interior point x. Just to avoid this discussion of left hand right hand derivative we shall take the interior point. Then first thing is f plus g is differentiable at x, at x. And the value of derivative that is f plus g prime at x is f prime x plus g prime x. You can see that there are going nothing much in the proof of this. This will follow simply by the corresponding theorem about the limit. Because if you write the, if you write there. For example, if you write this function of phi t replacing f by f plus g. It will be f plus g of t minus f plus g of x divided by t minus x.

So, it will be f t minus f x plus g t minus g x divided by t minus x. And if the limit of both since f and g both are differentiable the limit of each of those components will exist so that the limit of sum will exist. So, all these things follow whatever I say about derivatives all these things will follow by using the corresponding theorems about the limits, and of course some simplification and some adjusting of sums etcetera.

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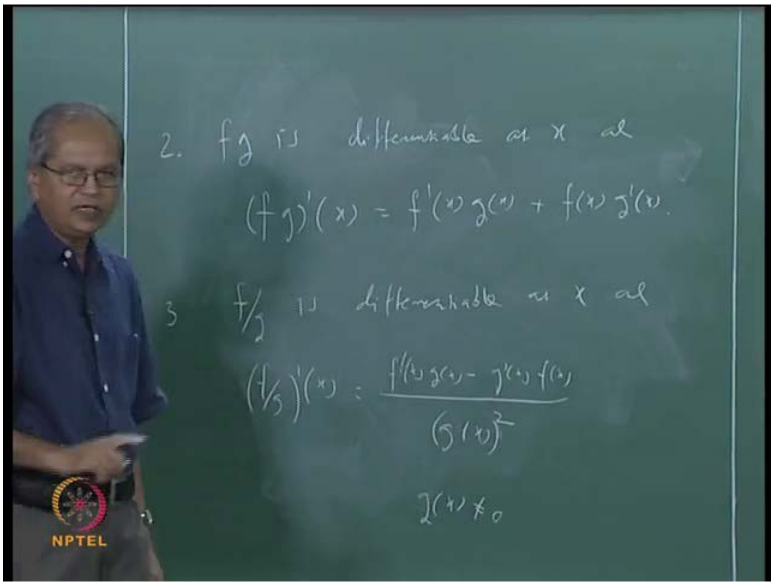

So, second part is that about the product. f into g is also differentiable at x, at x and derivative of f g at x that so call product formula. So, that is f prime x into g x plus f x into g prime x. Again here also there is nothing much involve. See if instead of this f you replace f t, f into g everywhere. So, it will be f t into g t minus f x into g x and if you remember what we do is that you add subtract a term f t into g x. And then just since the limit is with respect to t, wherever the terms involving only x are there you just take them as a common factor and then you will get this formula.

So, there is again there is nothing much in this. So, and find there is f by g is differentiable at x and the derivative of that f by g prime at x. f by g prime at x remember that this is different from the corresponding theorems on limits. The product of the derivative of the product is not same as the product of derivative. Unlike the limits, but you… of course, this follows by using the corresponding theorems on limits, but before that you have to do some work. Because we replace f by f t into g t etcetera you are not getting the product of these kind of fractions. So, that is where you need some work.

So, similarly, here for f by g so if you replace this by f t by g t and this by f x by g x.You have to simply and adjust the adjust the corresponding terms and then use the corresponding theorems about the limits. We shall not into that kind of details. Because I am sure you have done you have seen this kind of proofs in your undergraduate courses. If you have forgotten you can always look into one of those books. And so what is this formula? f by g whole prime x that is this is g prime x square then here you have f prime x into g x minus g prime x into x. This is g prime x square. Yes,

Student: g of x whole square.

Oh sorry, g of x whole square. I am sorry it should be g of x whole square. That is right. Okay? Now coming to the usual functions and their derivatives. Again I just said since this is the topic, which you should have done quite thoroughly in the undergraduate course. We shall not spend any time on that. How to find derivatives of some well known functions by using various formulas.

And the and doing things so called logarithmic derivatives etcetera, etcetera. Those types of things we shall not going to, we shall not discuss in this course. But, we shall just recall that we will need the, we will need the facts that derivatives of some well known functions again some well known function. Let us just recall a few things. For example, if f is a constant function then its derivative is 0. That is, that is again follows clearly from the definition. And further if let us say.

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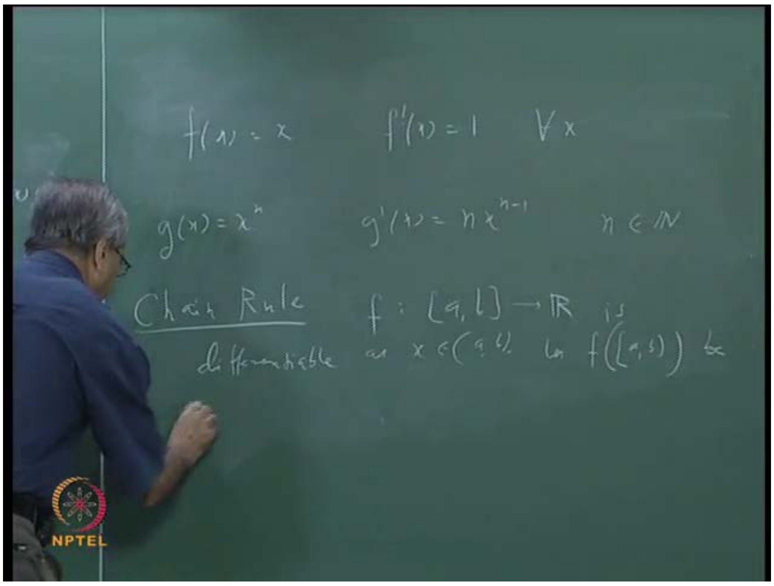

f x is equal to x, f x equal to x. Then that function phi t, which we will write that, will simply become t minus x divided by t minus x. So, that will be just one constant function one. So, in this case this is differentiable everywhere. And so f prime x will be equal to one for all x, then using this fact and repeat using this second thing here. We can write we can write suppose I take g x is equal to x to the power n.

Then you have to just use this repeated here x square x cube etcetera, etcetera, products of various thing. Using this repeatedly we can we can get that g prime at x is n into x to the power n minus. Of course, here I am assuming that n is a positive integer power. So, at is n. Of course, n is equal to 0 is also fine. If you take negative powers the you will have to use, you will have to use this. Of course, I should have mentioned here that this last formula is valid provided g x is not 0, but that is understood because at no point of time we are allowing the…

In the discussion derivatives, where we say derivative exist it means that it exist as a real number. We are not allowing the extended real line value for the derivatives. At least right now let us see if we need to change that in future. So, if you take n as a negative integral value then you will have to use this last thing also right. Then you can use this well known functions like things like derivative of e power x is e power x. Or log x is 1 by x and things like that, and various trigonometric functions and inverse trigonometric functions and all those things.

Let me just say one more thing here that or one theorem which is quite useful and which will be which needs to be used quite often. And what is called chain rule. This is about the derivative of composite functions. So, again let us say that suppose f from a b to R is differentiable at x in. Again let me take some open interval a to b. And suppose we consider one more function g. That is defined not on this interval a b, but on some intervals, which contains the range of this function f. On some interval which contains the range of this. So, let us say that let that this range I should write by f of a b let f of a b be content.

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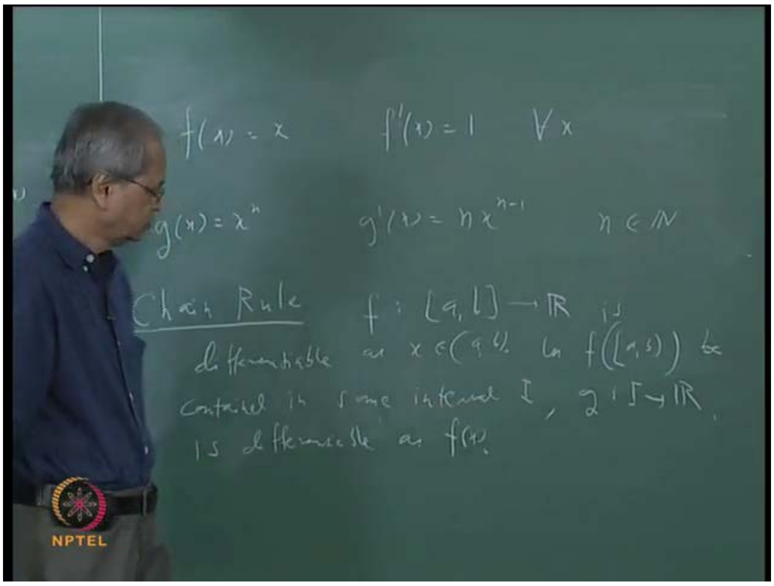

In some interval I and g is defined from I to R, g is defined from I to R. And since f of this whole interval lies inside this interval I f of x is a point in I. f of x a point in i and so g is defined at that point g is defined at that point. So g from I to R and suppose I assume that g from I to R is differentiable. Write f x at f x. Remember f x is a point in I. And g is defined on I then what this chain rule says is that. The composite function g composed with f that is differentiable at x and it tells us the value of the derivatives. I will continue here.

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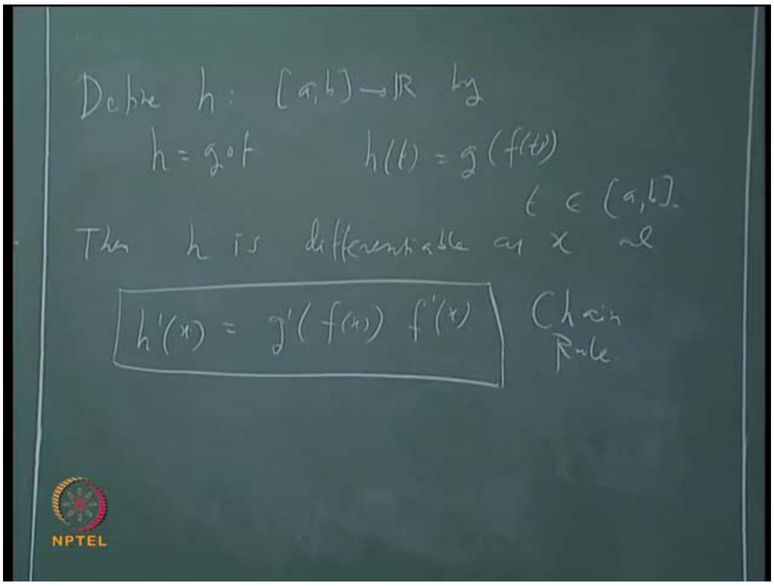

Let us say define h from a b to R by… Let me write like this h is equal to g composed with f, h is equal to g composed with f. Or if you want to write in the full form then let us say h of t is equal to g of f of t for t in. Then h is differentiable at x, h is differentiable at x. And h prime at x, that is the derivative of h at x it is same as g prime at f x, and multiplied by f prime at X.

There is see… Till now we have talked about this algebraic operations on the functions now this a different type of operation. Here we take the composition of two functions and then this size when is the composite function. Composite function here is h. h is g composed with f. If g and f are differentiable that is f is differentiable to x and g is differentiable to g is differential at f x. Then it is differentiable at x its value is given by this formula. This formula is the one, which is called chain rule.

Again I am sure you have seen this formula earlier and used it also. Actually just discussed the main idea in proving this without going into too many details of this. How does one go about proving this? What is done here is as follows. See f is differentiable at x let us again recall the definition what we have done is that we have defined this function.

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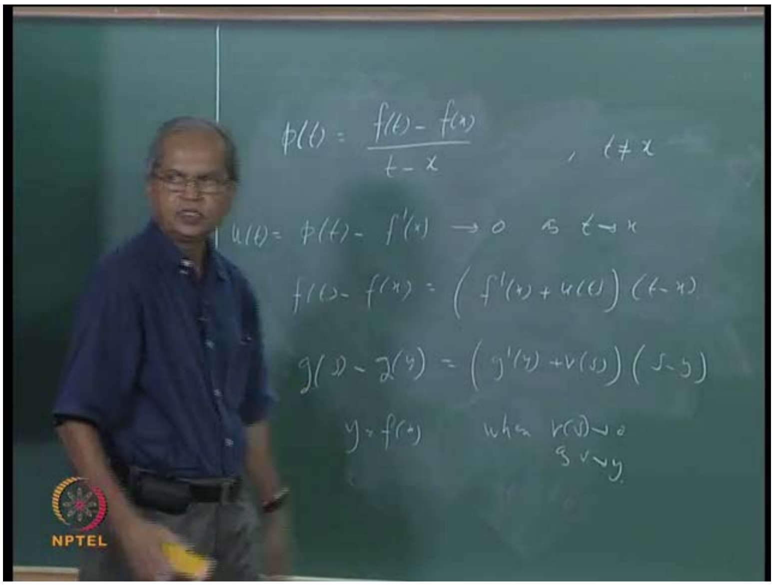

Phi t which was defined as f t minus f x divided by t minus x. This was defined for t not equal to x. So, what we can say is that… What we know if f is differentiable at x this phi t has a limit as t goes to x, as t goes to x. And that limit is nothing but f prime x that limit is nothing but f prime x. So, which is same as saying that if I look at the function phi t minus f prime x.

If I look at the function phi t minus f prime x. Suppose I call that function something suppose I call that function let us say u of t. Then u of t is that the u of t tends to 0 as t tends to x. Saying that phi t… if f is differentiable at x what we should have what we should have is that this function u of u should go to, go to, go to 0 as t tends to x. Now what again do. What I will, I will now do is I shall use this whole idea as follows. I will say that f t minus f x is the thing, but phi t into t minus x right and phi t is the thing, but f prime x plus u t right so I can say that… sorry f t minus f x.

This is same as that will say f prime x plus u t into t minus x, f prime f plus u t into t minus x and what is the… All the ((Refer Time: 34:33)) just written this whole thing. And with the property that u t tends to 0 as t tends to x. Now whatever I have done for f, whatever I had done for f I can do the same thing for g. Only difference is now g is not differentiable at x, but it is differential at f x.

Suppose I call that point f x is y. Then it is same as seen that g is differentiable at y and just to avoid confusion you stop using the variable t I shall use some other letter. Let us say s. So, what I can say is that just as f t minus f x let me call say g s minus g y, g s minus g y. Then g x minus g y since g is differentiable at y, if g is differentiable at y… let me take this y is equal to f x. We are assuming that g is differentiable at y then we can write g x minus g y similarly, as this will be g prime y plus some function of s, which goes to 0 as s goes to y, which goes to zero as s. Just as here we have taken u t there will be some function some other functions similarly.

Suppose I call that function v of s, small v of s into instead of t minus x this will be s minus y, instead of t minus x this will be s minus y. Where what was we know is that where v s tends to 0 as s tends to y. Now we have used this facts in simplify I am getting this. I shall just give one more step and may be remaining part of the proof you can complete. Now we want to show that h is differentiable. So, we should look at the corresponding fraction of about h. That is what we should look at is…

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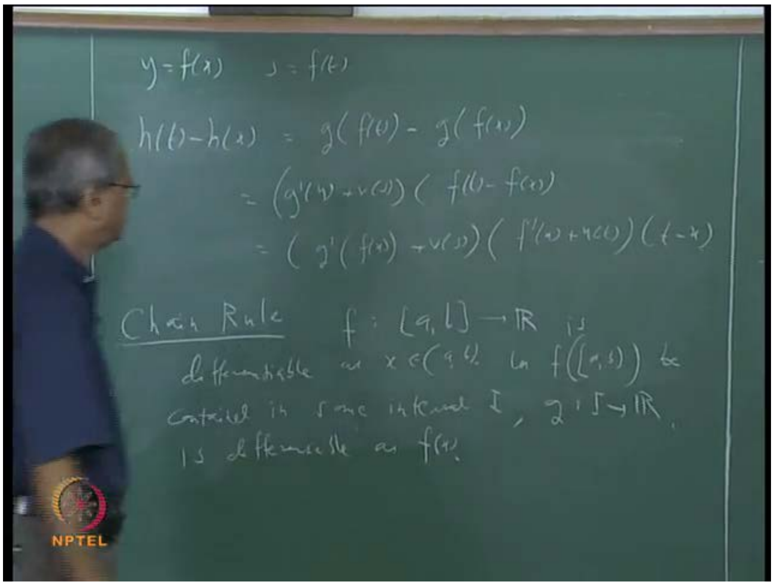

h t minus h x divided by t minus x. That is what we shall see. That is what we should look at. That is our aim is to show that h t minus h x divided t minus x. This has a limit as t tends to x. Alright this dividing by t minus x i will do little later. Now h t minus h x by that this is same as g of f t minus g of f x right. Let us just use this notation. What is the one notation the one notation is y is equal to f x and let us say s is equal to f t. Then this is nothing but g s minus g y. This is nothing but g s minus g y, this is nothing but g s minus g y and used g s minus g y is g prime g prime y plus v s into s minus y.

That is… this is thing, but g prime y plus v s into s minus y, where what is the property of this v s. v s goes to 0 as s goes to y. And y means f x. Alright now let us write, let us unpack this s minus y. s is f t and y is f x right. Now I will change this too… Is it clear? I will, I will rewrite this so this is s is f t and s y is f x. Now use this first thing that is f t minus f x is equal to f prime x plus u t into t minus x. So, this is same as this will remain as it is this is g prime y, for this y i will write back again f x. g prime y, that is g prime at f x plus v s into this f t minus f x is f prime x plus u t into t minus x. and now I think we have all the all the required things for the proof. So, consider that t is not equal to x then we can divide by t minus x.

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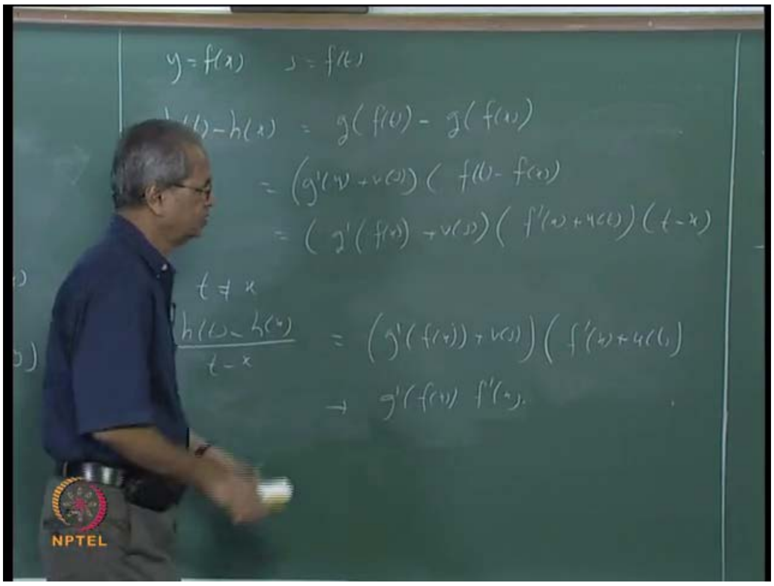

So, let us say that t not equal to x divided by this t minus x. So, what we will get is h t minus h x divided by t minus x. That will be same as this g prime of at f x plus v s into f prime at x plus u t. And to show that it is differential at x means. We should show that limit of this left hand side exist as t goes to x right. As t goes to x what happens look at right hand side as t goes to x u t tends to 0, as t goes to x u t tends to zero. So this limit of this becomes just f prime at x. What about this bracket? This is independent of t g prime f x is a constant. What about v s? Now here we have to use the finite whether the function is differentiable it is also continuous. So g is differentiable means g is, g is also continuous.

So, eventually we want to say v s goes to 0 and h goes to… That is if t goes to f x that is the finite we have to use. If t goes to... s goes to, s goes to y, which is same as that f t goes to f x, which means f is continuous, but that is true because f is differentiable at x. So, that is why v h goes to 0. As s goes to y or s goes to y means s goes to f x. So, this tends to g prime of f x into f prime at x. So, we have, so we have proved that h is differentiable and its limit is nothing but g prime of f x into f prime of x. Let us now just see a couple of examples of using these theorems that is both the earlier theorem about the sums products etcetera and this chain rule.

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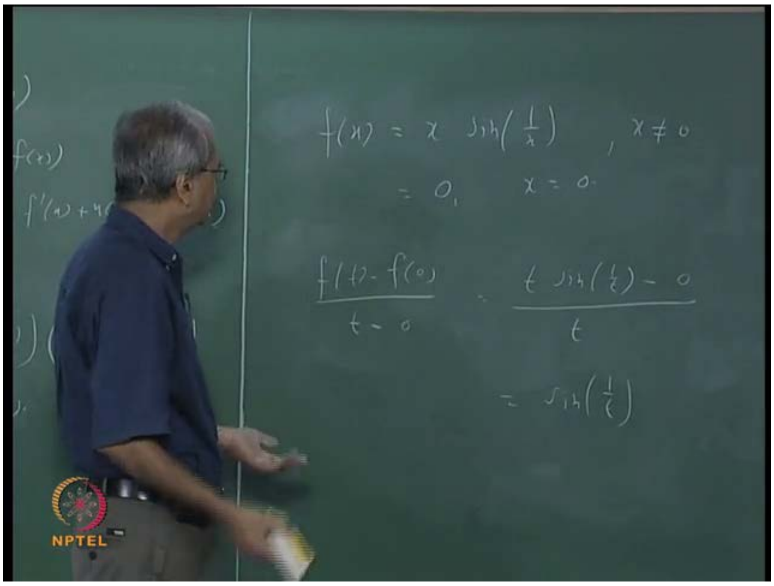

Let me just take this example first f x is equal to something that we have seen earlier. x sin 1 by x for x not equal to 0 and equal to as say 0 for let us say 0 for x equal to 0. Now if x is not equal to 0 there is no problem. That is… it is this function is differentiable everywhere if at all there is any problem we talk of because of this function sin one by x. But, again if you use the chain rule we let us assume that sin is differentiable everywhere and it derivative is cos of whatever is inside. So, the problem is about the function 1 by x then function 1 by x. The 1 is differentiable everywhere that is a constant function. So, you take one by x. So, that would be differentiable at those point wherever x is not 0 right. So, this product is differentiable every at all x not equal to 0 the only points that needs to be checked is what happens at x equal to 0 alright. So at x equal to 0 again you think of this f t minus f x divided by t minus x.

So, that is now we are looking at the derivative at 0. So, let us look at f t minus f 0 as divided by t minus 0. So, that is the thing, but t sin 1 by t minus f 0 is 0 divided by t. And that is the thing, but sin 1 by t. And the question is limit of this sin 1 by t exist as t goes to 0. We have already seen this limit does not exist, we have already seen this limit does not exist And so what is the conclusion? f is not differentiable at 0. What is the conclusion? F is not differentiable at 0. Remember that we have already shown that f is continuous everywhere, f is continuous everywhere. So, we have got one more example of the function which is continuous, but not differentiable. This function is continuous at 0, but it is not differentiable at 0. Now let me consider a small modification of that.

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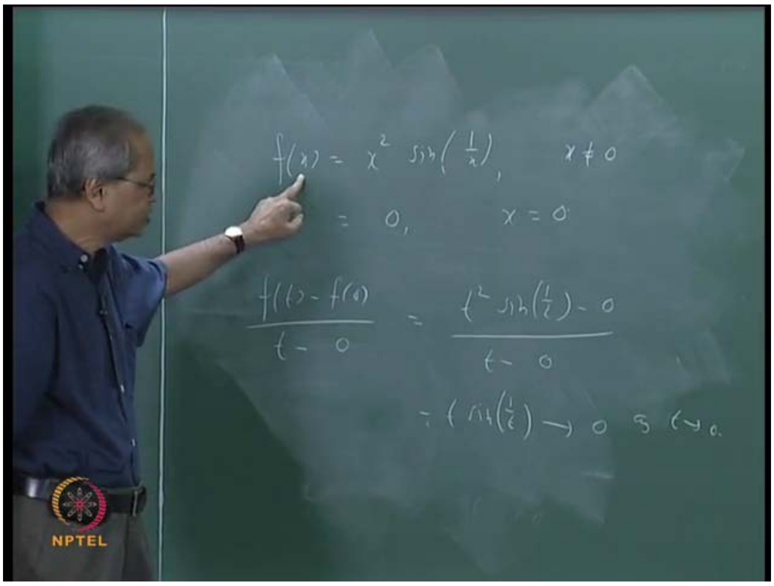

So, now I think of f x is of x square sin 1 by x suppose I consider x square sin 1 by x for x not equal to 0. And said the same this here it is say 0 for x equal to 0. Even you will see that for x not equal to 0 that is no problem. If is… because if x is not equal to 0 you have product of the two functions and this is again composite function. So, it is, it is differentiable and we can find a value of derivative if we want by using product rule and the chain rule. If necessary we shall do it later. So, the real point is to be consider is what happens at x equal to 0 right. Let us say again we do the same thing so suppose we take this f t minus f 0 divided by t minus 0.

Then this will be t square sin by t minus 0 divided by t minus 0, t minus zero. So, what is that? So, this is t sin 1 by t. And we have already seen that the limit of this as t goes to zero exist. And that limit is the thing, but that limit is the thing, but 0. So, this tends to 0 as t tends to 0 which means this function is differentiable this function is differentiable at x equal to 0 and its value is value if the limit is 0.

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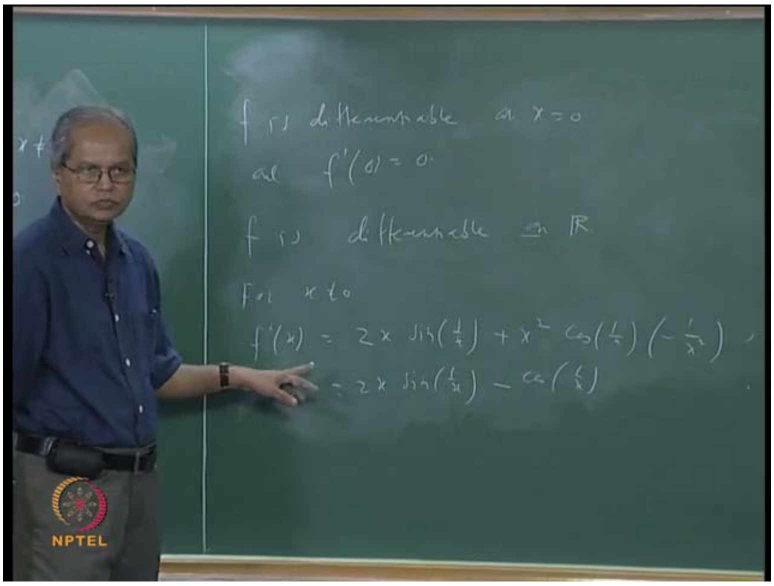

So, the conclusion is, so conclusion is f is differentiable at x equal to 0 and what about the value and f prime 0 is 0 right, f prime 0 is 0. As far as x not equal to 0 is concerned we already know that f is differentiable. So, we can say that f is differentiable for all x sin, f is differential on R. Again this is something let me explain I have not mentioned it earlier if is, if f is differentiable at all points in some interval. Just as if f is continuous at all points of interval. We say that f is continuous on a b. Similarly if it is differentiable at all points on some interval we say that it is differentiable on that interval. On that set in this case that set turns out to be R. So f is differentiable everywhere on R.

Lets us say suppose now I want to ask a question is this f prime a continuous at 0, is f prime continuous at 0. If that is to be answered what is to be do? Will have to find what is we know that f prime at 0 is 0. So, we will have to look convert f prime x for x not equal to 0 also and take the limit of that as x goes to 0 that can be done right. Let us say let us do this, what is for x not equal to 0 what is f prime? We will have to calculate this f prime by using two theorems proved earlier namely the product rule and the chain rule. We can do it quickly so what is f prime. So let us… It will be 2 x into sin 1 by x that is derivative of this into this. Multiplied by plus x square into derivative of this first it is cos 1 by x, and then multiplied by minus 1 by x factor.

So, this becomes… this will remain as it is so 2 x sin 1 by x and minus cos 1 by x right. Now if we want to know that whether f prime is continuous at x equal to 0, we now take the limit of this as x goes to 0, limit of this as x goes to 0 and so that that limit is same as 0. Now what is the answer here? Thus the limit of this exists as x goes to 0. Again as far as this first term is concerned there is no problem x sin by ((Refer Time: 51.32)) but what about the second term cos 1 by x. The limit does not exist, the limit does not exist so this function is differentiable everywhere, but it is not continuous at…

That is f prime is continuous at x equal to 0, is it clear? And what is the type of discontinuity? It is what we call discontinuity of the second type. Because the limit does not neither left hand limit nor the right hand limit will exist. So, this is a discontinuity of the second type fine. We will stop with this for today we shall continue with that with the discussion about these different types of discontinuity etcetera in the next class.