Real Analysis
Prof. S. H. Kulkarni
Department of Mathematics
Indian Institute of Technology, Madras

Lecture - 31
Continuous Functions of Compact Sets

So, we have so far discuss this property of compactness I mean definition examples and various equivalent conditions of compactness etcetera. We shall now discuss certain properties of continuous functions defined on compact sets. Of course, one property we have seen already that is the namely that the continuous image of a compact set is also compact. Let us start from that.

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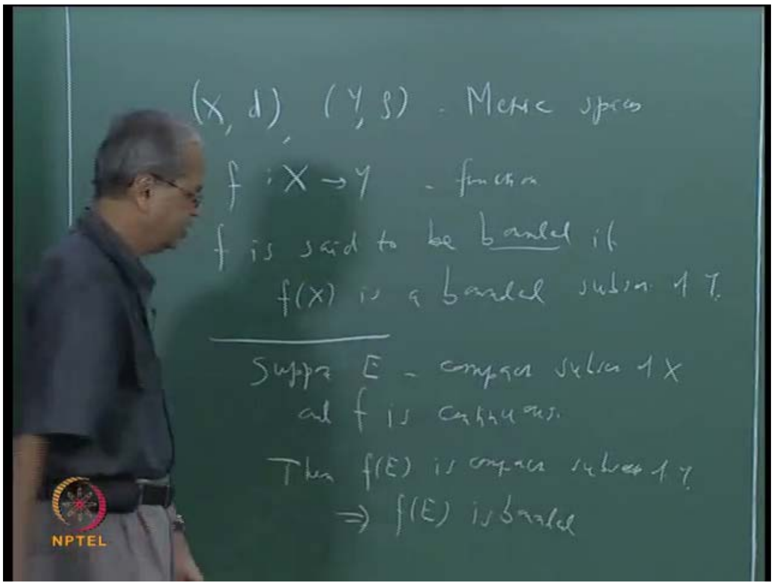

Let us also recall the following thing that is suppose we have let us say two metric spaces x d and let us say y rho are metric spaces. And suppose f from x to y is a function. We already know what is meant by a bounded set in any metric space. It is again recall bounded means its diameter is finite. And using that we can also say when will we say that the function f is bounded. We will say that f is bounded if its range. Range will be a subset of y if that is a bounded subset of y we shall say that f is bounded.

So, f is said to be bounded. I mean this is the usual definition. We have already used it said to be bounded, if f of x is a bounded subset of y. We may also consider some subset of x. And for example, a is a subset of x and say that f is bounded on a. It will mean that f of a is a bounded subset of y that is the usual terminology.

Then let us now say that what we have seen is already is this. That is suppose E is a compact subset of x and if is continuous and if is continuous. Then we already shown that if E is compact and f is continuous. Then f of E is also compact right, then f of E is also compact then, f of E is compact. Compact means compact subset of y. And we have already seen that every compact set is closed and also bounded. In fact we say something about it in fact it is what is called totally bounded. But let us not bother about that right now. It is at least bounded.

So, in particular what this means that… than f is… So, this in particular implies that f of E is bounded. f of E is of bounded and that is same as in f is bounded right? So, what we observed is that every continuous function defined on a compact set is bounded. Continuous function defined on a compact set is bounded all right. We can say something more if this space y is r if this space y that if it is a continuous real valued function defined on a compact set. Than what we can say is that not only that the function is bounded, but its bounds are attend. That is we can find a point at which the function takes its maximum value. And similarly, we can find a point at which the function takes its minimum value. So, let us let us take that.

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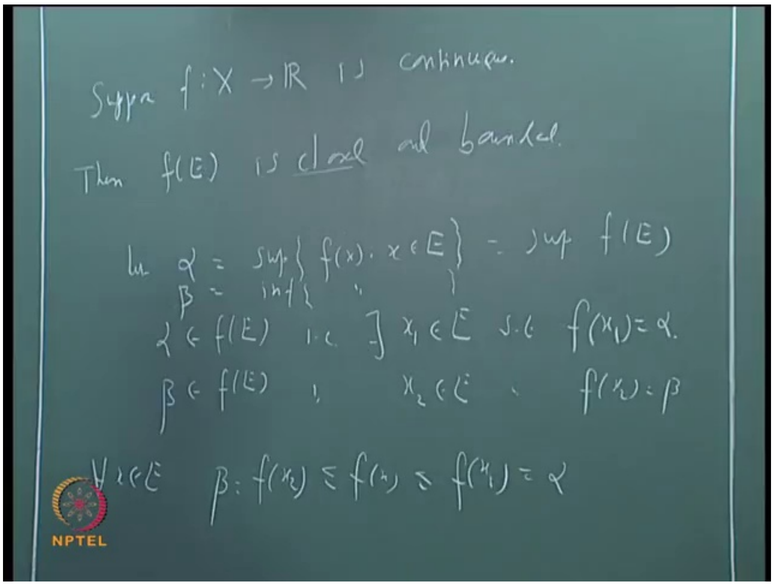

Suppose, now suppose f from x two R is continuous. And let us again take that E is a compact subset of x. Suppose E is a compact subset of x. Then f of E is closed as well as bounded right f of E is, f of E is compact and hence closed as well as bounded. So f of E is closed and bounded. And we are observed it in R there is no difference between bounded and totally bounded. This two concepts coincide in R. Since it is bounded we can say that there exists some supremum, there exist some supremum.

Let us say that let alpha belonging to, let alpha be the supremum of this f of E… six fine… Supremum of f of E is same as supremum of f x x belongs to E. Or which is better to still we can say what we can say same as supremum of f of E. That is the same thing right? And we have already seen that when you top of subsets of R then supremum is a point of closure right? Supremum of a set or least upper bound of a set is a point of a closure. So, if it is a closed subset then the supremum belongs to that set. Similarly, is the case of infimum right? So, this alpha since f of E is closed that implies that implies that alpha belongs to f of E.

That means that alpha belongs to f of E that is, that is we can say, that is there exists… Lets us say some x 1 in E such that f of x 1 is equal to alpha, f of x one is equal to alpha. Similarly, instead of the supremum suppose I take infimum suppose I take. Let us say suppose I take beta as infimum of this same set. Then this is also a point of the closure of f of E. So, we can similarly, say that beta also belongs to f of E. And so we can similarly, say that there exist some point x 2 in E such that f of x 2 is beta.

So now if you take any point x in E the value of f x must lie between beta and alpha or in between which is same as saying between f of x 1 and f of x 2. So, now what we can say that for all x in E since this beta is f x 2 this should be less than not equal to f x. And this should be less than not equal to f of x 1. And f of x 1 is alpha, right? That means that this two points x 1 and x 2 at x 1 if attends its minimum at x 2 if attends is maximum.

So, if it if f is a continuous real valued function defined on a compact set then it is bounded that is in fact boundedness is true whatever be y. In case of real valued functions, we can say something more. It is bounded and not only that the bounds are attend that is they are points at which f attends its maximum value as well as minimum value.

The way in which we come across this property is that see in the in many applications. This function is defined on some closed interval, closed bounded interval. And we know that is a compact set in the real line. So, every function defined on a close bounded interval will attend its continuous function will attend its maximum as well as minimum. Then let us also take one more thing suppose this map f its bisection suppose f is both 1, 1 and on 2. Then we know that inverse function exist. Now if f is continuous f inverse may or may not be continuous in general. But if x is compact then whenever f is continuous bisection its inverse is also a continuous function. So, that is a very important property.

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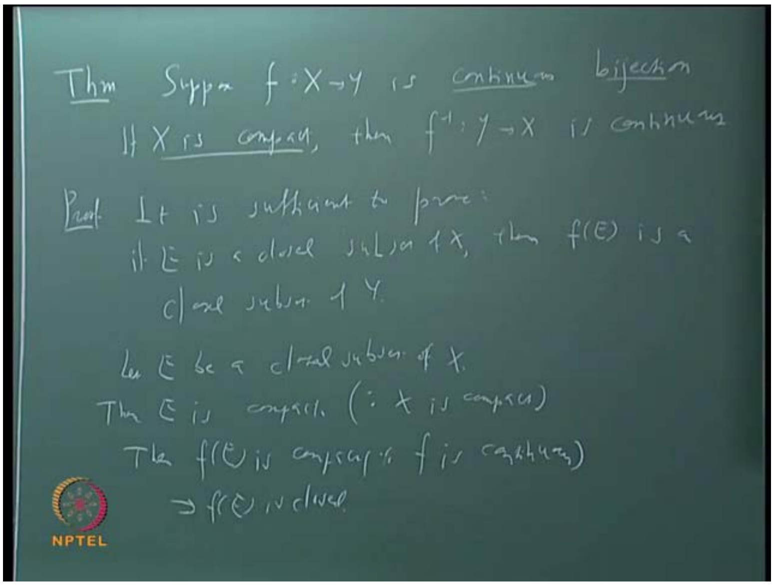

So, let us or important theorem rather let us do that as a theorem. So, suppose f from x to y is continuous bisection, suppose f from x to y is a continuous bisection. Then of course, once it is a bisection the map f inverse is defined, the map f inverse is defined, but it may or may not be continuous in general. But if x is compact than f inverse is also continuous. So, if x is compact then f inverse from y to x is continuous.

In other words we can say that… See we have already seen that is if a map is a bisection and if both f and f inverse are continuous. We have called such a map homeomorphism right? When f as well as its inverse are continuous we called it homeomorphism. So, what it says is that every continuous bisection on a compact metric space is a homeomorphism right? All right now what do we need to prove here? We need to prove that this is continuous.

Now, how does one prove that your map is continuous? There are several ways. You can either used that epsilon delta definition or we has shown that equivalently one can show that inverse image of an open set is open, right? Inverse set of open image is under map is open. Is it also clear to you that it is also equivalent to saying that inverse image of a close set is closed. Because closed sets are nothing, but the complements or opens set and under a bisection complements are also preserved.

So, it is enough to show that inverse image of a closed set is closed. But in, but inverse image under which map? We want to we want to say if f inverse is continuous so inverse image under f inverse. But is it same as saying that is, is it same saying inverse image under f inverse means nothing, but direct image under E f right? In you are taking inverse image under f inverse, f inverse inverse of some set which is same as direct image of under f of the same set. In other words what we need to prove is that if you take a close subset of x then its image is also closed. If we show that it will mean that f inverse is continuous map. Is that clear?

Once that is clear than you will select once this is clear than the prove is very easy. It follows from whatever we have proved earlier. So, let us let us first observe this it is sufficient to prove this, it is sufficient to prove that, it is sufficient to prove what if E is a close subset of x? Than f of E is a close subset of y.

So, let us start from this. Suppose let E be a close of x. But we have assumed that x is compact remember? And what can you say about a closed subset of a compact set it is also compact we have seen that closed subset of a compact set is compact. So, then E is compact. This is because we have assumed that x is compact right. Once we say E is compact what can we say of f of E? F of E is also compact because we are saying that f is a continuous function. Remember we have seen f is a continuous function. So, then f of, then f of E is compact. What is reason for this? This is because f is continuous. But once f of E is compact it follow that it is closed. We have already shown that every compact set is closed. So, this implies that f of E is closed.

Let us also recall that we have seen what is meant by saying that a function is uniformly continuous right? And we know that a function can be continuous, but need not be uniformly, uniform continuity is a stronger property. There we have seen examples of functions which are continuous, but not uniformly continuous Now what we want to say is that this sort of thing will not happen on compact sets on a compact set if a function is continuous then it must be uniformly continuous. So, remember all along we are try we are actually explaining how compactness is a very important property how it leads to many important and useful things.

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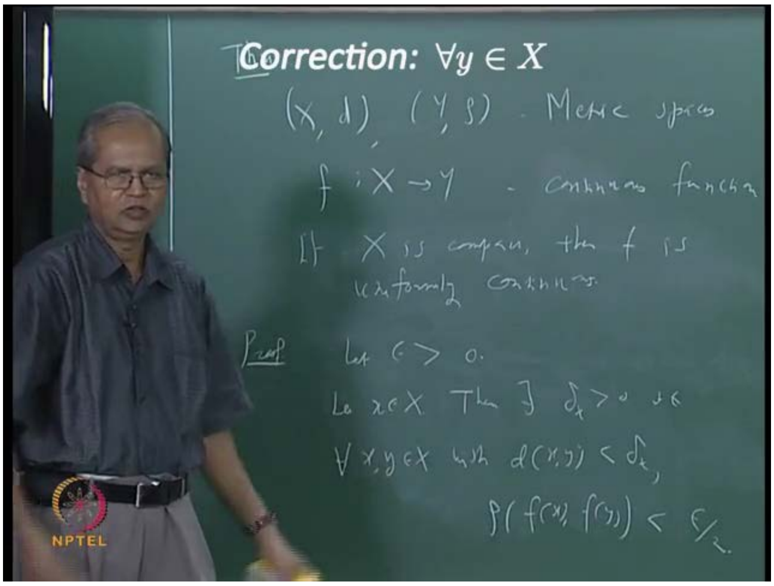

So, lets us now go to prove this theorem. Let me just read say this. Suppose now I take x, d the two metric spaces and f from x to y. Let us say continuous function. So, I want to say now is that. If f is continuous sorry if sorry if x is compact then f is uniformly continuous. Or in simple language every continuous function defined on a compact metric space is uniformly continuous.

All right there are several ways of proceeding with the proof. In fact there are many known proofs of this theorem. I shall now discuss the proof which is given in Rudin's book and when I you use some use the next problem sheet there I shall give you the certain steps in other approves as exercises. So, anyway what is needed to be proved that the to show that f is uniformly continuous? That given any epsilon there should exist some delta, which is independent of x. That is what we need to prove.

So, let us let us start with that. So, let epsilon be bigger than 0 and our aim is to find the epsilon find the delta, which corresponds this epsilon. Such that if you take any two points x and y in x with the distance less than delta. Then the distance between f x and f y should be less than epsilon all right. But what we know right now that f is continuous. So, given any x there will certainly exist delta, which works for that particular x.

Let us, let us start from that. So, let us say that let us take any x in x. Then since f is continuous at x there exists some delta, which does not required. So, that delta may depend on this particular x. So, then so, I will let me call it delta suffix x. Then there exits delta x bigger than 0. Such that for all x, y in x with d x, y less than delta x. Either one what this is same as saying that if y belongs to the open ball which centre at x and radius delta x. Then the distance between f x and f y should be less than epsilon.

This distance is rho. So, rho f x, f y is less than epsilon, less than epsilon. But I cannot take epsilon or any other small number. I can also say less than epsilon by 2 epsilon by 2 or 2 epsilon or epsilon by 3. It is any number I can take which is suitable for me to use in the subsequence steps in the proof. If this is not suitable we can change it later. So, for the time being I will take epsilon by 2. Now, see it is a usual way of using the compactness is the following. Because we know that how does one use compactness? We know that by compactness we mean that every open cover has a finite sub cover.

So, all proofs you would observe by know that, all proofs using compactness what they will do that for each x you construct some open set, which contains that x. Then family of all such open sets will cover x and then extract a finite sub cover from that. That is the, that is the idea. Now what I do is that, for each x I have this delta x. We take an open ball with centre at x and radius either delta x or again delta x by 2 or delta x by 3 we shall adjust that. So, for the time being I will start with delta x by 2.

So, suppose I take this open ball with centre at x and let us say radius delta x by 2. And take the family of all such open balls x in x. Then this is an open cover of x, this is an open cover of x. And since we have assumed that x is compact. This has a finite sub cover. Open your f x and hence and since x is compact has a finite sub cover, finite sub cover. Say that finite sub cover involves n points x 1, x 2, x n etcetera.

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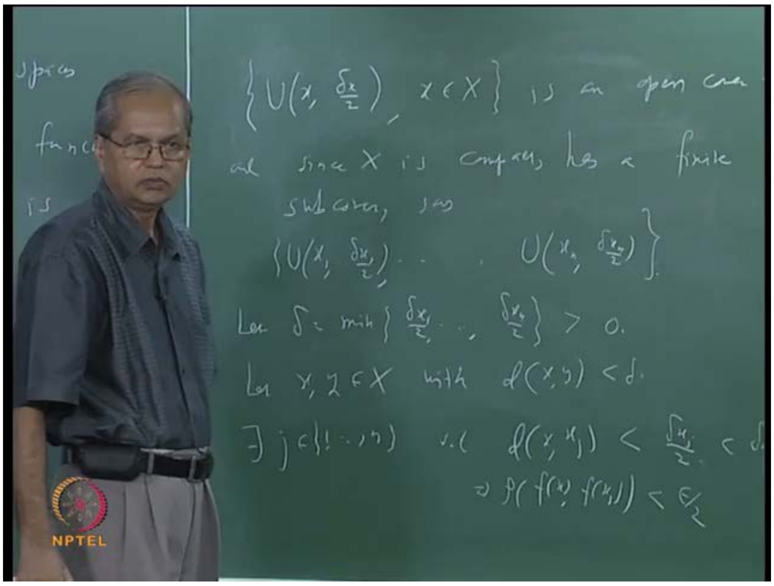

So, say u x 1 delta x 1 by 2 u, x 2 delta x by 2 etcetera. Let us say that ends such points u x n delta x n by 2. What does this mean? That if are taken given any point x in x any point small x in big x. That must lie in 1 of this balls. Each x that means given any x you should be able to find some j such that x belongs to u x j delta x j by 2. That is there was distance between x and x j should be less than delta x j by 2. For each x such j should exist.

Now let us take delta to be minimum of all this number. There are end such number delta x 1 by 2 delta x 2 by 2 etcetera. So, let delta be minimum of delta x 1 by 2 etcetera. Then see it is here that we are basically using that this a finite sub cover. So, this is a finite set and each of this number is strictly bigger than 0. So, the minimum amount them is also strict is in fact one of these numbers minimum is one of this number. So, that should also be strictly bigger than 0. If you take infinite set where each number is positive then the infimum over that cannot be positive. That can be, that can be 0. So, that is where we use the finiteness here.

So, we found a delta such that delta is bigger than this. And obviously our idea is to show that this is the required delta, which works for all x irrespective of x and y all right. Now let us see how that can be shown. Now let us take any two points x and y such that the distance between x and y is less than delta. So, let x y be in x with distance between x and y less than delta. And our aim is to show that the distance between f x and f y should be less than epsilon. Let us, let us use whatever we have done so far.

So, since x is in x we consider for this x there exists some x j, there exists some x j such that distance between x and x j is less then delta x j by 2 right? So, we can say that there exist j in this set 1 to n. Such that distance between x and x j is less than delta x j by 2. And of course, delta x j by 2 is obviously less than delta x j. And hence distance between f x and f x j that will be less than epsilon by 2, that will be less than. Because we have seen that if the distance between x and x y is, if distance between any two point is less than delta suffix x. The distance between corresponding f x and f y should be less than epsilon by 2.

So, this implies distance between f x and f x j. This is less than epsilon by 2. Remember our object was to show that distance between f x and f y is less than epsilon by 2. So, one way of achieving that is to show that distance between f y and f x j is also less than epsilon by 2. That we can do if we show the distance between y and x j is also less then delta x j. So, considered distance between y and x.

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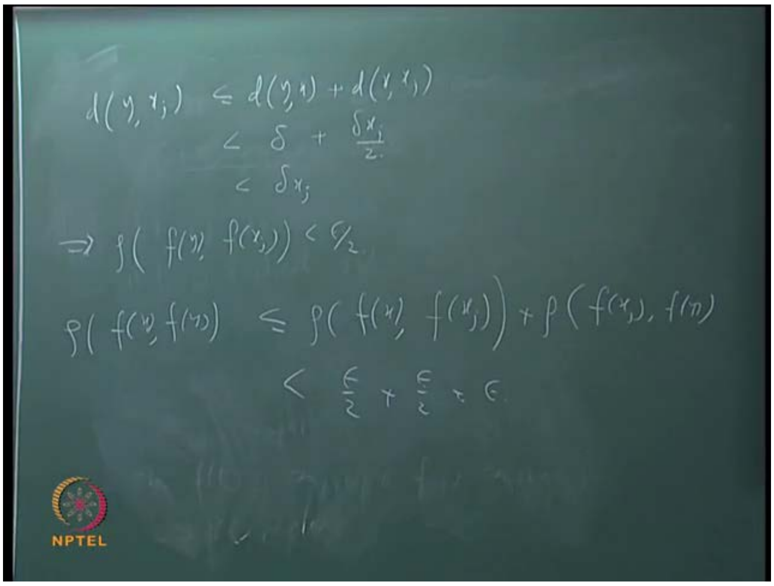

Distance between y and x. Obvious thing to do here is that since we know something about distance between x and x j. And we know something about distance between x and y and we want to know y and x j. Just use ((Refer Time: 27.07)) equative. So, this is the one equal to distance between y and x, plus distance between x and x all right. What do we know about this distance between y and x that is less than delta, right? And distance between x and x j that is less than delta x j by 2 right, distance between x and x j that is less than delta x j by 2 all right. Now how is this delta and delta x j are related? It is minimum of this all this.

So, I can say that this is also less than or equal to delta x j by 2 right? So, the sum is less or equal, less than delta x j right? And if… and hence now we have know that the distance between y and x j is less than is, less than delta x j. and hence the distance between f x f y and f x j should also be less than epsilon by 2. So this implies distance between f y and f x j is less than epsilon by 2

Now I think we have everything that we require. So, compare consider now distance between f x and f y. So, this is less than not equal to again by ((Refer Time: 29.13)) equative distance between f x and f x j. And plus distance between f x j and f y. And we have shown that each of the term occurring on the right hand side is less than epsilon by 2. That is this is less than epsilon by 2 that we have shown here and that the last term is less than epsilon by 2 that we have shown here. So, this whole thing is less than epsilon by 2 plus epsilon by 2 that is equal to epsilon.

So, we started with an epsilon bigger than zero. And we obtained a delta with the property that if the distance between x and y is less than delta. Than the distance between f x and f y is less than epsilon. And this delta is independent of x. We have taken any two x and y in x and we have shown that if the distance between any two x and y is less than delta than the distance between f x and f y is less than epsilon right? So, that shows that f is uniformly continuous right? Remember we are used the compactness of x here to obtain this finite sub cover to get this delta. And we have used the continuity of f here that is there may will exist delta, which depends on x.

Obviously we have used several things using compactness. Obviously if the compactness is not there then we should expect that this conclusions do not follow right? But of course, saying that this proof uses compactness and hence the conclusion does not follow is not a correct argument. Because we can always think that there may be some other proof. So to show that the conclusion does not hold. There is only one way. And what is that way? You must use a counter example.

So, in this case what does it mean that give a counter example. If x and y are metric spaces and f is a continuous, f is continuous. If x is not compact then continuous but not uniformly continuous. Similarly, in the earlier theorem if some x to y is a continuous bisection but f inverse is not continuous and similarly, the earlier properties for each of those properties see whatever we have discussed today various properties of continuous functions defined on compact sets try to drop the compactness and get a counter example.

So, that will be a set of exercises for you. Now we shall pay some attention to the functions defined on subsets of the real line. And let us see how some of these ideas apply there. Of course, already we have seen the functions which good the real line, but now we shall also think of functions, which are defined on subsets of real line.

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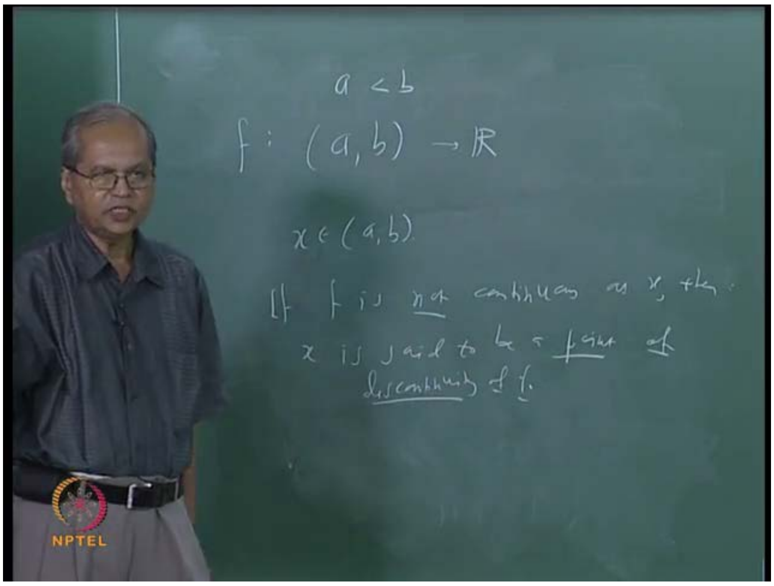

So in other words for the conditions that we all discussed it is convenient to take open intervals a, b. Let us say some a less than b and take the open interval a, b and consider f defined from a, b to R. And suppose we look at say x in a, b. We have already defined what is meant by saying that f is continuous at x, f is continuous at x. And if f is not continuous at x we say that x is a point of discontinuity of f right? So we will say that if f is not continuous at x than x is said to be point of discontinuity of x f.

What we want to discuss now is that… What are the points of discontinuity of a given function f and what are the types of discontinuity. Or in what way the this can happen that. In exactly how many ways x can f can be discontinued on x or f can failed to be discontinuous, f can fail be continuous at x. Remember that we have seen that when we are talking about continuity. If x is an isolated point then f is always continuous at x. But that will not happen in this case because this is an open interval and x belongs to the point in the open interval. So, x is not an isolated point x is always a limit point. And what we have seen further is that.

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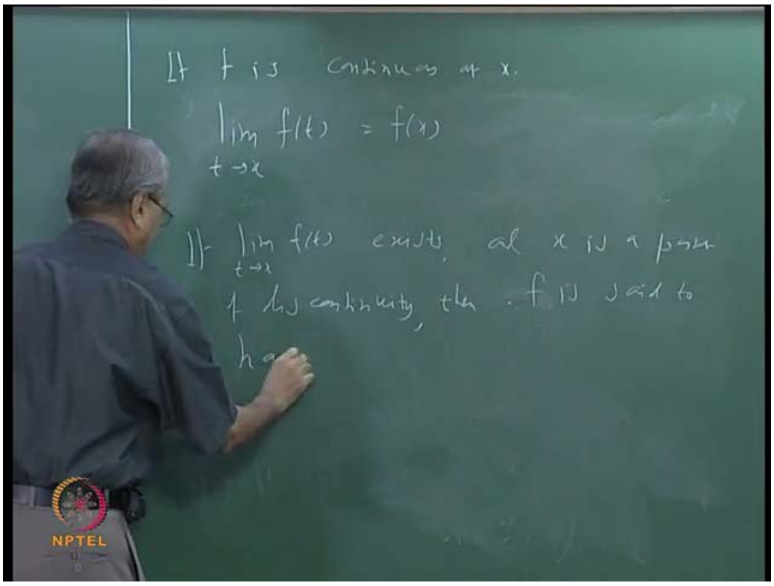

If a point is a limit point. Then f is continuous at x is equivalent to say that limit of since we have taken up let us say limit of f t as t tends to x. This limit should exist and its value should be same as f of x. That is if f is continuous at x. All right, but if it is not than one of the two things should happen. Either that this limit does not exist or the limit exist, but its value is not same as this. If its value is not same as this.

The second type that is the limit exist, but its value is not same as this. That part is called that is if limit t tends to x f t exist. Suppose this limit exist then what are the ways in which f can fail to be continuous? Only way is that that its value is not same as f x. Then that kind of discontinuity is called removable discontinuity. So if limit t tends to x f t exist and x is a point of discontinuity.

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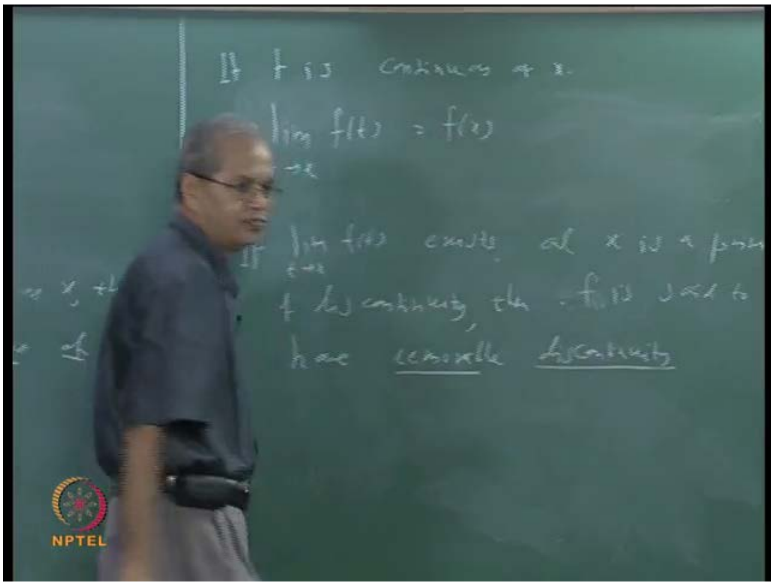

Then x is say let us say f is said to have removable discontinuity, removable discontinuity at x. So, this is in other words what we are doing is that we are classifying discontinuities in our words we are, we are considering what is called types of discontinuity types of discontinuity.

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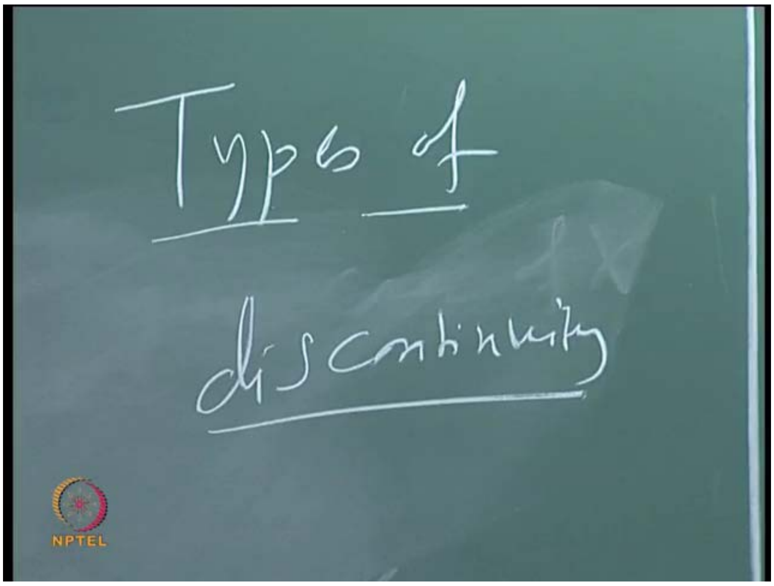

So this is one type namely removable discontinuity. Why it is called removable? Because if this the case what we can do is that we can redefine the function at x. And make its value equal to this limit and the new function will be this will be continues. That means we can remove the discontinuity by changing the definition of function at that point x. And that is why it is called removable discontinuity. And in for all practical purposes removable discontinuity is as good as consider the point of continuity. So we simply ignore the removable discontinuities of the function all right.

So we only talk of those discontinuities, which cannot be removed in this fashion. That means what that means this limit does not exist, that means this limit does not exist. What we shall do further is that we want to further classify those discontinuities. And to do that we will also define what is called you must you must have heard have these things in your undergraduate course also, but let us recall again. What is called left hand limit and right hand limit of a function? Lets us again use ruddiness notation.

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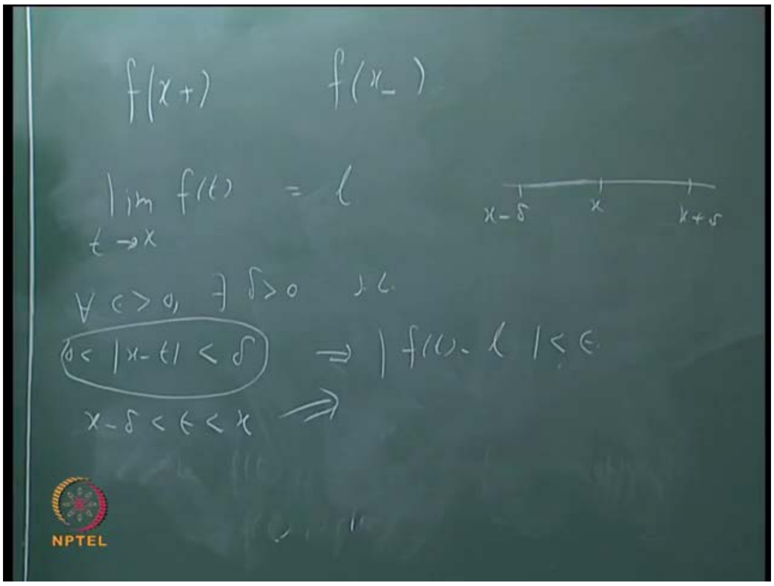

f of ((Refer Time: 38.59)) f of x plus and f of x minus, f of x plus and f of x minus. Let us first recall this suppose limit of f t as t tends to x. Suppose that limit is equal to l now what is the meaning of this? We have we have seen that this means that for every epsilon bigger than 0 there exist delta bigger than 0. Such that mod x minus t less then delta. Let us say 0 less than mod x minus t less than delta. This implies mod f t minus f x is less than epsilon.

And we have seen the geometrical interpretation of this. This means that mod x minus t less then delta means that t lies between x minus delta and x plus delta. That is you take this interval say suppose this is x this is x plus delta and this is x minus delta. t lies between x minus delta to x plus delta. If t lies in this interval than f t should lie between f x minus epsilon to f x plus epsilon. Now instead of this instead of letting t lie between this whole interval x minus delta to x plus delta.

Suppose I say that t lies only in this interval. What I say that t lies in the interval x minus delta to x. And for that t suppose this happens mod f t minus f x is less than epsilon. That is instead of this what I say that if so happens that for every epsilon bigger than 0 there exist delta bigger than 0 such that for 0 less than I shall say that for x less than x minus delta less than t less than x. That is for all t such that x minus delta less than t less than x.

This implies that mod f t minus f x is less than epsilon sorry I am not f x. We are talking about limit being l mod f t minus l less than epsilon. But we do not know anything about what happens if this t in this half. If this happens we say that this l is a limit of f t as t goes to x from left. And that is this notation. We say that l is equal to f of x minus left hand limit right? That instead of letting t lie between x minus delta to x plus delta we say that this is true.

Mod f t minus l is less than epsilon is true for those t, which lie between x minus delta and x. It may or may not be true for this part, it may or may not be true for this part. In a similar way we can define what is meant by f of f plus. Instead of taking x minus delta less than t less than x. We take x less than t less then x plus delta. So, whatever is that number that number will be called the right hand limit. That is limit of f t as t goes to x from right instead of this minus sign here will put the plus sign and this number l will be denoted by f of x plus right?

Another way of saying that is in terms of sequences what we have seen is that… See suppose, forget about this minus suppose limit of f x t tends to x is equal to l. We have seen that is equivalent to saying that if you should take any sequence t n converging to x than f of t n should converge to l. f of t n should converge to l. Only modification in this left hand limit will be that instead of letting the sequence t n to be lying anywhere. I will take only those sequences t n such that every t n is less than x. Only those sequences t n suppose the whole thing is happening in the interval a to b. And suppose x that if how we started right.

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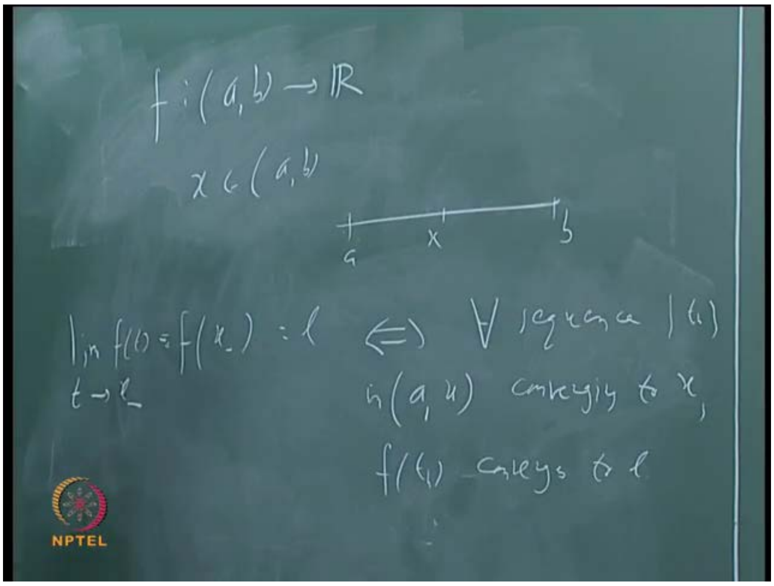

We took f from a, b to R. Than x is some point in a, b. So, suppose this is the interval a, b and x is some point here. Than I take only those sequences which lie in a to x, which lie in a to x. So, suppose you take only those sequences which lie in a to x and suppose for every sequence t n converging to x, f of t n converges to l. Than that l will be called left hand limit. So, we can say that, so we say that f of x minus. Or which is same as limit of our notation limit of f t as t tends to x from left, t tends to f from left. That is, this is equal to l. This is if and only if that this is same as saying that for every sequence t n, for every sequence t n in the interval a to x, for every sequence interval a to x.

Converging to x, converging to x f of t n converges to, converges to l, f of t n converges to l. And similarly, for the right hand limit we can say that if you take any sequence t n lying in this part x to b. And if it converges to x then f of t n converges to l. If that happens than we select that number l is the right hand limit all right. Now it is clear it is obvious to say that if the limit, if this limit let me remove this if this limit exist.

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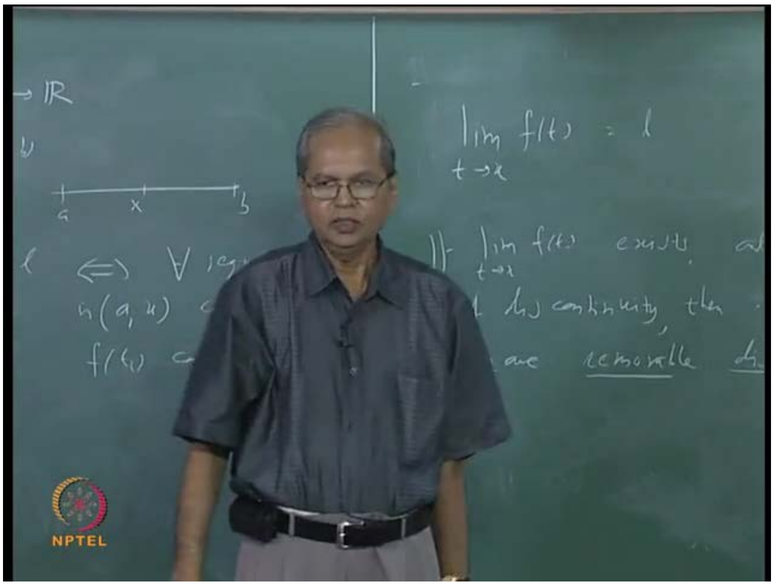

And that limit is same as l that limit is equal to l than the right hand limit as well as left hand limit both exist and both are equal to l right? But it can happen that the left hand limit exist right hand limit also exist, but they are different, left hand limit exist and right hand limit exist and they are different.

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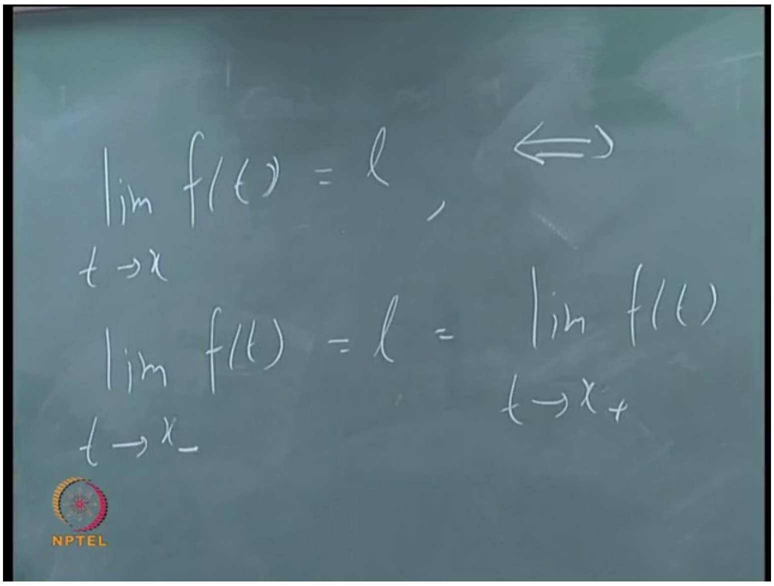

So, let me just write this. If limit of f t as t tends to x exist and that is equal to l then f of x minus that means f of x minus exist or let me write it this one. Then limit f t as t tends to x minus that exist. And that is equal to l and that is also same as limit of f t as t tends to x from right. So if the limit exist then the left hand limit as well as right hand limit also exist. And they have the common value and that common value is same as the value of the limit. Conversely if both the limits exist if both the left hand limit also exist and the right hand limit also exist.

And if they have they have the same value then the limit exists and the value of the limit will be same as the that common value right? So I can say that this is so if and only. But what can happen is that the left hand limit exist and right hand limit exist and the values are different. Then obviously means that the limit does not exist, then obviously means that the limit does not exist.

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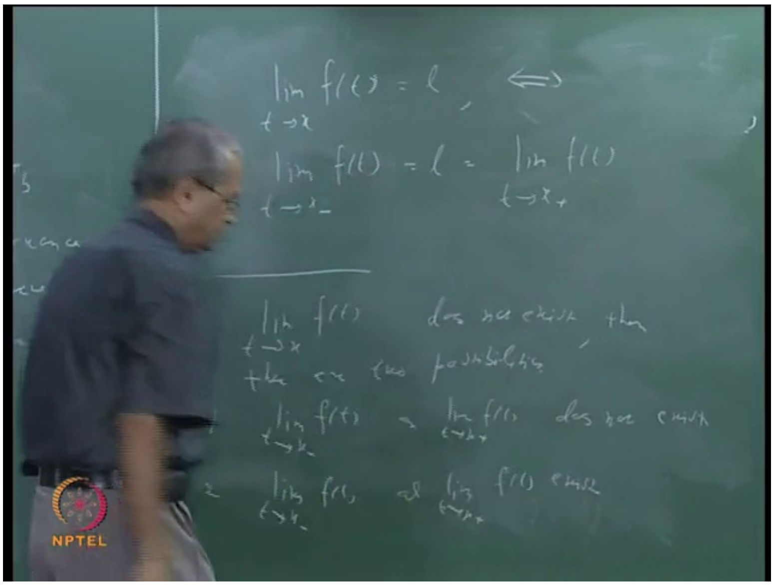

So, lets us say that the limit f t. Let us say as t goes to x. Suppose this does not exist. That means x is a point of discontinuity of f. This if the limit does not exist this means x is a point of discontinuity of f. Now we can say that it can happen in how many ways there are two possibilities. Either the left hand limit does not exist or the right hand limit does not exist or off course both do not exist, or what is the other possibility that a left hand limit exist right hand limit exists and their value is different.

So, if this does not exist than let us say there are two possibilities, there are two possibilities. What is the first possibility one is that limit of f t as t tends to x minus. Or limit of f t as t tends to x from right does not exist or both. But that is ((Refer Time: 50.03)). When we say this order it means both is also equal to ((Refer Time: 50.06)). Or what is the second possibility that limit of f t tends to x minus. And limit of f t as t tends to x from right both exist. And they are different, and their value is different.

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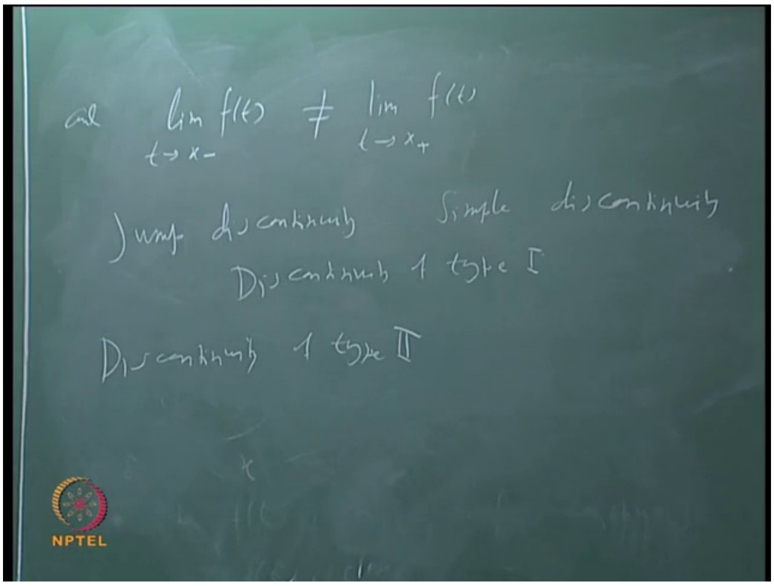

And limit of f t as t tends to x minus is different from limit of f t as t tends to x plus. When this happens. That is the left hand limit exist right hand limit exist and their value is different that kind of discontinuity is called jump discontinuity. Jump discontinuity or simple discontinuity or it is also known as discontinuity of the first type. So, this is, this called, this is called jump. There are different name jump discontinuity also known as simple discontinuity or also known as the discontinuity of type one.

Let us again ((Refer Time: 51.59)). What is meant by jump discontinuity or simple discontinuity or discontinuity of type one? It means left hand limit as well as right hand limit exist, but the values are different. That is jump discontinuity. If one of this limits do not exist that is called discontinuity of type two. That is this part, that is this or this limit does not exist that is called discontinuity of type two. We shall see the examples of this both types of discontinuity of course, we already seen that examples we shall recall to that examples. And demonstrate that which are the kinds of discontinuities that we have come across of the real valued function defined on an open interval. This we shall do in the next class.