Real Analysis
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Lecture - 30
Characterization of Compact Sets

So, we will continue our discussion of compactness or towards the end of the last class, so I said that in the real line a set is compact if and only if it is closed and bounded, and we shall, we shall prove that today, but instead of proving that directly we shall prove a theorem which is more general than that and that will come out as a consequence of it. We shall use several statements which are equivalent to the compactness and use of that is that we can use anyone of those statements, to prove or check whether a given set is compact or not.

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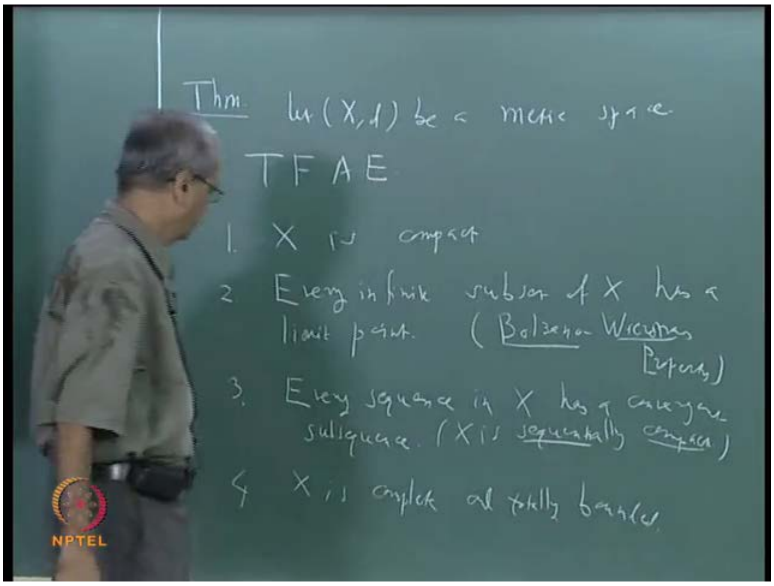

So, that theorem is as follows these are you can say various characterizations of compactness, so let us say let X t be a metric space then what we want to say that the following statements are equivalent. That this is the standard short form used for that T F A E; that means, the following are equivalent, following statements are equivalent and what it means is that if one of the statements is true all the statements are true if one of them is false all are false. So, first of that is X is compact, so that is why we call this correct resistance of compactness, second is that if you take any infinite set in X then that has a limit point.

So, every infinite subset of x has a limit point every infinite subset of X has a limit point this is sometimes called Bolzano Weierstrass property. If a metric space has this property that every infinite subset has a limit point that metric space is said to have a Bolzano Weierstrass property. So, what we here is saying is at compactness is equivalent to Bolzano Weierstrass property.

In case of metric space then third is every sequence in X has a convergent sub sequence every sequence in X in X has a convergent sub sequence this Bolzano Weierstrass property is, for example is a term used in Siemens. Similarly, this property is described by saying that X its sequentially compacted if a metric space has this property that every sequence has a convergent sub sequence we say that X is sequentially compacted. So, this is another way of saying the same thing is that X is sequentially compact. So, what this theorem says that in case of metric spaces compactness, and sequential compactness these two things are the same.

This is not true in arbitrary topological spaces in arbitrary topological space compactness and sequential compactness is different. But, let us not go into that and lastly X is complete and totally bounded and totally bounded, now when we want to prove the equivalence of all the statements one can proceed in the various ways. For example, one can show that 1 is 1 if and only if 2, 2 if and only if 3 etcetera or one can prove 1 implies 2, 2 implies 3, 3 implies 4 and then 4 implies 1 or any cycle like that you can chose any cycle like that and go over to prove. But, even before going to the proof of this let me say how the Heine Boul theorem follows form this, so let me write that as a corollary.

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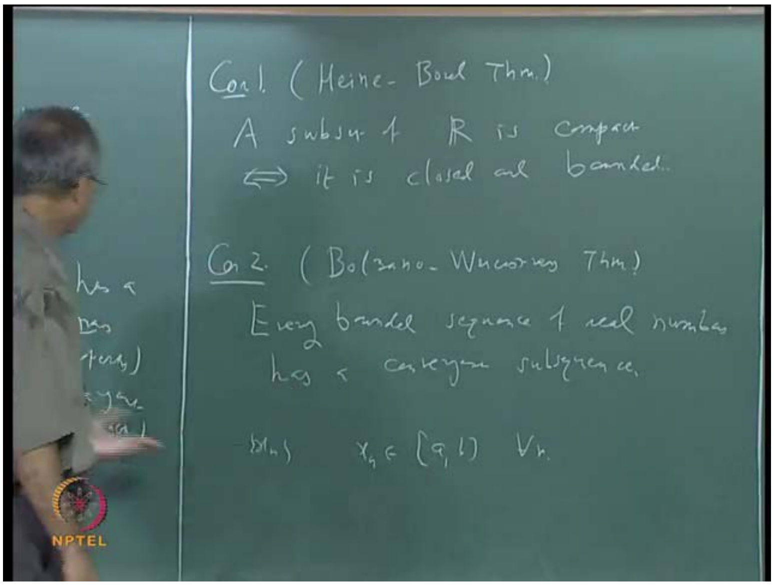

Let us say corollary 1 that is Heine Boul theorem, so what that theorem says that if a subset of real line with the usual metric it is compacted if and only if it is closed and bounded. So, a subset of r is compacted if and only if it is closed and bounded, now how this follow from this theorem of course we have already seen one thing that if a set is compact then it is closed and bounded. That is true in any metric spaces only issue is to prove the converse, but we already know that r is a complete metric space and we have shown that a closed subset of a complete metric space is complete.

So, if you take a subset which is closed then that is complete and we have also shown that in r every bounded set is totally bounded, in r every bounded set is totally bounded. So, if a set is closed and bounded there it is complete and totally bounded, so that is this last sentence if X is complete and totally bounded 4 implies 1 will say that X is compact. So, if you prove this theorem that will follow, that will follow let me also recall one more well known theorem that we have proved in the case of sequences.

We have shown that every in fact that is that was call Bolzano Weierstrass theorem at that time, let me just write that was call Bolzano Weierstrass that is namely that every bounded sequence has a convergent sub sequence every bounded sequence of real numbers. So, if real numbers has a convergent sub sequence of course we have proved this earlier using the ideas of limit superior limit inferior etcetera. But, you can see that this also follows immediately from this because what we can say that if a sequence is a convergent sequence.

Suppose X is at a sorry, X n is at bounded sequence then we can always find an interval let us say interval a b such that each X n lies in that interval, if X n is bounded we can say that X n belongs to some interval a b, X n belongs to some interval a b. Now, this set is closed and bounded and hence compacted and once a set is compacted we have saying that every sequence in a b must have a convergent sub sequence. So, this also follows from this theorem, so this is a very powerful theorem it gives various characterizations of compactness and it has lots of uses also.

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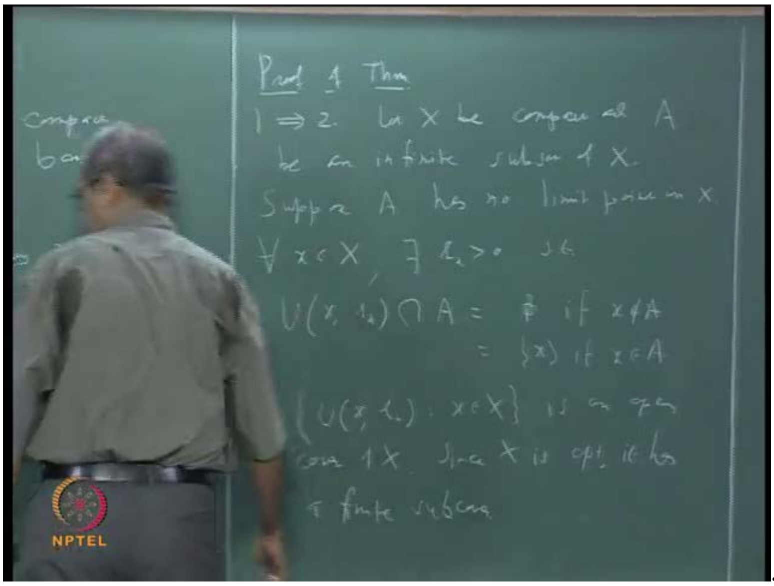.

So, let us go to the prove will theorem as I said we can start proving in one cycle let us, let us attempt this way 1 implies 2, 2 implies 3 etcetera if we get it to some problem let us see. So, let us start with one implies two one implies two means what we assume that X is compact and show that every infinite subset has limit point. So, let X be compact and suppose that the infinite sub set is a let us say A be at infinite subset infinite subset of X, now if this subset of this set A suppose 2 is false then it means that this subset does not have a any limit point.

Now, if A does not have any limit point what does it mean suppose if you take any x in X then that x is not a limit point, that X is not a limit point. So, suppose A has no limit point in x, suppose A has no limit point in X that means what if you take any point x in

X any point x in X that x is not a limit point of A. Now, what is that mean if you look at the definition of bit point we say that if you take any open ball with center at x that should contain at least one point from A other than x itself, other than x itself.

So, if something is not a limit point it means there exist an open ball, we does not contain any point of A other than x itself. If at all x belongs to A if x does not belong to A then no point from A, if x belongs to a then just singleton x intersection with that ball with a will be just singleton x. So, this means that means for every x belonging to X there will exist a ball with radius r that r may depend on x. So, I will say, I shall denote that by r suffix x for x in x 3 exist, r suffix x bigger than 0 such that suppose you take an open ball with center at x and radius r, this r suffix x this does not contain.

This does not contain any point other than, other any point of A other than x itself if in case x belongs A. So, you can say that let us, I did that this intersection A will say that this is empty if x does not belong to A, this is empty if x does not belong to A and this is equal to singleton x, if x belongs to A. Now, if you take a collection of these open balls see this U x r x obviously contains x, so suppose you take the collection of all these open balls for every x in X then that is open cover of x, that is the open cover of x. That is you take all such open balls U x or suffix x for x in X, this is an open cover of x, this is an open cover of x, an open cover of x and since we have assume that the x is compacted.

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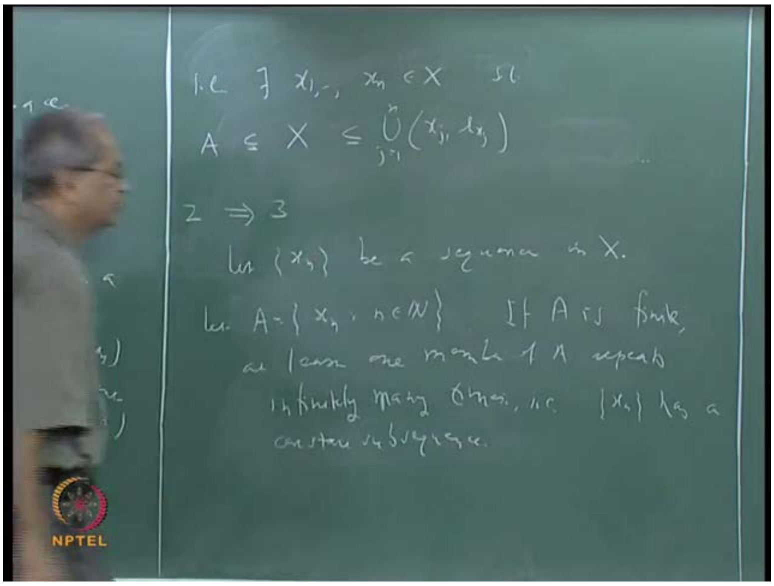

So, this must have a finite sub cover, this must have a finite sub cover, so is an open cover of x and since x is compacted this is usual, since x is compacted it has a finite sub cover I will remove this part, what does this mean this finite sub cover, this finite sub cover of this means what this should exist some end points. Let us say x 1, x 2, x n such that x is containing union of these n balls.

So, that is let us say that is there exist x 1, x 2, x n in x such that X is contain in union let us say x j and then r suffix x j, j going from 1 to n, and A is obviously a subset of X. So, A is also contain in this, so A is containing union of a finite number of balls like this, but each of this ball can contain just one point of A at the most and we started with saying that A is an infinite set.

So, that is a contradiction, there are only finite number of balls and we say that A is contained in their union, but each of those ball can contain at most one point of A. So, that will say that A must be finite and that is a contradiction, so that that proves that 1 implies 2. Let us, now attempt 2 implies 3 that means what we should assume that every infinite set has a limit point and from that we will try to prove that every sequence in x has a convergence sub sequence.

So, our assumption, now is 2 that is every infinite sub set of x has a limit point, let us say that let x and b a sequence, let x and b a sequence in x we have to show that this sequence has a convergent sub sequence. If we want use to namely that every infinite set has a limit point, we have start if we want to use that we have to start with some set not a sequence. But, does not matter every sequence also will lead to a set that is namely the range of that sequence, that is you consider the set of all these points x n, that is suppose let us suppose called this set as A, let A be equal to x n, n belonging to N. In other words, what we would have described usually because we know that every sequence is a function.

So, this nothing but what you can call range of that function you take the set of all those points which form the sequence there are, of course two possibilities that A may be finite or A may be infinite. If A is finite is it clear that at least one number, one member of A must repeat infinitely many times, in the sequence if the range of a sequence is finite at least one, of course more than one also may repeat. But, at least one will repeat infinitely many times, you take that particular sub sequence that is obviously convergent sub sequence. So, let us edit if A is finite, if A is finite at least one member of A repeats infinitely, at least one member of A repeats infinitely many times in other words we can extract a constant sub sequence out of x n.

You just take that wherever that, for example suppose x 1, x 3, x 7 they are all same you just take that particular sub sequence, so that is we can say that that is x n has a constant sub sequence. So, if A is finite there is no problem, if A is infinite what can we do then we use the set, we assume that every infinite subset must have a limit point. If A is infinite then A must have a limit point in x, in particular limit point is a point in the closure it is a limit point means its distance from A is 0, which means you can always find a sequence in A which converges to that point and we can choose that sequence in such a way that it is a sub sequence of this, so let us say that.

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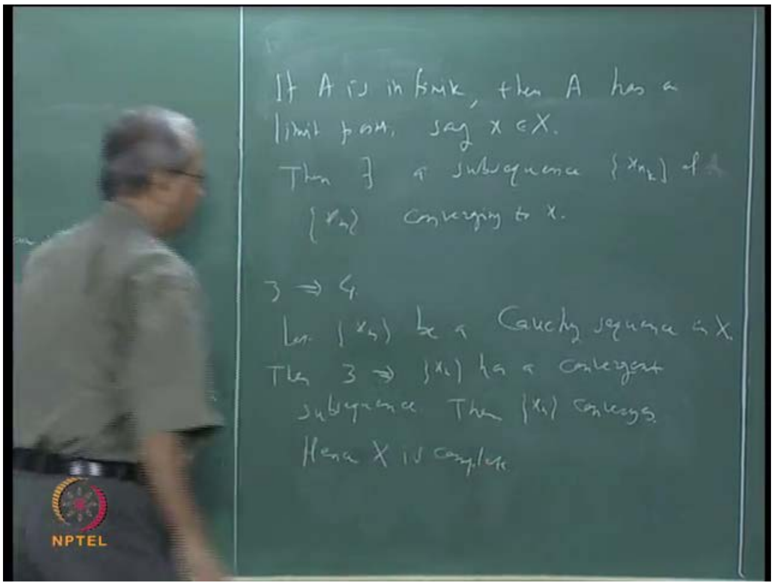

Let us complete this argument if A is infinite, if A is infinite then A has a limit point suppose I call at limit point x, say x, x belonging to X. Then since remember we have already shown that if you take any point in the closure then you offers closure of a set, then you can find the sequence of elements of that set convergent to that that point. So, if a limit point is in particular point in the closure, so we can always for example if you take any open set with x as the center then it should contain some point in A other than x itself.

So, suppose first you take a radius let us say 1 by n then you will find some element, here you will find some element, here such that the distance between x and that element is less than 1 by n. Suppose that element is, let us say some x 20 then you take the, you take let us say next number then you can choose next x and k which is where n k is bigger than that 20. Basically, what we need is to choose a sub sequence we need, we need this indexes n 1, n 2 etcetera strictly increasing, but that can be done that is not, that is not a big problem.

So, what I will be simply say is that then we can construct a sub sequence of x n converging to x then there exist sub sequence x n k of x n converging to x. So, we have proved in both the cases if A is finite for A is infinite in both cases x n has a convergent of sequence. So, we have proved 2 implies 3 that means or using this terminology Bolzano Weierstrass property implies sequential compactness. Now, let us go to this 3 implies 4 again remember what is that we need to prove that assume that every sequence has a convergent of sequence and using that we need to prove two things.

First is that x is complete and second thing is x is totally bounded, let us look at this first to show that x is complete what do need to prove, we need to prove that every Cauchy sequence is convergent, every Cauchy sequence is convergent. So, let us start with that, so let x n be a Cauchy sequence in x, x be a Cauchy sequence in x, but our assumption 3 is that every sequence has a convergence of sequence. So, in particular this Cauchy sequence also has a convergence of sequence, so let x n then so let us say 3 implies that x n has a convergence of sequence.

We have observed long ago that if a Cauchy sequence has a convergence of sequence then the sequence itself is convergent either we have proved it in the class or its one of the problems its. So, if a Cauchy sequence has a convergent sub sequence then the sequence original sequence itself converges, so then x n itself converges. So, this shows that X is complete, this shows that X is complete, so we have shown that every Cauchy sequence is convergent, hence X is complete.

So, what remains we need to show this last thing that x is totally bounded, x is totally bounded. Now, to show that x is totally bounded let us recall the definition what is mean by saying it is totally bounded that given any epsilon, there should, X we should be able to cover X by a finite number of balls with radius epsilon. So, suppose that is false, suppose that is false then what does it mean that if you take there should exist some epsilon such that you cannot cover X by a finite number of balls with radius epsilon, so let us, let us start with that.

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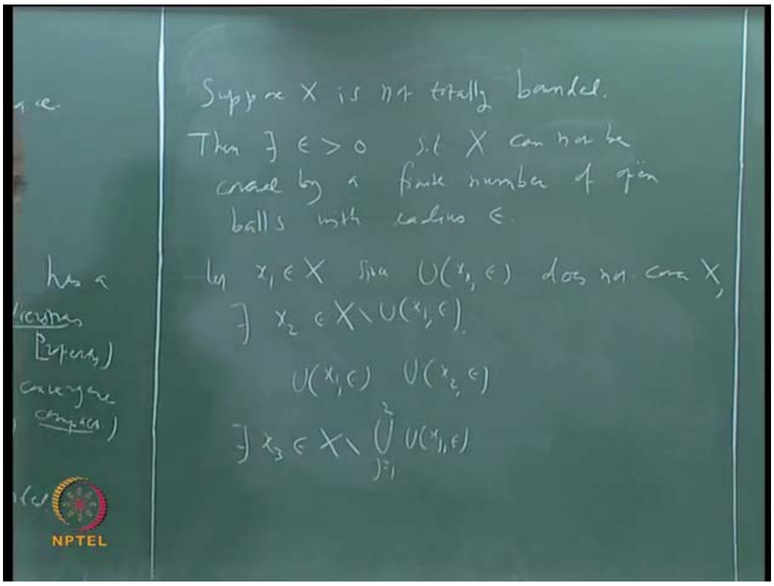

Suppose X is not totally bounded, not totally bounded then there exist epsilon, there exist epsilon such that whichever finite number finite family open for radius epsilon you take that will not cover X. Then there is epsilon such that X cannot be covered, let us say X cannot be covered by finite number of open balls open balls with radius epsilon. Now, we proceed as follows, using this we can construct a sequence which cannot have any convergence of sequence that is the idea, so how will we construct.

We start with any arbitrary element x 1, let x 1 belong to X, let x 1 belong to X then we know that this u x 1 epsilon cannot cover x, U x 1 epsilon cannot cover x. That means there should exist some x 2, there exist some x 2 which should be outside this, so we can let us just write this argument since U x 1 epsilon does not cover, x 3 exist x 2 in x minus U x 1 epsilon there is, there is should exist some x 2 which is outside this ball.

Now, what I do is that I will consider U x 1 epsilon and U x 2 epsilon that is consider these two balls, U x 1 epsilon and U x 2 epsilon these two balls also cannot cover X because our assumption is that X cannot be covered by a finite number of open balls. So, that means I can find some x 3, I can find some x 3 which is outside both of these, which is outside both of these.

So, again the say by the same argument what will be the argument here since U x 1 epsilon and U x 2 epsilon this is a finite number of open balls that does not cover X. so there is must exist x 2 sorry, there must exist x 3 which is outside union of these two balls. So, we can say let me just write it here, so there exist x 3 in x minus union let me say j going from 1 to 2 U x j epsilon, so we have conceded.

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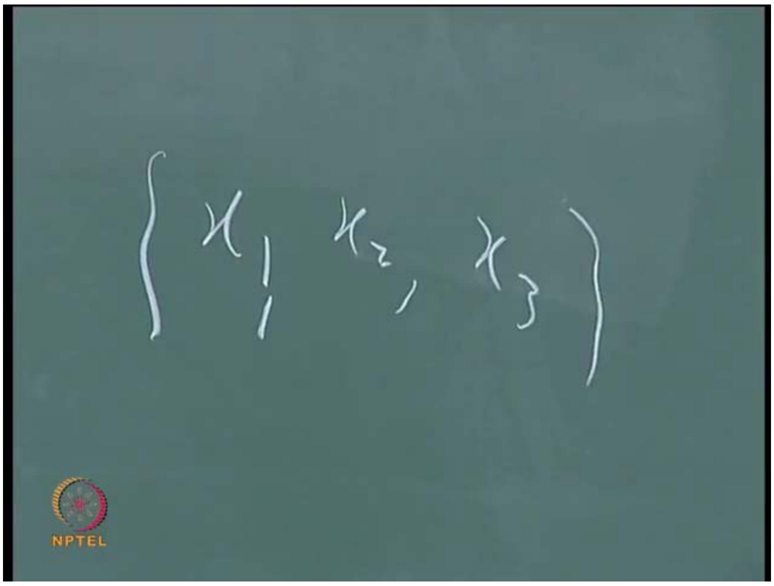

We have, now at this stage we have constructed the first three elements of the sequence x 1, x 2, x 3, we can construct it as similar way. But, before proceeding further just observe what is the property of these three elements x 1, x 2, x 3 it is that the distance between any two of them is bigger than or equal to epsilon because why distance between x 1 and x 2 is bigger than epsilon because x 2 is outside U x 1 epsilon and why distance between x 3. Either x 2 or x 1 is bigger than or equal to epsilon because x 3 belongs to neither to this ball nor to that ball, so that is important.

So, we can say proceeding in this way we can construct a sequence suppose we have constructed n such balls then union of those n balls also cannot cover x. So, there will be exist some x n plus 1 there will exist some x n plus which is outside union of those balls in this way by induction a sequence can be constructed.

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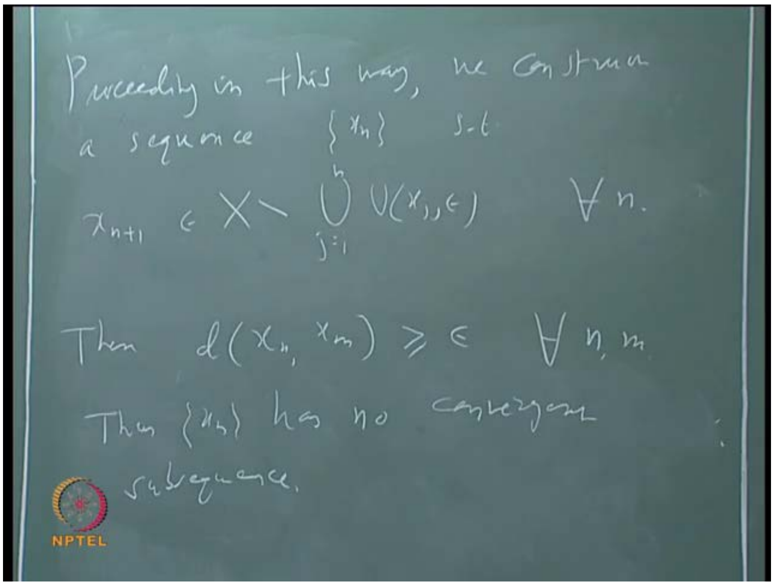

So, I will simply say proceeding in this way, so proceeding in this way we construct a sequence x n such that what is the property of the sequence, that x n plus 1 does not belong to any of those n balls with center at x j and radius epsilon for j going from 1 to n. So, let me just write the last thing such that x n plus 1 belongs to x minus union open balls with center at exact at epsilon j going from 1 to n and this happens for all n. So, what it means is that if you look at instead of x 1, x 2, x 3, now if you look at x 1, x 2, x n plus 1 then the distance between any two of those elements is bigger than or equal to epsilon. Since this is at each stage we have used this property that x is not, x is not covered by a finite number of open balls of with radius epsilon that is the property which we have used again and again.

So, finally what is the property of this sequence then it is the distance between any two elements of those sequence will be bigger than or equal to epsilon. So, I can say that then distance between x n and x m is bigger than or equal to epsilon for all n and m, for all n and m. Now, can such a sequence have a convergence subsequence obviously not because if you take some epsilon which is less than for example, suppose we take epsilon by 2. Then there will not contain there will not exist any n 0 that which will contain all elements of that sequence.

So, such a sequence will not have any convergence subsequence then this x n has no convergence subsequence, convergence subsequence and that contradicts this sequential compactness. We were trying to prove for using 3 our assumption was every sequence has a convergence subsequence and have shown that if x is not totally bounded. We can construct a sequence which has no convergence subsequence, so this must be false, so x must be totally bounded, so what is left. Now, so we have proved 1 implies 2, 2 implies 3 and 3 implies 4, so what is remaining is the last thing 4 implies 1.

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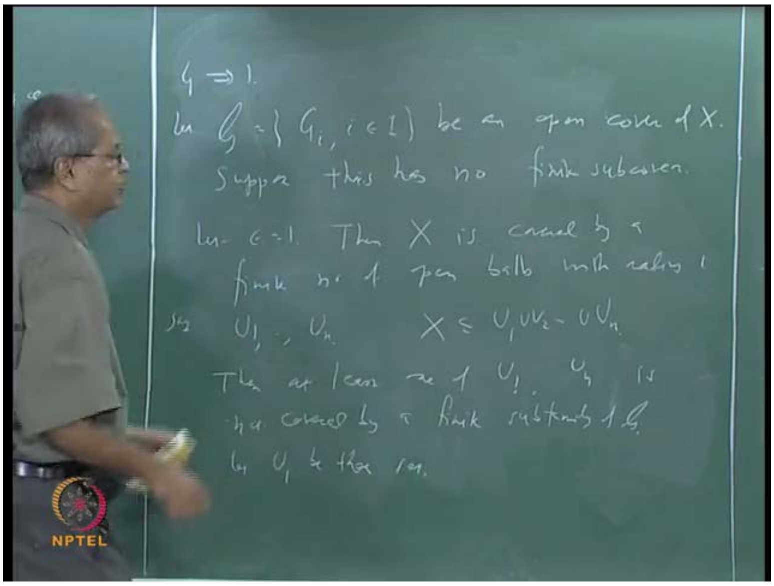

So, what is that I showed, now that X is complete and totally bounded using that we have to show that X is compact. Now, to show that x is compact let us proceed in the usual way how does one start showing that something is compact think of some open cover takes any open cover of x and show that it has a finite sub cover. So, suppose, so let us, let us let G i, i belonging to i be an open cover of x, now suppose this has no finite sub cover, suppose this has no finite sub cover than we will get a contraction to these things suppose this has no finite sub cover.

Now, to get a contradiction we have used this file that x is complete and it is also totally bounded I will use this part first, since X is totally bounded we know that if you take any epsilon X can be covered by a finite number of open balls with radius epsilon. This is true for any epsilon I will start with let us say epsilon is equal to 1, let epsilon is equal to 1 then X is covered, then x is covered by finite number of open balls, open balls with radius 1.

Suppose those balls are suppose I call those balls U 1, U 2, U n see that is each of this ball is let us say centre x 1 radius 1 than centre x 2 radius for the time being I do not bother about the centers what is important each of these balls is of radius 1. So, x is contained in the union of this finite number of balls, so x is contained in 2 1 union U 2 etcetera, union U n. Now, the next step is crucial suppose it is, so happens that each of these balls U 1, U 2, U n has a finite sub cover from this has a finites let us give some notation for this because this is something that we are going to use again and again.

Suppose I call this script G, that script G is of some family opens it which covers x, suppose you find a finite sub family which covers U 1 again some finites family which covers U 2 and some finites of family which covers U n. Than what will that mean suppose you take union of all those finite sub families you will again get a finites of family and that will cover X because is contained in union of this. But, we are started with the assumption that that is not true that this has no that that will mean that this cover as a finite sub cover, but we are started with the assumption that is not true.

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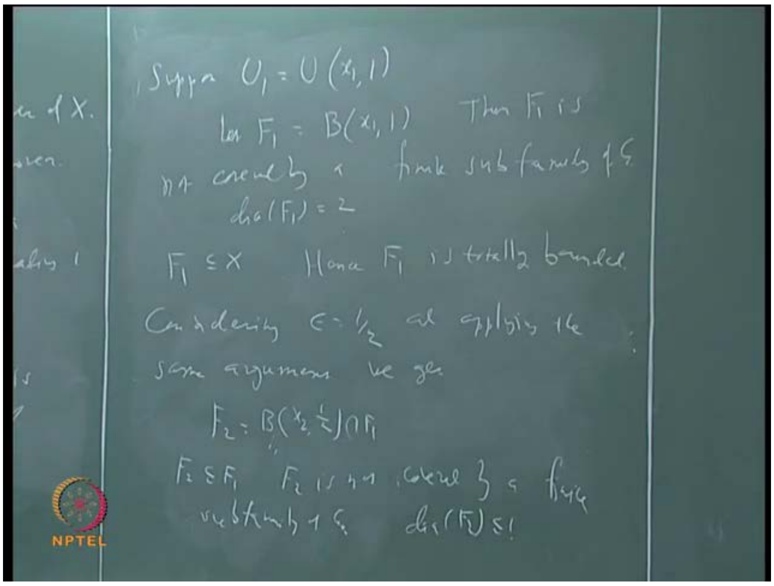

So, what must happen that at least one of these sets, at least one of these sets does not have a finite sub cover, at least one of these sets does not have a finite sub cover from this that is the first crucial step. Then at least, at least one of U 2, U n does not have or lets us say U 1, U 2, U n is not covered by a finite sub family of G of finite, I will just call that, I will just assume that U 1 is that set it does not matter can be U 3 also U 7 also.

Whichever is a set, that set I will call as U 1, so we can assume that, so let us say is not covered by, let us assume that that set is U 1, let U 1 be that set, let us again recall what is U n, U n is an open ball.

Let us say that suppose U 1 is nothing but u let us say x 1, 1there is an open ball with some centre and radius 1 and what is the property that it is not covered by a finites of family of this script G, finites of family of this script G. Now, what I will do is that I will take a close ball I will take a close ball with the same centre and the same radius. So, let F 1, let F 1 be the close ball with the centre at x 1 with radius 1, so that is then F 1 is also not covered, F 1 is also not covered by a finite sub family of G because F 1 contains U 1.

So, then F 1 is not covered by finite sub family of G, there is one property of F 1 what is the second property, I will also want you to note that diameter of F 1 is 2 diameter of F 1 is 2 the radius is 1. So, diameter is 2 there is one more property of F 1 namely that F 1 is contained in x F 1 is a subset of everything is happening inside x and x is totally bounded and we observed yesterday that every subset of a totally bounded subset is again totally bounded, where subset of a totally bounded subset is is again totally bounded, so F 1 is also totally bounded F 1, F 1 is contain in x.

So, F 1 is totally bounded hence F 1 is totally bounded, is totally bounded, so next what I will do is instead of taking epsilon equal to 1. Now, I will consider epsilon is equal to half there is, now I will consider open balls with radius half open balls, with radius half and whatever I did for x and x 1 the same argument I will now do for F 1 and half F 1 is not totally bounded. So, it cannot be covered by a finite number of sorry, F 1, F 1 is totally F 1 is totally bounded, so it can be covered by a finite numbers of open balls with radius half, finite numbers of open balls with radius half.

Now, each of those open balls with radius half if each of them had a finite sub cover from G that it will mean that F 1 also will have a finite sub cover from G. But, that is false that means there exist at least one ball with radius of which has no finite sub cover from G, which has no finite sub cover from G and that open ball will now consider F 1 itself as a metric space. I will consider F 1 itself as a metric space I will not consider open balls with which in x, but open balls inside F 1 open ball that means I will take elements only from F 1, I will take elements only from F 1.

So, suppose we do that then I will get a next subset F 2 there is a F 2 will be contained in F 1, F 2 will be contained in F 1. What is the property of F 2, F 2 does not have a finite sub cover from G and diameter of F 2 will be less not equal to diameter of F 2 will be half of because it is a radius, it will be half of this diameter. So, let us say so we can say that considering epsilon is equal to half, considering epsilon is equal to half and applying the same, argument applying the same argument we get F 2, F 2 will be a close ball with. Let us say center at x 2 and radius half which is of course this ball is not a ball in x, but it is, it is a ball in F 1. If you want to emphasize that, we can say or you can say that it is a ball in x 2 intersection F 1 it is a ball even that is also fine it does not matter.

This intersection F 1 which every you do the point is that this F 2 have the property that F 2 is there are three properties, first is F 2 is closed, F 2 is contained in F 1 and if F 2 is not covered by a finite, F 2 is not covered by a finite sub family of G, finite sub family of G. Diameter of F 2 is 1, 1 or less than or equal to 1, diameter of F 2 is less than or equal to 1 and F 2 is also totally bounded F 2 is also totally bounded, now I hope it is clear to you how to proceed. Now, next stage I will take radiuses 1 by 4, next stage I will take radius 1 by 4 and apply the same argument to F 2.

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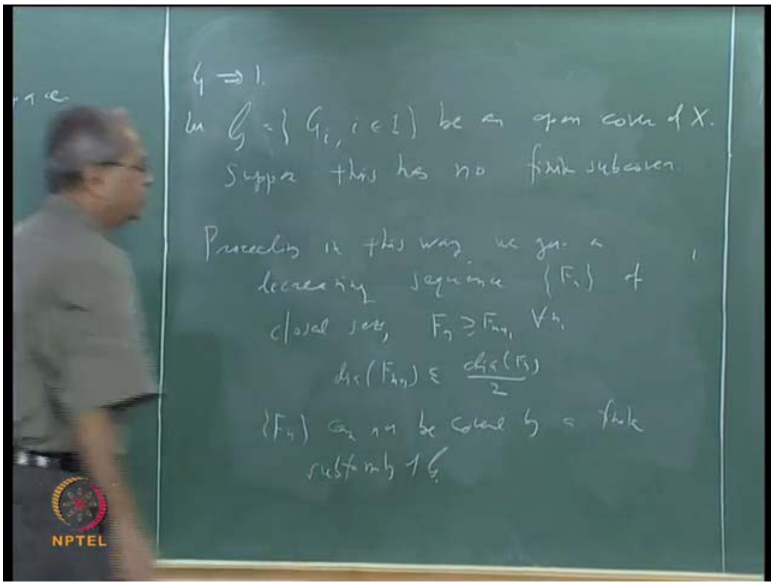

So, suppose I do that what I will get, I will get the next F 3 that is contained in F 2, that is contained in F 2 and if 3 is also not covered by any finite sub family of this set of this cover G. So, suppose we proceed in this manner what is going to be the ultimate result you will get a decreasing sequence of closed sets, you will get a decreasing sequence of closed sets with each of F n has all those properties each of F n has all those properties each F n has all those properties, so let me just continue, write this requirement we simple say that proceeding in this way we get a decreasing sequence of closed sets.

Proceeding in this way, we get a deceasing sequence let us say F n decreasing sequence, F n of closed sets. Let us again say decreasing means what that F n contains F n plus 1, F n contains F n plus 1 for all n then diameter of F n plus 1 is less not equal to diameter F n by 2. We can say exactly how by some 2, 2 the 1 by 2 to the power n minus 1 or whatever that is not very important, so diameter of F plus 1 is less not equal to diameter F n by 2 this I believe. If you continue like that you will get this whole thing less not equal to 1 by 2 to the power n 1 by 2 to the because diameter of F 1 was 2.

So, diameter of F 2 is less not equal to 1 etcetera, so in general you will get diameter of F n plus 1 is less not equal to 1 by 2 to the power n. Most important none of these F ns can be covered by a finite sub family of G and F n cannot be covered by a finite sub family of G, for this script G. Now, we have all the ingredients required to complete the proof, remember once again we have assumed that, we have assumed 4.

So, till now we have been using the total boundedness of x, we have not yet used the completeness, we assume that x is complete and totally bounded and in that, remember we have proved Cantor's theorem for a complete metric space. What do we prove that if we take any decreasing family of closed sets such that if the diameter of F n tends to 0 then its intersection consist consists of just a single turned point. So, we can say that the Cantor's intersection theorem intersection of F n consist of just single point.

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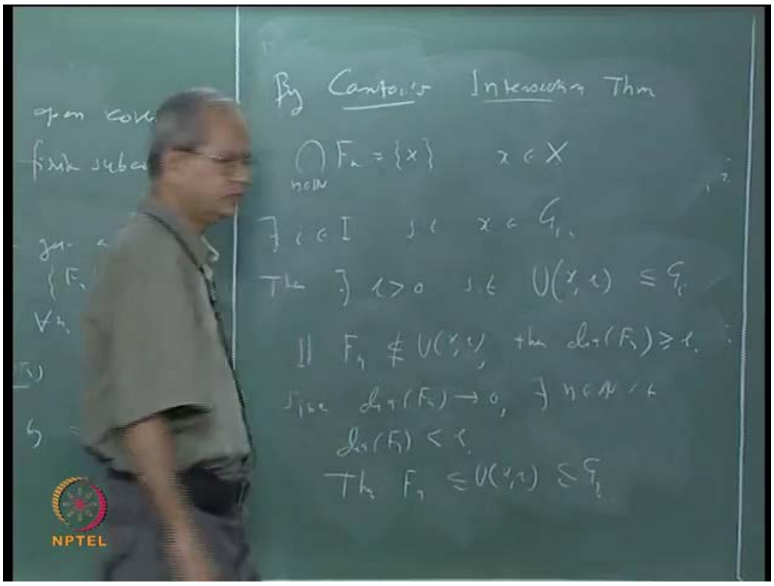

So, by Cantor's intersection theorem, remember it is important, here to know that X is complete because there we can only use Cantor's intersection theorem. It is not truly incomplete spaces so by Cantor's intersection theorem intersection of all this F n is just a single turn set X. Now, you will guess how we will complete the proof, now this x is some element in x and we have started with assuming that this is a cover of x. So, x must belongs to one of this sets, here x must belongs to one of this sets here, so we can say that suppose like all that sets some G n.

So, there exist we have to say that there exist i in i, i in I such that x belongs to G i, but this is an open set this is an open set. So, if x belongs to G i there should exist some r such that open ball with center at x and radius r must be inside G i, so then there exist r bigger than 0 such that open ball with centre at x and radius r that is contained in G i. Now, look at the relationship, now remember that x belongs to each of these F n x belongs to each of these F n. So, if there are two possibilities, if F n is contained inside this remember x belongs to each other.

So, intersection of U and F n should at least contain X, there are two possibilities either F n may not be inside U x r or F n may be some elements of F n may be outside the U x r. But, if some elements of F n are outside U x r then diameter of F n, diameter of F n must be at least see remember let us just draw the picture, see this is x and this the radius R, this is the radius R. Now, suppose you take some set F n, some set F n, F n contains x, F n contains x and suppose F n contains some point outside U x r also then is it clear that diameter of F n must be bigger than or equal to r by 2.

There is because here there are, now bigger than or equal to r because there is some point distance between this point and this point is bigger than or equal to r. So, if F n is not inside let us say if is this argument clear if F n does not is not contained in U x r, if F n is not contained in U x r then diameter of F n must be bigger than or equal to r y. Let me again repeat because x belongs to F n and if x contains, and if F n contains some point outside this ball then it means that it contains 2 points distance between whom is bigger than or equal to 1.

Now, can this happen for every F n, we know that diameter of F n tends to 0, we know that diameter of F n tends to 0. So, we can always find some large value n such that diameter of F n is strictly less than R, but if that is the case then that F n must be completely contained inside this. So, we can say that since diameter of F n tends to 0 we can always find some F n, so we can say there exists in fact we can say more there exists some n 0 such that for all n bigger than or equal to n, n 0 diameter of F n should be less than r.

But, that is not very important it is a decreasing family if diameter is less than r for some F n, for all n bigger than or equal to that it will be less than that because it is a decreasing family. So, let me simply observe that there exist n in n such that diameter of F n is strictly less than r, but if that is the case that F n must be a sub set of U x, then F n is contained in U x r. Remember this U x r is containing G 1, U x r is containing G 1 can this happen because we observed it no F n can be covered by a finite sub family of script G.

Whereas, we have showed that for large values of n, if F is contained in just one set, if F is contained in just one set from here, so that is a contradiction. So, that proves that if X is complete and totally bounded then x must be compact it that clear, so that proves the equivalence of these conditions. We have already seen that theorem implies in particular Heine Boul theorem, namely that in a real line a set is compact if and only if it is closed and bounded. It also implies Bolzano Weierstrass theorem namely that every bounded sequence of real numbers has convergence sub sequence. We will stop in that for today.