Real Analysis

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Lecture - 32
Types of Discontinuity

We were considering the real valued functions and the types of discontinuity, so we shall continue with that discussion.

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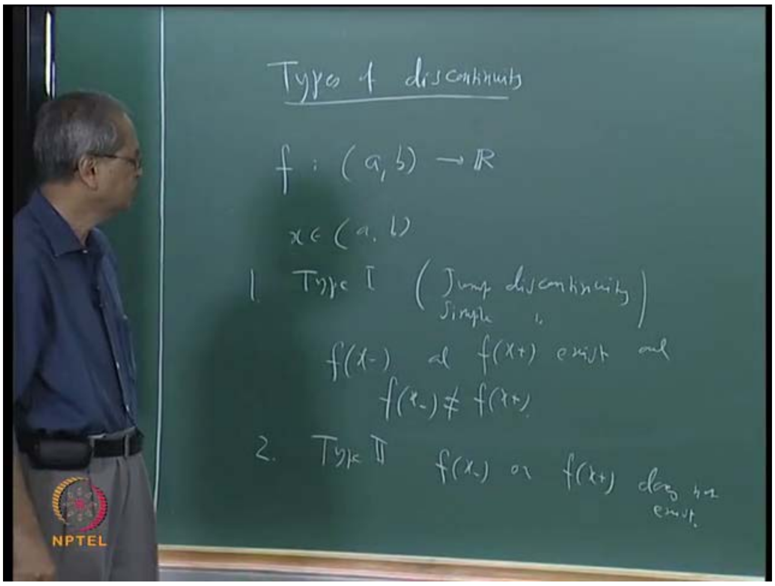

Let us recall that we had done following thing, let us write types of discontinuity. So, for this we have to consider a function f defined on open interval a, b to R and x some element a, b. And we had said that x is a discontinuity of let us say type 1 or another way means jump discontinuity are also called simple discontinuity. So, this means that the limit, left hand limit and the right hand limit both exist and their values are different.

So, this means f x minus and f x plus exist and the values are different f x minus not equal to this f x plus. And the second type, which is called discontinuity of type two, this means that either f x minus or f x plus or both do not exist. So, we can simply say this means f x minus or f x plus does not exist. Before proceeding further let us see some examples of this kind of discontinuities. Again let us take the well known examples, which we have considered earlier several times. For example, let us take f of x as integral part of x. Now we can take any interval in fact we can take f defined from R to R.

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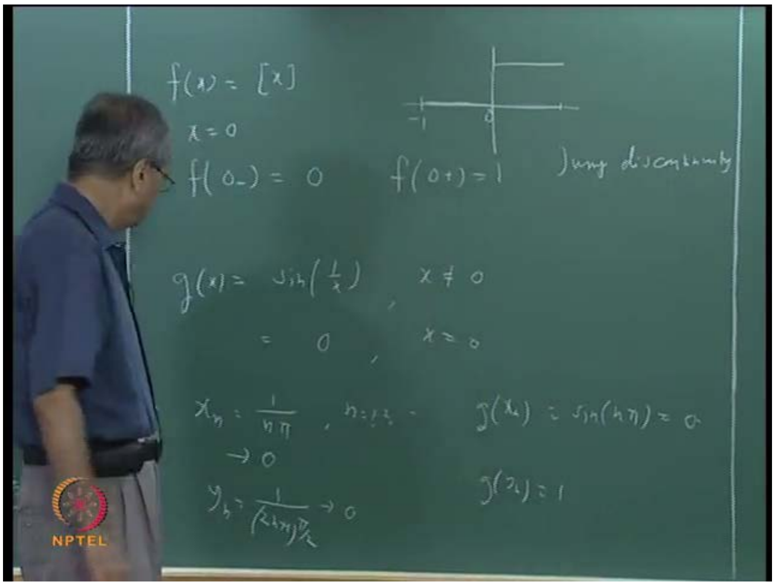

So, suppose I take let us say x is equal to 0 then, we know that near 0. Again let us say by remember that integral part of x is the greatest integer naught not greater than x.

So, in this interval minus 1 till 0, its value will be 0, integral part of x will be 0 and in the interval 0 to 1 its value will be 1. And so, it is clear that if you take f of let us say in this case f of 0 minus, if you take any number here its value is value of f is going to be 0 and hence it is a constant function here.

So, this limit will be 0 and on the other hand f of 0 plus for that we are considering the interval 0 to whatever delta you take, whether the integral interval 0 less than t, less than delta. Whenever delta is less than 1, the function is going to be constant and its value is going to be 1, so the right hand limit here is 1. So, this is a discontinuity of type 1 or and you can now see why it is called jump discontinuity because the function x jump here.

Of course, in this particular case it so happens, that the value of the function at 0 is same as this, but that may or may not be the case, the value may lie somewhere in between also in general. So, this is the discontinuity of type 1 or jumped discontinuity. Let us take another example that is also something we have seen earlier. Let us say g of x is equal to sin 1 by x. Of course, this will not define the function when x is 0. So, let us give some by strictly speaking it is not necessary to define the values of the function that x is equal to 0, if I am considering the limit left hand limit or right hand limit.

Since we are talking of the types of continuities, let us take some value, suppose I take the value 0, this value really does not matter. Now, in case of this you will see that neither g of x plus nor g of x minus exist, that is the neither the right hand limit exist nor the left hand limit exist. Now, how does one show that, I have said earlier also that the best way to show that the limit does not exist is that you construct two sequences both converging to 0. And limit of g let us say suppose, you call those sequences x n and y n in such a way that, x n and y n both converge to 0.

In this case and g x n and g y n converge to different. Only thing is that if we want now, we have to restrict this sequences x n and y n both either to the right of 0, if I consider right hand limit and goes to the left of 0, if I am considering the left hand limit. Let us take one of those things suppose, we are considering right hand limit then, in fact even this is also something I think we have seen earlier. For example, suppose we take x n as let us say 1 by n pie and suppose I take n is equal to 1 2 3 etcetera then, all x n are bigger not equal to 0 all x n.

So, this sequence lies on the right of 0 and what about g of x n, g of x n is sine n pie, so g of x n is sin n pie, so that is 0. And in a similar way and of course, x n tends to 0, and g of x n also tends to 0, but what we can also do is that I can let us say another sequence y n as 1 by let us say 2 n plus 1 that is pie by 2. Then this is also tends to 0 and g of pie n that will be sin 2 n plus 1 pie by 2 so that will be 1for all. So, x n and y n both tend to 0 and g x n tends to 0 g y n tends to 1 and remember x n and y n both go to 0 from right. So, this shows that a right hand limit does not exist.

So, this shows that g is 0 plus does not exist and so this is a discontinuity of type two. And of course, we can also show that g of 0 minus also does not exist by similarly, constructing two sequences, but if the objective is to show that it is discontinue of type two, it is not essential. If you show one of this does not exist that is enough to conclude that the discontinuity is of type two. In this particular case in fact both do not exist. Let us also recall another example, which we had seen earlier in several contexts namely this as called as Lebam function, it is h x is equal to 1 if x is rational and 0 otherwise.

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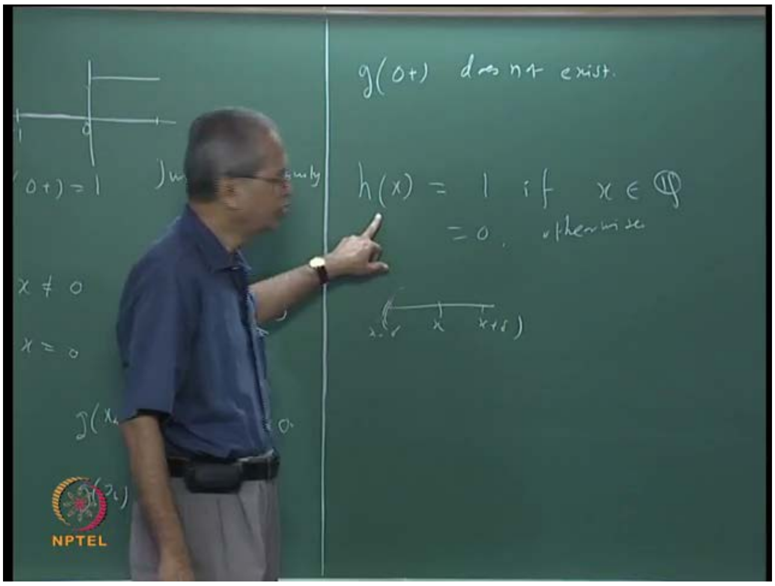

It can also be described as the characteristic function of q. In case of this function you will see that whatever x you take, if you take any interval say x minus delta to x plus delta, if you take any interval like this then, any such interval will contain rationals as well as irrationals. So, what you can always do is that you can construct a sequence regardless what whether x is rational or not. You can construct a sequence of rationales converging to x.

In that case h of x n will converge, suppose you take a sequence x n such that each x n is rational and x n converges to x then, h of x n will converge to 1. In a similar way you can also construct a sequence, let us say y n of irrationals converging to x, in which case h of y n will be 0. So, that means this will have a discontinuity of type 2 for every x.

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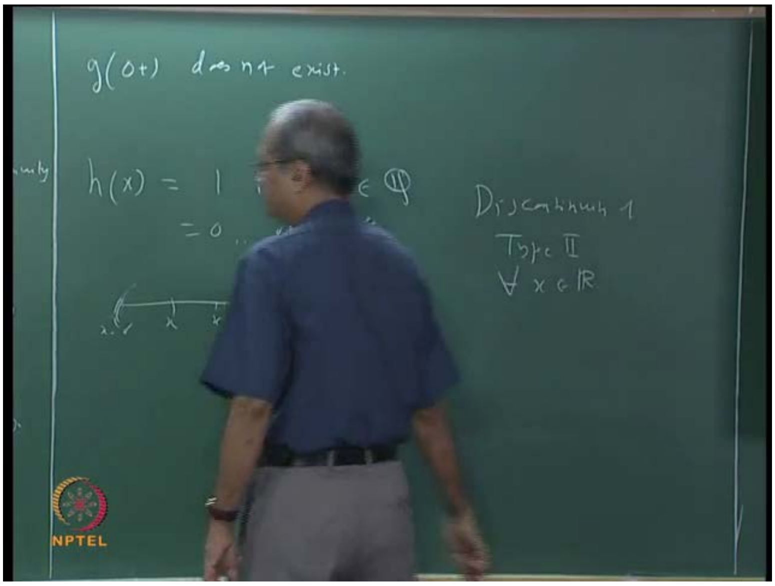

So, h x are discontinuity of type 2 for all x in R, whether in case of this functions we have shown that the discontinuity of type 2 is that x is equal to 0. For every other x it is continuous this function sin 1 by x, for every x naught equal to 0, it is a continuous function. At x equals to 0 it is discontinuous and that is of type 2 whereas, in case of this function at all integral values it has a discontinuity of type 1, at all other values it is continuous again.

And in case of this function, it is discontinuous everywhere and has a type 2 discontinuity. Now, we shall consider a class of functions again this class is something that you are familiar with what are called monotonic functions. And we will see that as far as these types of discontinuities are concerned, monotonic functions have some very interesting property, namely that monotonic functions do not have this kind of discontinuity.

For example, we have seen this example integral part of x, which I had shown just now, that is how monotonically increasing function. And we have seen that only discontinuity of that is discontinuing of is of type 1. What we shall show is that something like this is true for all the monotonic functions. So, let us begin by recalling the definition of a monotonic function.

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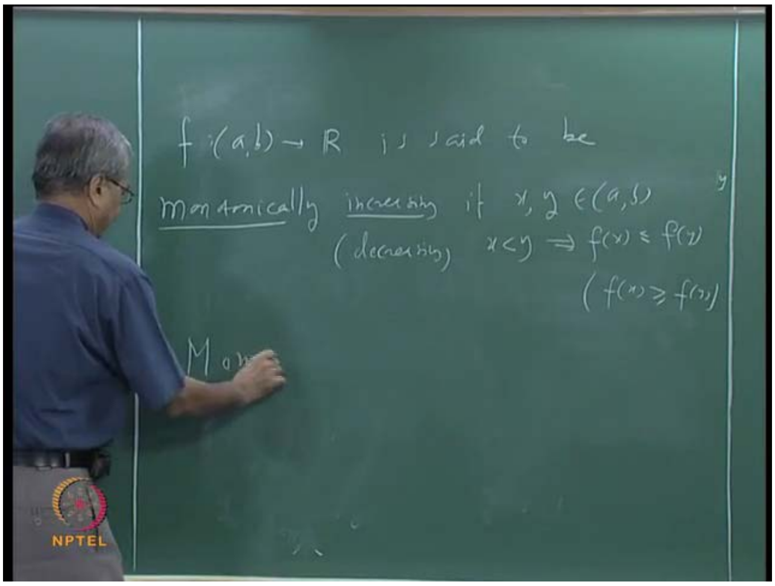

Again it is convenient to define all this things in an open interval. So, let us start with that so f from a, b to R is said to be let us say monotonically increasing, if suppose we take two numbers x and y in a b and then x less than y, this should imply f x less not equal to f y. We have defined what is meant by monotonically increasing sequence. Sequence is special case of functions, so it is similar. And we have seen that monotonically increasing sequences have some special properties as far as convergence etcetera is concerned.

So, similar thing is true for a monotonically increasing function when it comes to limits and continuity of course. We can similarly define what is meant by monotonically decreasing function. Only thing is that this inequality will be reversed, this will become f x bigger naught equal to so that I can simply write it here decreasing, and so this will be f x neither bigger nor equal to f y. And similarly, we shall say it is strictly monotonically increasing, if this inequality is strict that is f x less than f y and similarly, strictly monotonically decreasing.

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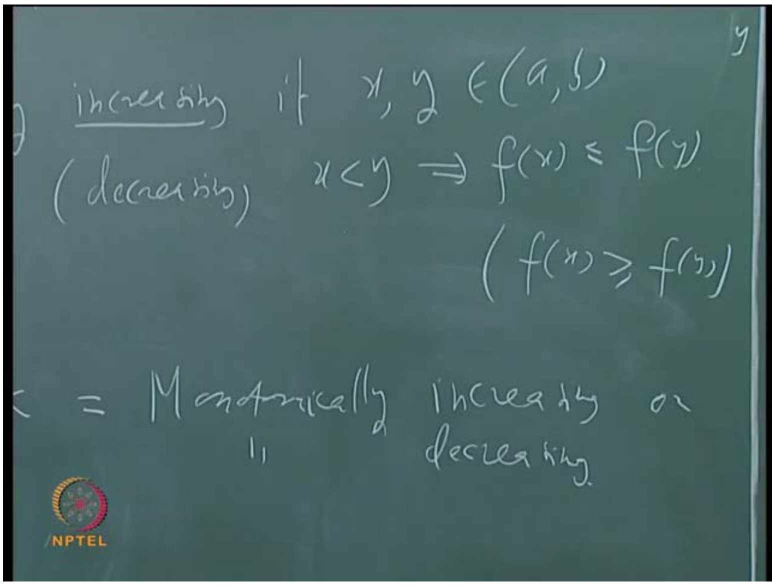

Finally, what is meant by a Monotonic function? Monotonic function means it is either monotonically increasing or monotonically decreasing. So, monotonic this means monotonically increasing or monotonically decreasing. Well you know several examples of monotonically increasing or decreasing functions. All the functions which are not monotonically increasing or decreasing so we shall not go into that. All ready we have seen a given one example namely that integral part of x that is an example of a monotonically increasing function.

So, let us go to the end result what we want to prove is that, if a function is monotonic that is the ultimate idea. We will need some preparation for that what we need. We want to prove is if a function is monotonic then, it cannot have the discontinuity of type 2, so let us start.

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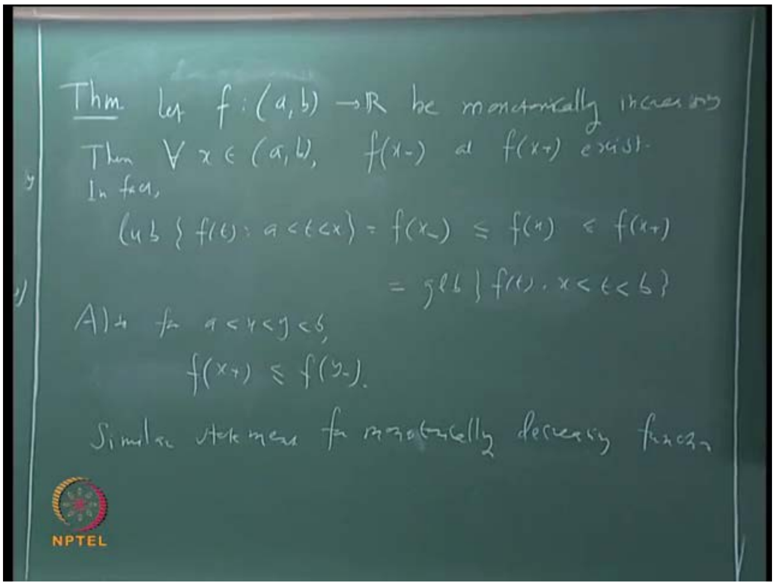

Now, let us begin with a monotonically increasing function. So, let us say that f from a, b to R being monotonically increasing. Then, what we want to show is that, it cannot have this discontinuity of type 2 and to show that it is enough to show that for every x both this limits exist. Suppose we show that that is enough. So, then this set for every x in a b f of x minus and f of x plus exist. In fact we can say something more, we can also say what those left and right hand limits will be?

What we can say is that this left, let us say this is the interval a, b and suppose this is x somewhere. Then, if you take the set of all values of functions between a to x and take their supremum then, that will be left hand limit of x. That is another words what we can say this LUB of this, let us say f of t where, t is between a less than t less than x. We can say that in fact this is what should happen. This is f of x minus that is left hand limit will be supremum of all values of taken over these values of f t over this.

And similarly, I will f of x this will be always less nor equal to f of x and f of x will be always less nor equal to that right hand limit f of x plus. And what will be f of x plus for f of x plus you take this interval x to b and take all possible values of f on that interval and take their infimum, that will be the right hand limit. So, g l b that is infimum of f t where, t varies between x n, x less than t less than b.

And we say something more, if we take say two points x and y with x less than y then the right hand limit of x. We all ready know that right hand limits as well as left hand limit exist. Right hand limit of x will be neither less nor equal to left hand limit at y. So, also we can say that for a less than x less than y less than b, f of x is less not equal to f of y minus. And I have stated this for monotonically increasing function, similar theorem for monotonically decreasing functions.

For decreasing functions what will happen is that this l u b etcetera that will be interchanged. So, you can suitably write the theorem for decreasing functions, I will not write the complete statement. So, I will say similar statement for monotonically decreasing function. So, let us go to the proof of this, let look at this set first and why the supremum of this should be same as f of x minus.

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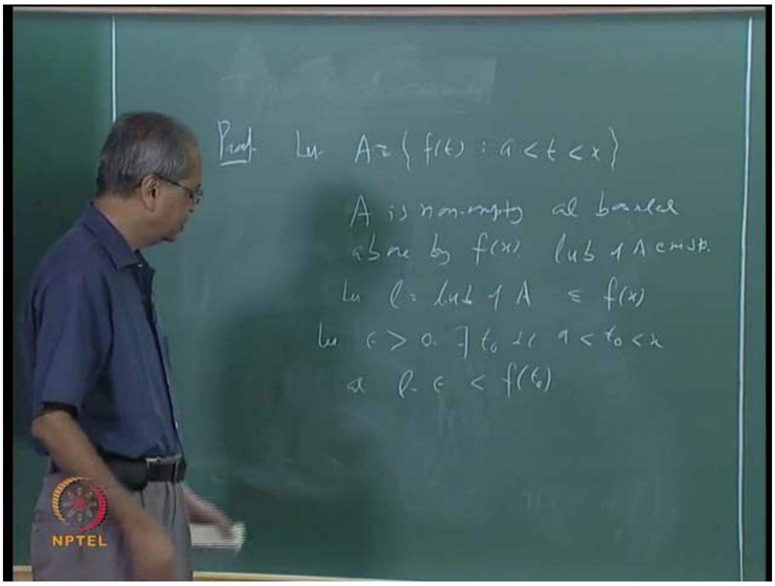

So, let me give some name to this set, suppose I call this set A. Let A be equal to set of all this values f of t, such that t varies over a less than t less than x. Obviously A is non empty, you can always there is some t between A and x, so that f of t lies here. Is it also bounded above because for every f t is neither less nor equal to f x. So, f x is an upper bound for this so A is non empty and bounded above by f x. So, l u b of A exists, it is a non empty set, which is bounded above.

So, the l u b axiom l u b exist therefore, l u b of A exist and not only that l u b will be less not equal to f x, f x is an upper bound. So, least upper bound will be less not equal to f x. Let us now give some notation here for l u b also. Suppose, let us say that let one be equal to l u b of A then, this one is less nor equals to f x and, we want to prove that l u b is same as this f of x minus that is we want to know. Put it one is same as f of x minus, we have all ready shown that one is less nor equal to f x.

Now to show that is a limit let us use the usual epsilon delta definition, so let us say that let epsilon be bigger than 0. Then, we will make a mutual argument, if you take any epsilon bigger than 0 then, l minus epsilon is not an upper bound of this set because l is the least upper bound. So, there should exist some element here, which is bigger than l minus epsilon. So, let me call that element t naught, so let epsilon bigger than 0 then there exist t naught such that a less than t naught less than x and l minus epsilon is less than f of t naught. Now, what is to be done after this is clear? See f is a monotonically increasing function remembers that.

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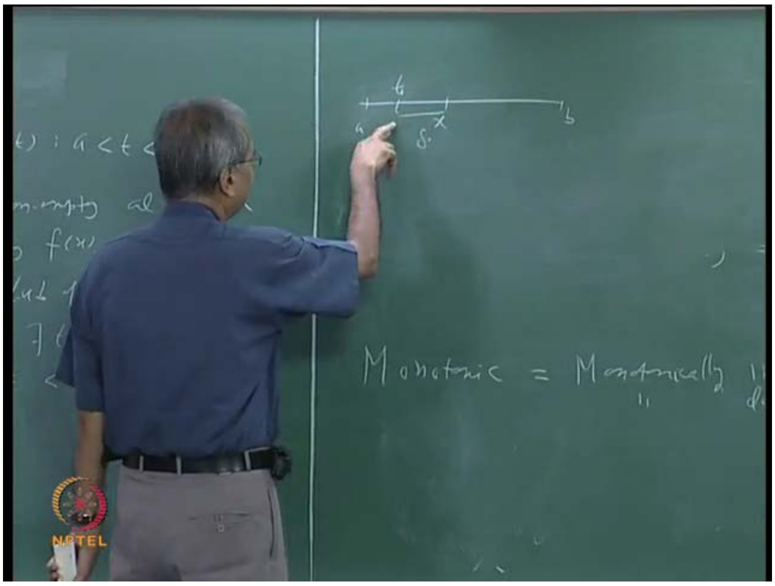

What is the situation here that is we have this interval a, b x is let us say one of the points here and t naught is somewhere here between a, x naught. And f of t naught is bigger than l minus epsilon f of t naught is bigger than l minus epsilon. Now, is it clear that if you take any points say t naught to x then, f of t will be bigger naught equal to f of t naught because f is monotonically increasing and hence bigger than l minus epsilon.

So, suppose I take this distance as delta then for that particular delta for all t lying between x minus delta to x. We will have f of t bigger than equal to f of t naught and hence bigger than l minus epsilon and obviously less than or equal to l n.

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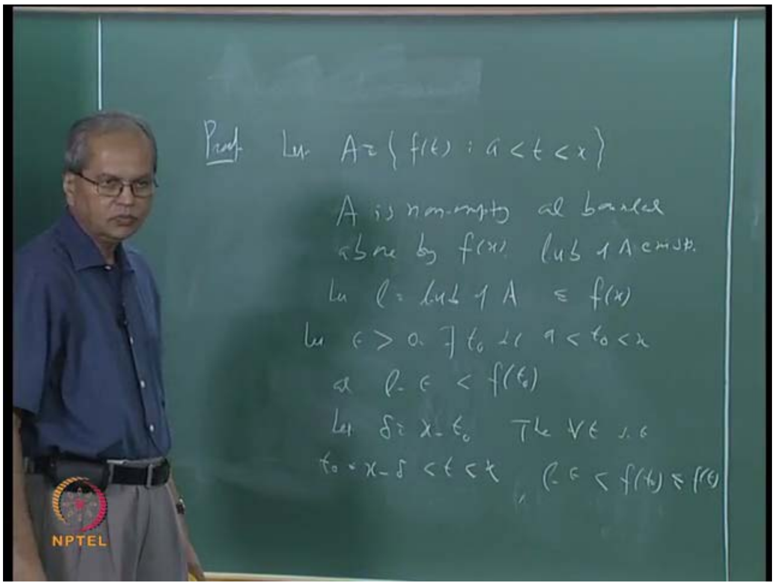

So, let us just complete this so let delta b equal to x minus t naught then, for all t such that x minus delta less than t less than x, x minus delta is the nothing but t not. So, what we have is that l minus epsilon is less than f of t not and the less that is not equal to f of t. And see t is somewhere here between t not to x. So, f of t is because on about to f of t not is bigger than l minus epsilon.

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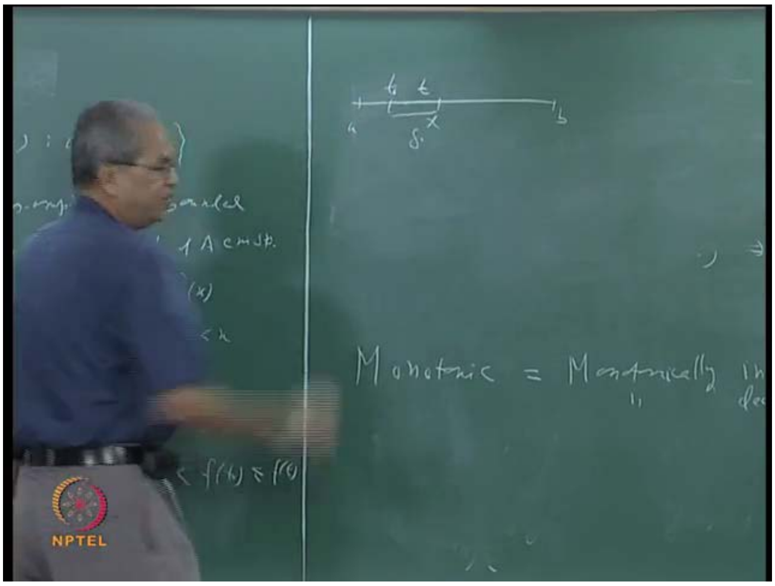

And f of t is anywhere less that is not equal f because l is the supreme amount.

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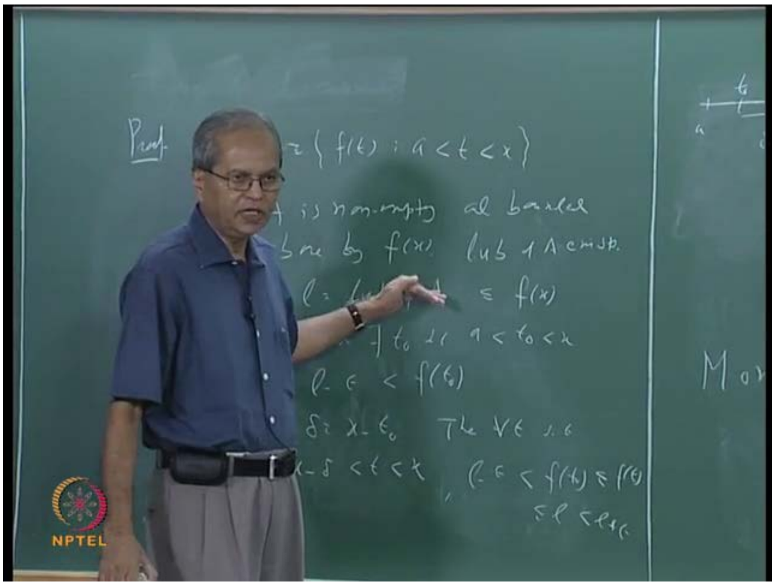

So, f of t is epsilon to become l, which will be anywhere less than l plus epsilon. So, what it show that if t lies between x minus delta x then f t lies between l minus epsilon to l plus epsilon. Now, this is same as saying that l is the left hand limit of f t s t equals to x t, so this process that therefore, l is equal to f of x plus.

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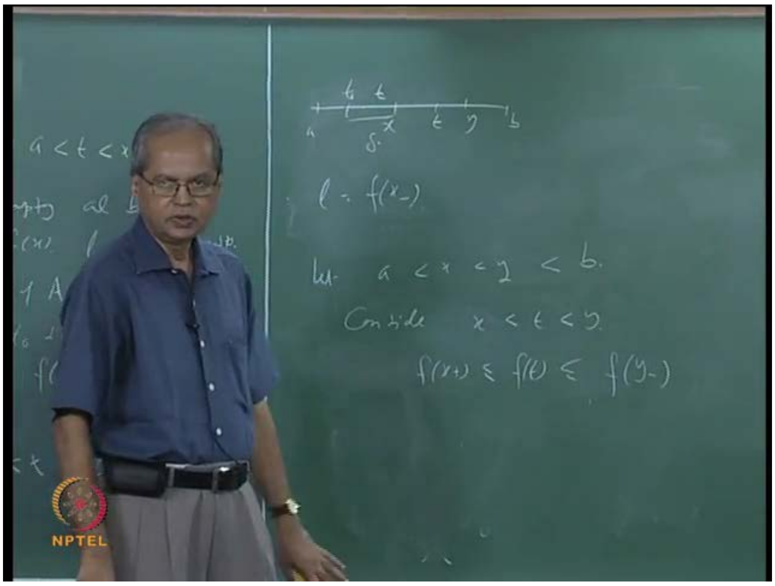

So, what did we prove? We proved this equation that is, this is what we had called l and now we have put it to l is equal to f of x minus that is left hand limit. We have put l is equal to f of x minus and we already shown that l is less than or equal to f x. So, all these we have proved. Now what are you decides the severally you can proved this inequality. Just as here I have taken a set a as f t a less than a t less than A, you can take the set d at say f t x less than t less than v then, take its infimum and show that, that is the thing, but the right hand limit.

In the same way for example, if you that let us say m is the greatest flour bound of that because m plus epsilon, that will not is the lower bound. So, there we see similarly, some t naught. Since the x ideas are more or less very similar, I will go to the details of that proof. So, prove that part yourself so that f of x plus is same as this number and then that number is bigger not equal to f x. Now what remains this last thing, we need to show that if x is less than y then f of x plus is less that is not equal to f of y minus.

So, now consider this let a less that x less than y less than b of course, this proof is more or less trivial, if you just look at the earlier it. It follows from what you could earlier. See if x is less than y I can consider some number t lying between those two. So, consider x less than t less than y then, remember one thing suppose I look at f of y minus. What is f of y minus? By this result f of y minus is the supreme of ft lying between a less than t less than y.

If you repulse x by y so it clear that f of y minus should be bigger not equal to this ft. So, f of y minus will be bigger not equal to f. What about f of x plus? f of x plus is by this it is greatest lower bound of this set ft x less than t less than b. So, is it clear from this that t is bigger than x; this t is bigger than x. So, ft should bigger not equal to f of x plus ft should bigger not equal to f of x plus.

So, we have proved all the things required in this theorem, so in particular we can say that what follows from this is the, I will continue here.

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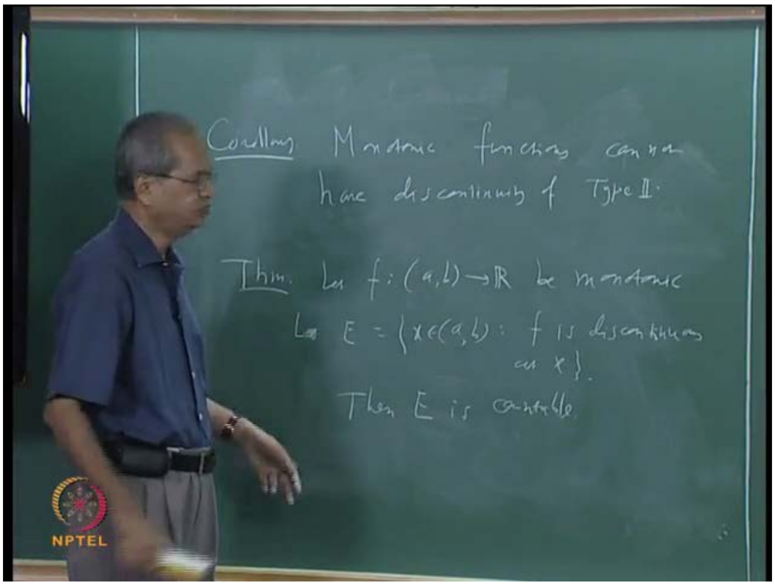

The statement monotonic functions cannot have dis-continuity of type 2, which is am a saying that all the discontinuity must b of type 1, that is there must be jumbled discontinuity or simple discontinuity. You can say that there is nothing new here. It does not have discontinuity type of two means what, the left hand limit and right limit always exist and that is what we have proved for every x left handed right hand limit always exist.

We can say something more we can say that if its subralonic function and if you look at this set of all discontinuities of f then, that set cannot be very big, there is that set has to be countable. Again if you look at the function inertial part of a x for the discontinues is the integral part of x, it is the set of all integers, that is a countable set. So, if you take any monotonic function and if you take the set of all discontinuities of that function then, that set must be countable set, which will also follow from what we have proved just now.

So, let us write that is a theorem, let f from a, b to R be monotonic. Again here monotonic means protonicaly increasing or decreasing and let us say that let e be the set of all discontinuity of f. Let us write the full form so e is the set of all x in a, b such that f is discontinues at x then, e is countable that is what want we want to see. So, this shows that is set of all discontinuities of a monotonic function is a countable set usually in ((Refer Time: 33:51)) countable sets are considers small sets.

So, these extremely noted in a set of discontinuities of a moronic function, it cannot be a big set.

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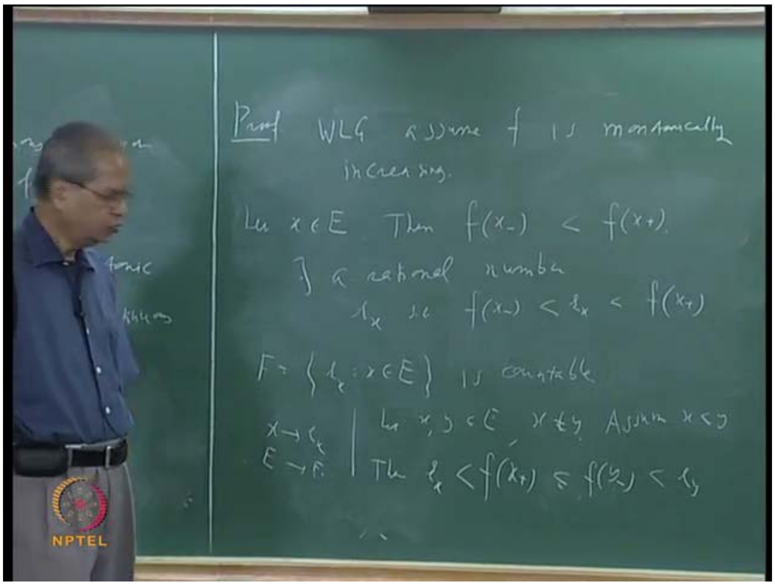

Let us take the proof of this for example; you cannot find monotonic function, whose set of discontinuities coincide with another set. Similarly, you could not find a monotonic function, which is discontinues everywhere that is not possible because the set of discontinuities has to be a countable set. We won’t to prove something for a monotonic function and as we know monotonic means only can increasing or monotonically decreasing.

So, let us take where all the cases other will be similar. So, as usual we shall say that without loss of journalist assume f is monotonically increase and then let us say that e is the set define like this. So, we would not show that e it’s countable. Let us take some x in e, as x e means in previous also in this interval a, b. Then, we know that f of x minus and f of x plus exist, that is there, but it is a point of discontinuity, so they cannot be the same.

If they are not the same then either one is that is not equal to 1 is strictly less than the other, which is we should be less than, which we know that f of x minus is always based on about to f of x plus. So, if it the function is discontinues at set x then f of x minus plus b strictly less than f of x plus. Then f of x minus must be strictly less then f of x plus.

Now, we do the usual ticks here. See these are two real numbers, we do not know what kind of real numbers are there, but these are two distinct numbers.

And we already know that between any two distinct numbers, there exists a rational number. So, we choose one as rational number. Since it is going to several, choose any one of them and since it is going to depend on x, I shall call that number or suffix x. So, there exist rational number R suffix x, such that f of x minus strictly less than is R suffix x and strictly less than f of x plus. Now after that what is to be done is clear. Suppose I collect all such rational numbers, for each x I have one rational number.

So, suppose I have taken these set of all R suffix x, the x belongs to e. This is of set of rational numbers and hence it is countable. This is countable. Remember for each x, we have chosen one rational number and obviously that set is going to be countable. Now, does it say immediately that e is countable? We are not shown that for example; it can happen that, two different x make a response to the same rational numbers. This function x going to R x, if you show that this one then that is fine.

If you show that x going R x that is one then there will be a bisection between e and this set. And then this is countable and hence e is also countable, but that it we have not shown. Now what we only know is that map x going to R x is on two because we are collected only those numbers here. Let us give a some name to this set, suppose I called this set, let us say f then, there is a map, which goes from e to f namely x going to R x that is x going to or suffix x.

This is a map, which takes e to f. We know that f is countable, but we know that this map is a on two, but that defines that e is countable. And that will not say that e is countable, but if you show that it is one then it will mean that it is bisection and if f is countable that will also give that e is countable. Now, what is required to show that this map 1 1, we must show that if you take two different numbers x and y then the corresponding R x and R y are different.

Suppose, you take two different numbers x and y then corresponding R suffix x and R suffix y are different. So, for that we are going to use this result next part of this. So, suppose you take two difference x and y, showing particular we can as show that x is less than y. To show that part let us say that let x, y be in e and x not equal to y. Obviously if x not equal to y, either x is less than y or y is less than x. It does not matter, which one you call x or which one you call y.

So, let us assumed that x is less than y then, what we know is the following. We know that then f of x plus is that is not equal to f of y minus then, f of x plus is that is not equal to f of y minus. What can you say about R x or R suffix x? R suffix x is must be less than f of x plus and what about R suffix y? R suffix y must be bigger than or equal to this not bigger than strictly bigger than remember or R suffix x is strictly bigger than f of x minus. So, R suffix y must be strictly bigger than f of y minus. So, now can R suffix x and R suffix y be equal.

So, what we have shown that if x is less than y, R suffix x is strictly less than r suffix y. So, that shows that this map which takes x to r suffix x is 1 1, it is already on two so there is a bisections between e and f and f is a countable set. It is a sub set of a rational numbers, so e is a countable set. So, I need to take a summary of this monotonic function, what we are shown is that monotonic functions cannot have discontinuity of type 2.

All the discontinuous of R of type 1 and the set at which the functions is discontinuous, that set must be countable. Now let us also discuss one more thing, which something that we have discussed about the sequences, but we are not yet discussed about the functions. In case of functions, we have seen that after defining the limits or convergence of a sequence. And discussing various properties, we extended those things to extended real numbers system.

And we talked about when do we select a sequence goes to infinity or minus infinity etcetera. Now, similar things we also want to do for the functions.

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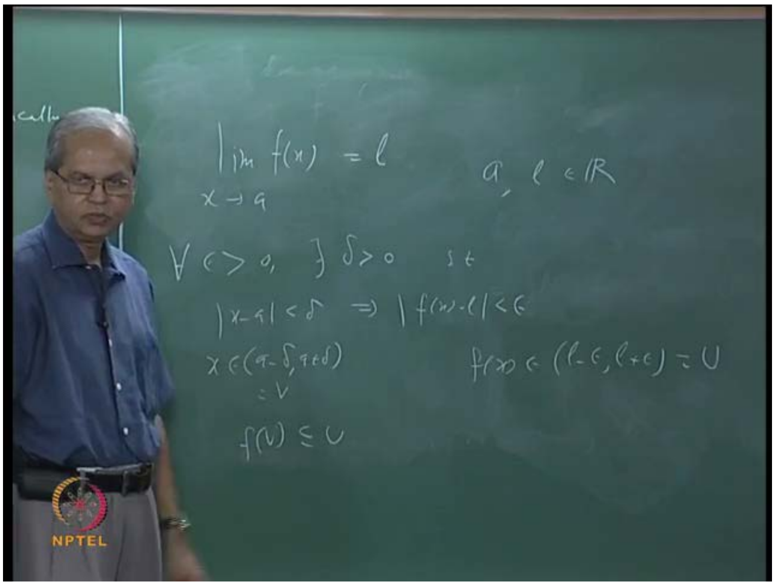

Till now we have talked about this what does meant by saying that limit of f x as x tends to e is equal to l. And of course, we have done it in case of any matrix space and a function if goes from x to y also, but let us know say that f is a real valued function. And it is define on some intervals on the real line and it is a real valued function. So, we have no problem in saying when a, l both are R, both are real numbers. What we want to do next is that, we want to consider whether e or l or both can be extended real numbers.

That means whether l is plus infinity or minus infinity or similarly a plus infinity or minus infinity. And the way we are going to do it is as follows. Let us first we call what is the meaning of this. Let me recall what we have said is this for given any epsilon because we said that there exist delta bigger than 0, such that for all y lying between x minus. So, for all x lying between a minus delta to a plus delta that is mod x minus a less than delta, this implies mod f x minus l less than epsilon.

We have a seen another interpretation of this or different way of writing the same thing saying that mod x minus a is less than delta, it is same as that saying that x lies in the interval a minus delta to a plus delta. That is let us say x lays in the interval a minus delta to a plus delta. Suppose I called this interval as v then, v is a open interval containing a or in other words it is an open neighborhood of a.

Similarly, saying that mode f x minus l is less than epsilon, that is same as saying that f x belongs to l minus epsilon to l plus epsilon. Again this l minus epsilon to l plus epsilon is in you can say open ball with center at l. So, it is an open set containing l so suppose I called it as u then, this u is a open neighborhood of l. What is relationship between v and u? If x belongs to v, f x belongs to u. Is that same as saying that f of v is contending u because what is f of v, f of v is a set of all f x for x in v.

And so every set f x must be in u so saying that whenever x is in v, f x is in u that is same as saying that f of v is in u. So, I can recast these four definitions in terms of these neighborhoods. So, what I can say is that for every epsilon is bigger than 0, there exist delta big etcetera. What I will say is it for every neighborhood of l, for every neighborhood open neighborhood let us say u of l, there exist a open neighborhood v of a, such that f of v is containing u.

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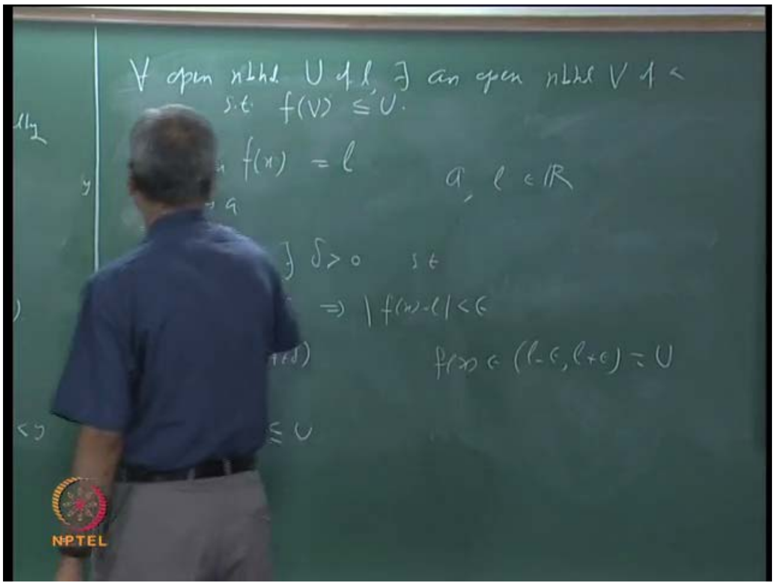

So, I will say that instead of this definition I will say that this means for every open neighborhood U of l, there exist an open neighborhood v of a, such that f of v is contending u. This is the meaning of this limit of the f x, basically we have just read it on the same definition using the idea of open sets or open neighborhoods. Now why did I do that is want to extend this to the case when a and l can be in or R star. What is R star?

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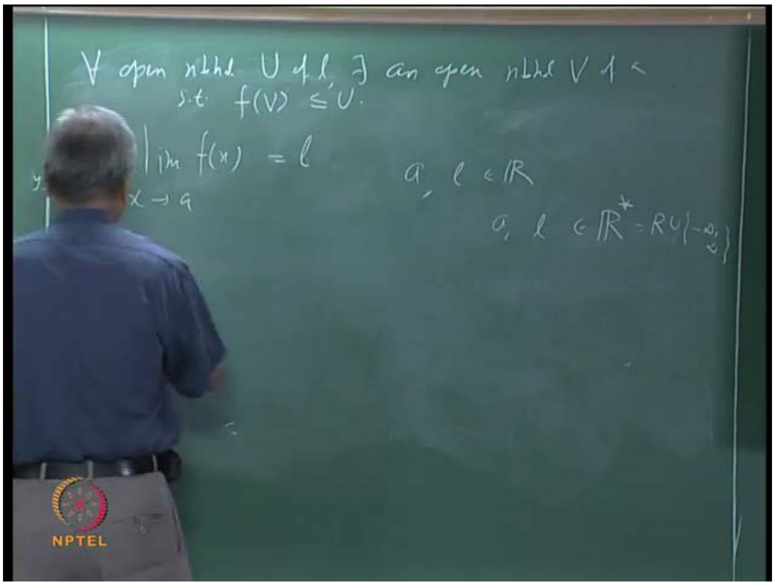

R star is R union these two symbols minus infinity and plus infinity with the usual operation, which we have turned about earlier. Now, if I want to extend this definition what I will do? This I will just use this definition I should only say what is meant by neighborhood of plus infinity and neighborhood of minus infinity. If I say that then the same definition will go and that is fairly easy.

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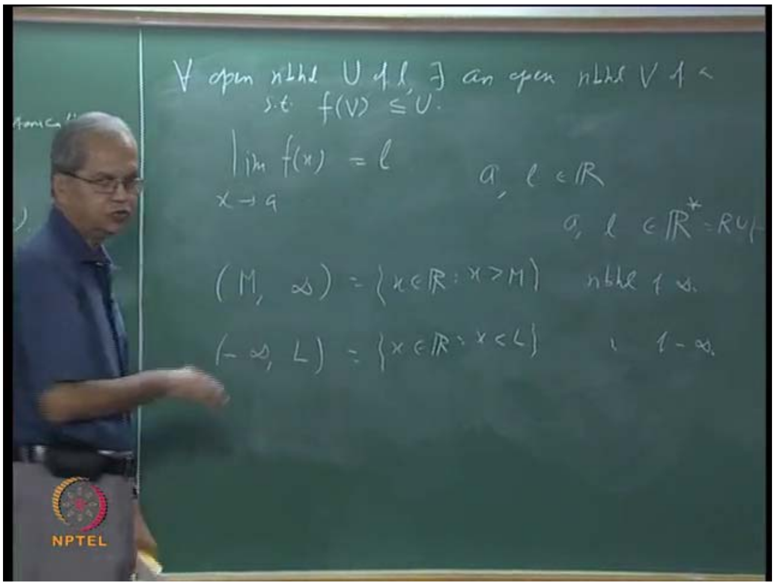

I will say that if you take any interval of this form m to infinity take any real number m take any interval form m to infinity then that is called a neighborhood of plus infinity.

What is this set, this is the set of al real numbers such that set if all x in R such that, x is bigger than m, we say that this is a neighborhood of infinity. Similarly, if I take minus infinity to m, that is called the neighborhood of minus infinity. Let me use some other reduce of m, so suppose I take see minus infinity to l.

So, this is the set of all real numbers such that, x is less than l, this we regard as neighborhood of minus infinity. Now, once we fix this what we have been by neighborhood then, we just use exactly the same definition. Now, a and l are both can be either real number or the plus inferior of a and l both now can belong to the extended real, that means one of them can be plus or minus infinity or both can be plus or minus infinity.

All that you do is that, just use this definition and understand it neighborhood of plus infinity means intervals like this and neighborhood of minus infinity means intervals like this. Let us let us just take one illustration of how these definitions will translating to the usual definitions that you're familiar with.

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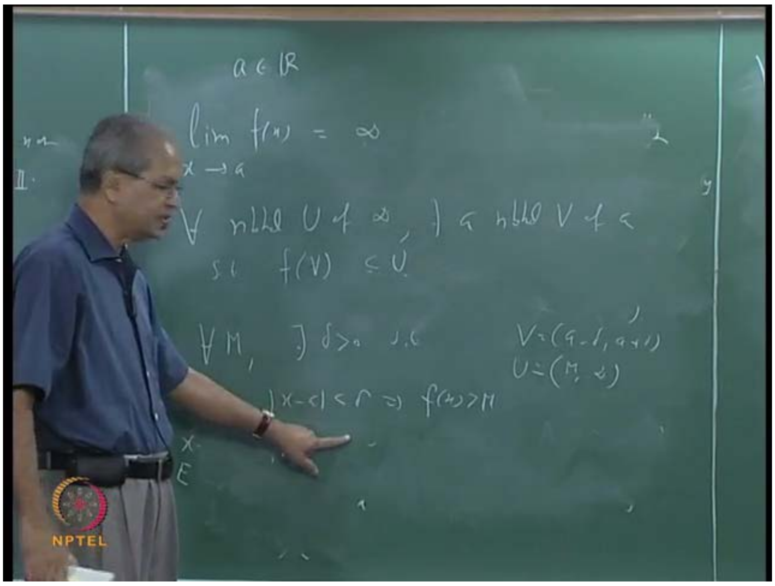

So, for example, lets us say a is a real number, but l is plus infinity that is suppose I want to say what is meant by this limit of f x as a x tends to e its infinity. So, a is a real number and l is plus infinity. So, what will be a definition, for every neighborhood of u of l, there should exist neighborhood v of e such that, f of v is contending u of this infinity. Now, but what are the neighborhoods of infinity we have say that neighborhoods of infinity of this type.

So, saying that for every neighborhood u of infinity, which same as say that for every m, there should exist v. v is a neighborhood of a and if is a real number neighborhood of the form a minus delta to a plus delta. So, we can say that for every f, there exist delta bigger than 0 such that, v is let us say v is a minus delta to a plus delta and u is m to infinity. So, what we have to say that f of v is contending u that means, whenever x is in this neighborhood f x is in this.

So, f of v is contended u that is same as that whenever mode x minus a is less than delta, this implies effects is bigger than m. That means given any number m, large or small it does not matter, you can always find a delta such that, whenever a mode x minus a is less than delta if x is bigger than m. This is the meaning of saying that limit of f x is x tends to a is infinity. Similarly, you can take all possible combination either l is infinity, x is real number or l is minus infinity, and a is the real number or a is infinity, and l is a real number etcetera or both are plus or minus infinity.

You can write a definition just take this as basic definitions and use these that neighborhood of plus infinites interval m to infinity, neighborhood of minus infinity is interval minus infinity to l. And then just translate all those definitions in this forms, you will get all the usual definitions in usual way, in which this kind of things are written in analysis books. We will stop with that.