Real Analysis
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Lecture - 03
Finite and Infinite Sets

Well we continue our discussion about the relations that we started yesterday, and let me remind in particular we discussed 2 special types of relations. Namely equivalence relations that 1 type and second was partial order, and let us discuss a few more things about the partial order.

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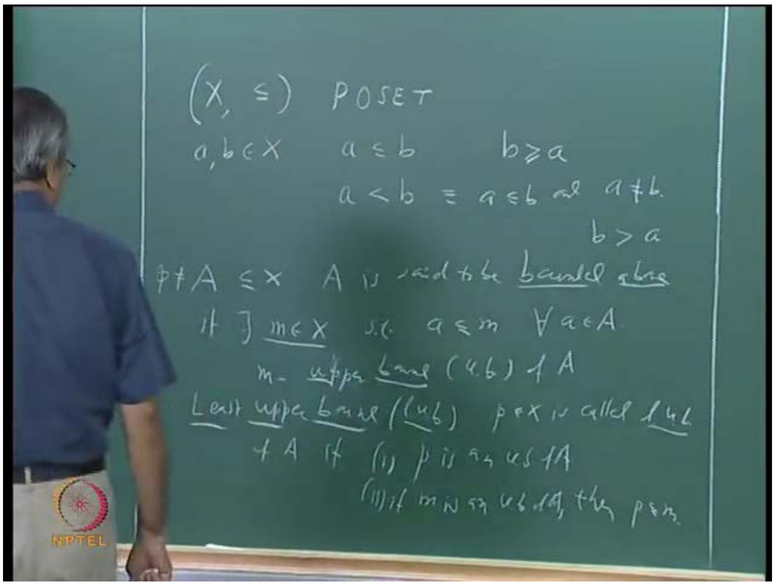

Let say that x is a set and less than or equal to is partial order 1 x. So, this is partially order set by then let me emphasize once again that though, we have using this notation less than or equal to it does not mean it is the usual less than or equal to order among the numbers. It is any partial order this is just a notation and this is something which is fairly common in this study of order of relation. Usual partial order is denoted by this simple and the partial order about the numbers the usual less than or equals to order is once special case of this partial order.

We shall also use the usual notations, which we use in case of the usual partial order that you know. For example, suppose we have 2 elements a b in x let us with a less than or equal to b we will also this is same we can also denote the this as by this b bigger than or equal to a. So, this will be just the thing, but reverse by one it basically means the same thing just a different notation. Similarly, a strictly less than b by this will be a less than or equal to b and a not equal to b a not equal to b. And similarly, and this will be we can let out this also by this also by this b bigger than a, if there is some m in x such that a is less than or equal to m for every a in A.

And then this element m that is called upper bound upper bound will use standard perform u, b upper b remember. Of course, in general the set may be not bonded above in general the set may be not bounded above which is bounded should have upper bound. Also notice here m it is not said that m belongs to a that upper bound did not be an element of a it is an any element of x, upper bound did not be a.

Second thing is that a upper bound did not be unique for example, suppose it takes element in x which is bigger than or equal to this m, then that will also satisfy this because of the transitive relation. Suppose let us say we have say m say n is less than or equal to n, then since a is less than or equal so any other n which is bigger than or equal to m will also be an upper bound. So, upper bound is not unique, so we will also discuss other similar concept what is called least upper bound, it is called l u b is the standard short form least upper bound means, what of course, the first level should be in upper bound because it should be smaller than any other upper bound right that the thing. So, least upper bound is so let me see at number.

Suppose, let us say I will call that number p not number any element p in x is called this least upper bound let us say l u b of a, if two things first thing is that p is an upper bound. What this mean? It means a less than or equal to p for every small a in that is the first thing it satisfy this property. Second thing is the if you take any upper bound for example, an m if m is an upper bound of a then m then p is less than or equal to m, that is ((Refer Time: 04:57)) second property is this if m is an upper bound of a, then p is less than or equal to m.

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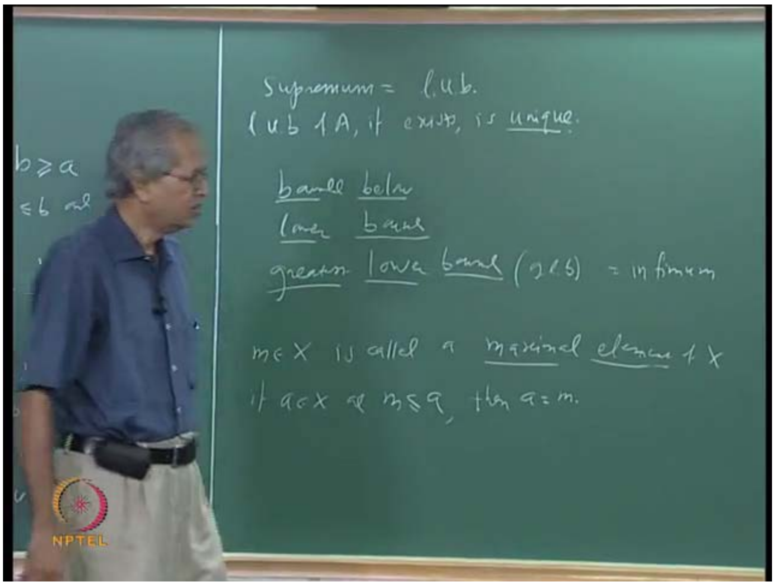

This least upper bound is also called as supremum just the same concept, but different let us say l u b and we shall be using the both terms quite frequently. Now, is it clear to you that in general an upper bound way will be not unique, but least upper bound will be unique always that clear for p and q are 2 least separate bounds then by this second property. Since, p is the least upper bound and q is the upper bound you must p less than or equal to q.

Similarly, reverse the routes so q less than or equal to p and by the anti symmetry property p must be equal to q. So, even though upper bound may or may not be unique least upper bound be always be unique. Of course, remember that non of this exist what all that we are saying this is if least upper bound exist it must be unique. So, let just mention that least upper bound of the if exist is unique.

In a similar way we can define what is meant by saying that is bounded below, what is meant by lower bound, what is meant by greatest lower bound. In the similar way we can show that in general lower bound may not be unique, but greatest lower bound will always be unique let us, since all those things are similar to whatever we have discussed in this case we shall not go into the detailed discussion around here. So, that means what is that I am talking about first a fall what is meant by bounded below, lower bound greatest lower bound greatest lower bound. And again just at this l u b will be the standard short form of least upper bound g l b is the standard short form of the greatest lower bound.

And corresponding to this supermum greatest lower bound is also called as infimum, and in a similar way you can show that if infimum exist it must be unique. Now, let us also discuss one more concept is very similar to this upper bound etcetera, it is called maximum element its slightly different from upper bound, let me talk maximum element in x, let us say that m belonging to x is called maximal element. It means that is nothing bigger than this m in x, one other way of writing this x if a belongs to x and m less than or equal to a. That is if at all nothing is bigger than or equal to m then it must coincide with them, then a equal to m.

Now, try to understand that there is a difference between maximal element and an upper bound, in a upper bound. For example, suppose n is an upper bound of x what will that mean it will mean that a is less than or equal to m for every a this is not what we are saying for maximal element, we are not saying something like that. We are not saying that a is less than or equal to m, for every m all that we are saying m is less than or equal to a then a must coincide with that that is that is nothing strictly bigger than m in x that all. That is called maximal element i think if we take examples it will be clear let me recall this example.

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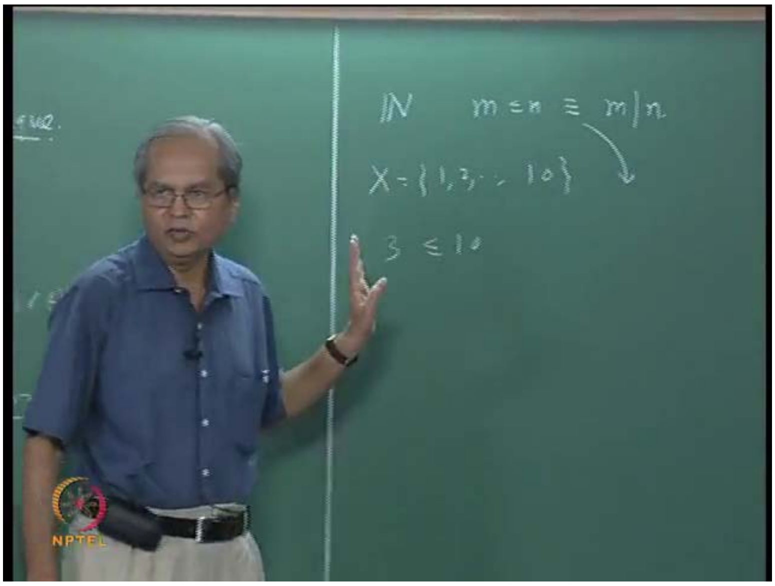

We are taken the example of this of natural numbers, and in this we define the order m let me use the same location again m less than, or equal to n. This means that not in the usual sense of the less than or equal to this m divides n. This means we had seen this is the partial order this is the partial order, now this time whatever you will do is that instead of taking the whole set n, I will take only let say this side of end. Suppose I take as x 1 2 let us say up to 10, and with this order with this special order, take first natural numbers consider the order is less than or equal to m divides n. Does this set have any any maximal elements? 10 is the maximal element is 10 an upper bound?

Student: ((Refer Time: 11:40))

10 is upper bound of x think, think carefully, what is the definition of upper bound? For the upper what we require that every element here must less than or equal to 10. For example, suppose I take the element 3 is 3 true.

Student: ((Refer Time: 12:00))

Is this true that means 10 is not an upper bound 10 is not an upper bound this clear, but 10 is a maximal element. So, there is a difference between maximal element upper element and this happen, when it is a partial order because certain elements do not be comparable at all. For an upper bound we require it should be bigger than or equal to every element in set, that it must be comparable with every other element and bigger than or equal to, but for the maximal element we are not saying that, but is the then only the maximal element are there any maximal element, what about 9?

Student: ((Refer Time: 12:41))

Are you saying that is this true? Are you saying this that is not true 10 is not bigger than right 10 is not bigger than 9 in this order so is that any element which is greater than 9 it in this set x.

Student: ((Refer Time: 13:07))

So, that does method 9 is a maximal element 9 is a maximal element is that clear 9 is maximal element, what about 8? It is an maximal element right now do you understand the difference between the maximal element and the upper bound. Now, this should be very clear on the mind because this concept are important right, this is because this is not an total order in this x we have two elements, which will be not comparable to each other at all. And if that is the case if the elements are not comparable then there will no upper bound.

Student: ((Refer Time: 13:53))

Yes we are resolvable in fact you can list all the elements we already see 8, 9, 10 all of them are maximal element you can verify the maximal elements you can verify the 6, 7 this also the maximal elements, what about 5?

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Let us take one more example of this type.

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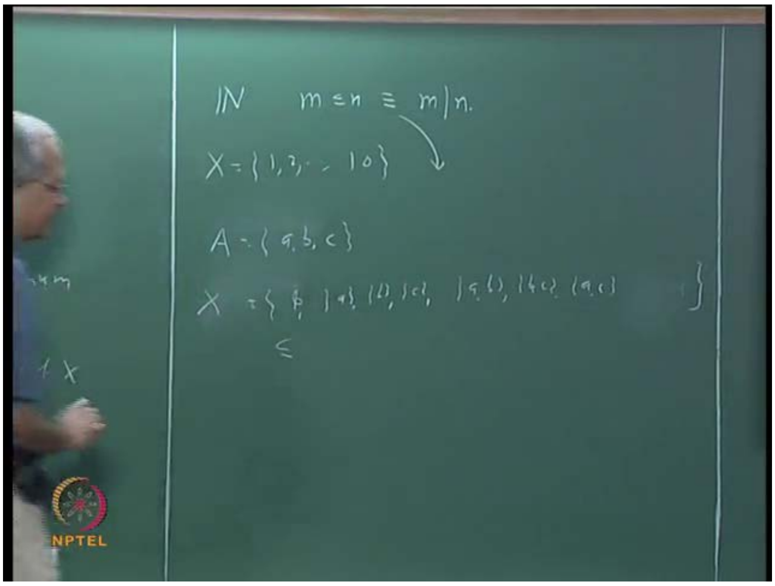

Let us start with set which contains 3 elements and will consider first this part set of this, how many elements this power elements will have?

Student: ((Refer Time: 14:47))

So, what are those 8 elements there are eight elements we can list and then these 3 set a b b c a c and then the last set a b c, which have we are accounted for all. What I will do them instead of taking the hole set we already seen this is a partially ordered set, with step inclusion. That this is the partial order on these set now instead of this I will take this set x as I will remove this last set from here, now is that clear that let me first before going to let me just go back to that this. Does this set have a upper bound to this order and what is that? This is upper bound because this set is even any element it is less than or equal to this, that is an upper bound and what I planned is precisely remove the element. That is I will just remove this element from here it is this is not 2 to the power this is x with respect to the same order.

Now, does this have an upper bound?

Student: ((Refer Time: 16:20))

Which is an upper bound only on the set, I am not considering anything outside this right fine this hole a b c upper bound for this, but I am not considering let us say elements outside this. So, inside this elements there is no upper bound. Does it have a maximal element?

Student: ((Refer Time: 17:00)) No.

Let us look at this take at this set a, b is there any set in this which contain in this a, b other than this right no other set is bigger than this right, this is an maximal element right. And the same thing can be said about b c etcetera all these 3 are maximal elements what about this they are not maximal elements. So, is this clear to now the difference between upper bound and maximal element. See usually upper bound to the set a itself then it is called maximal element, right? If an upper bound lies inside the set itself then it is called maximal element of a, but there is a difference between maximal element, and there is a difference between maximal element and maximum element. It should be infinite this the first thing that should be very clear to you when in your mind.

Then we will go in the since the discussion will be very parallel to what we have discussed about the maximal element, I shall not go into the details of that in a similar way you can define, what is meant by minimal element, what will be the minimal element will say that I will say instead of this m, let us say l is will be called minimal element if instead on this m.

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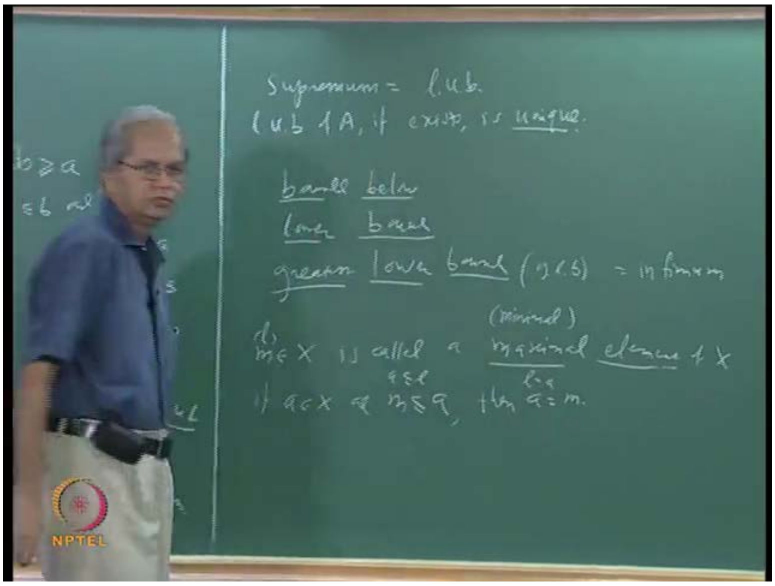

Let us not equal to we will have a is not equal to m then that should imply l is equal to a then is there is anything less than given element, then that should element should co-inside this that is a that is the definition of minimal element and minimal element is not same as lower bound it is not same as list element. And similarly, you can construct examples of that I shall not going to detail the discussion of that because that is very much similar to whatever we have discussed about the maximal element. Now, we discuss one very important question, when does a partially order set have a maximal element, and answer to that is given by a very well known principle of set theory it is called…

Zorn's lemma and this is lemma, which will be used in many proofs of very well known theorems, but anyway before going to that let us first see what exactly this lemma says. This says when does a partial order said have a maximal element, so let us say that let x be a partially ordered set. So, if every change in x, if every change in x as an upper bound remember chain is totally ordered set, suppose you take a totally order sub set of x. So, suppose every sets totally ordered sub set as an upper bound, suppose you take every then x has a maximal element.

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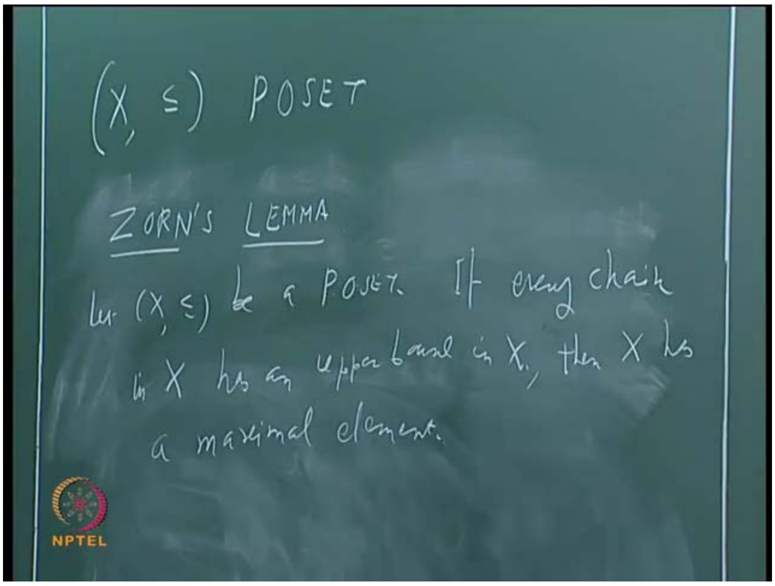

We shall not going to the proof of this ((Refer Time: 21:26)) the part that is because it is actually one of the axioms of set theory, we do not prove it. In my first lecture I told you about action of choice and action of choice, one version of that is that if the you take a family of non empty sets then the product is also non empty. Another of course, begin also restated it new by making use of the concept of choice function etcetera, it can be shown that the action of choice and Zorn's Lemma are equivalent.

That means if you assure action of choice you can Zorn's Lemma and conversely if you assure Zorn's Lemma you can through action of choice, but we shall not discuss any of this proves that is because this is not the course on set theory, where those kind of things will take lot of time. We are just basically removing certain things in set theory we shall we shall require in the real analysis course, and let me again say that this Zorn's Lemma is used in many proves of the standard theorems not only in analysis, but in many other courses.

For example, in linear algebra in order to prove that every vector space has a basis you have to make use of this Zorn's Lemma, whenever you learn that proof you will say that your required Zorn’s Lemma. Now, let go to the next term topic we will I will suspend the discussion about the partial order of for the time being. We now want to say something about the number of elements in a set, and how that is how one goes about it so again.

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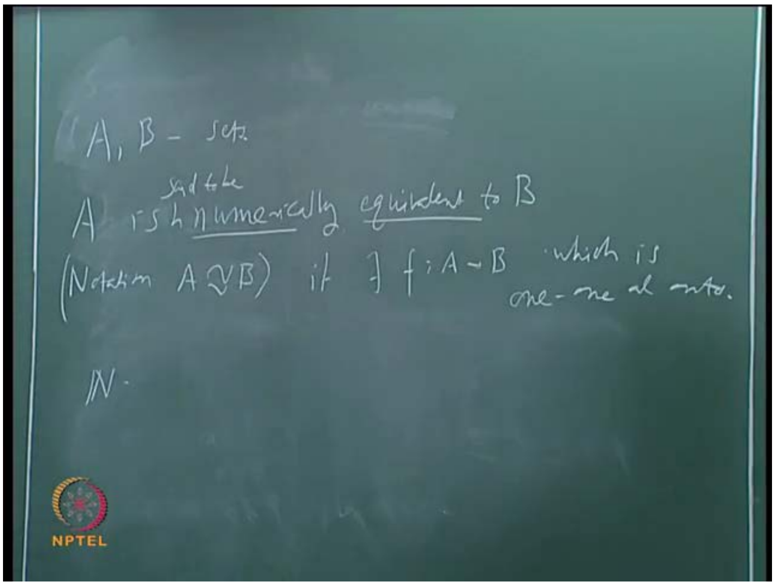

Suppose, let us say that A and B are two sets empty or non empty do not bother about it right now, then we define what is define by sail that A is numerically equivalent to B. And we shall use this notation for that notation, we shall use this notation A is numerically equivalent to B, if there exist a bisection between A and B that is this should exist a map from A to B which is 1 1 and 1 2.

So, let us say that A is said to be numerically equivalent to B, this is the notation if there exists f from A to B, which either you can say 1 1 or 1 2 or a bisection. Basically this notation which says that A and B have the same number of elements, that is the meaning of saying that A is numerically equivalent to B. Now, let us take this set for natural numbers, let me also remind you we has we have define what is meant by a function.

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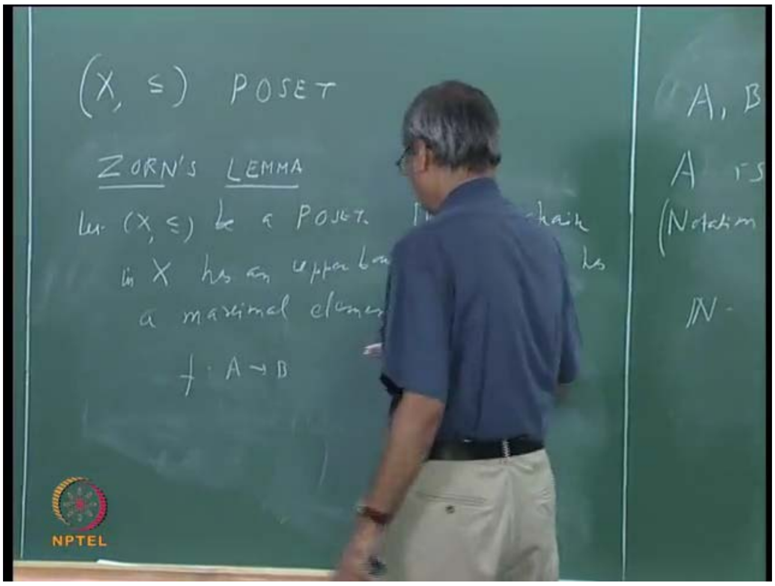

A function in general a function is a rule, which takes suppose you are taking to sit function a rule which takes from A to B, A is called a domain and B is called the co-domain. If this domain is the set of all natural numbers, then that function is called a sequence right. Sequence is nothing but a function whose domain is the set of all natural numbers, then you set B can be anything, it can be real numbers rational numbers complex numbers any other objects matrixes.

If it is a matrixes to be sequence of matrixes if it is function it will be sequence of functions sequence of real numbers etcetera. This is concept of sequence which we shall discuss in more detail little later. For example, when you take a sequence of real numbers what is meant by a convergent sequence etcetera, but here all the time the a require you to know right now is that if you have a sequence…

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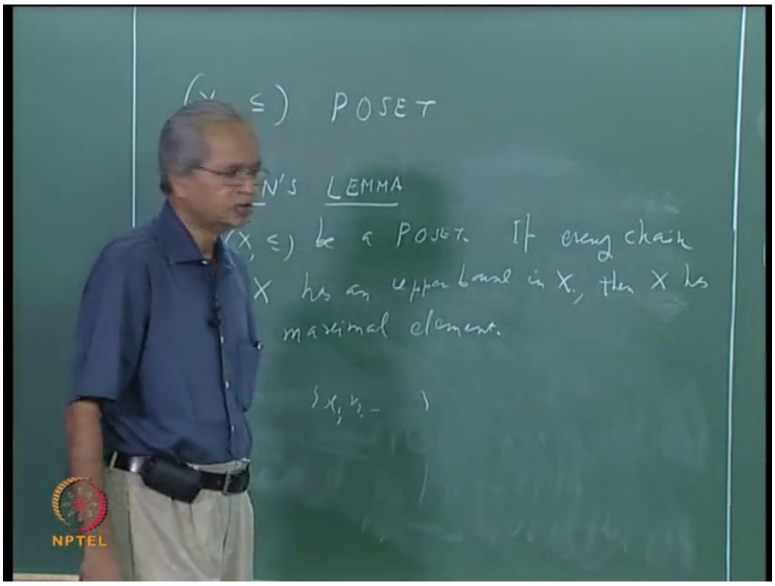

Then you can arrange the elements of the sequence like this x 1 x 2 etcetera right, that is you can label the elements.

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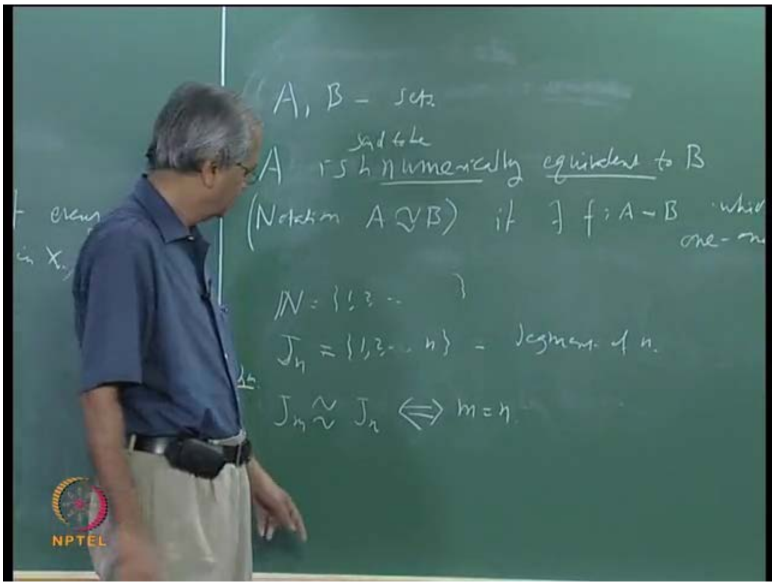

The sequences x 1 that is x 1 thing, but f of 1 x 2 is the thing, but f of 2 etcetera. Now, coming back to this numerical sequence. Suppose, this is the set N that is the set of all natural numbers, suppose we take only the first n natural numbers the we take the first n natural number, this will denote that say we will give some name for this suppose I will call it j is suffix n that is 1 2 etcetera up to n this is called segment of n.

Before proceeding further let us make one very obvious observation obvious sense certain sense. Suppose, I take 2 such elements J m and J n, J m and J n when will this happen J m is numerical equivalent to n if and only m is equal to n. So, that is the first observation if and only m is equal to n, let me just write this as a theorem. Again I will not going to the discussion the proof of this proof is fairly easy. In fact if m is equal to n there is nothing to prove it is identity math, you can take J into J this identity map that will be 1 1 .

But only thing is that if J m is cumulatively equal to J m then proving that m is equal to m that will require some work, but it is also not very difficult for another way of showing the same thing is if m is different from n, then there can be no bijection between say J m and J that is little easy to show. But that will also take some time, but I suggest that you try to do it on your own. If there is any difficulty then we shall discuss that in the class. Now, then let us go to next concept.

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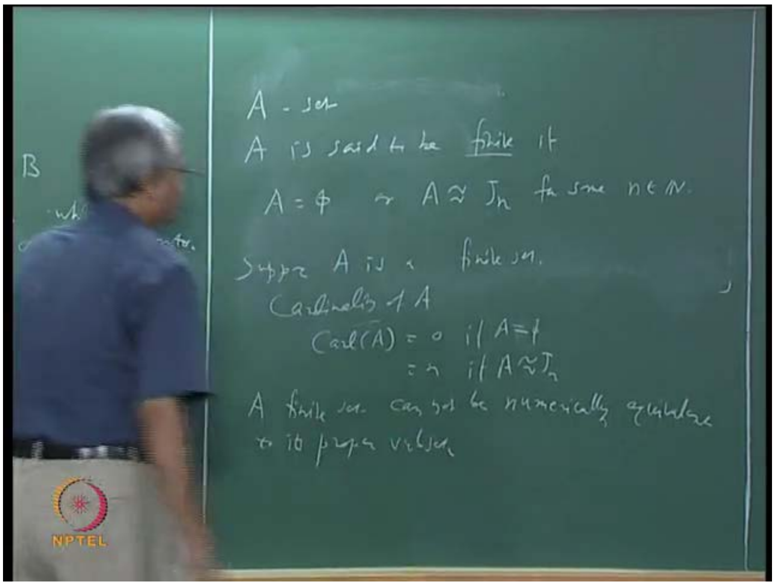

Suppose, we take as any set A as any set then A is said to be finite, if A is empty or A is numerically equivalent to one of this elements segment, if A is numerically equivalent to one of this segment, if A is equivalent to say n some. Now, for such finite sets suppose A is a finite set, will define what is meant by cardinality of A and instead of writing this full cardinality, we shall use this short notation cardinality of card A. This is this we shall define as this number n, if it is non empty if it is empty we will define it to 0.

So, cardinality of A to be defined as 0, if A is the empty set is equal to n, if A is numerically equivalent to j and yes.

Student: ((Refer Time: 30:26))

A is equal to empty set thank you, if A is equal to empty set and it is n if A is numerically equivalent to J. And can A be numerically given to different show segments is it possible that a is numerically product of J 4, J 7 no right. So, that means this come for n will be unique if once A is of finite in fact that is what is this observation, that is what is this observation J m is numerically equivalent that will happen only if and only m is equal to n.

So, given a finite set it can be numerically equivalent to only one of this J n exactly one of this J n or else it should be empty or else, all right? And what we have called is cardinality of A that is rough is rough not roughly it basically missed the number of elements in the set cardinality of A is nothing but the number of elements in the set, what is definition? It is that number n such that A is such that there is 1, 1 correspondence between A and J. One thing follows from this and that is the following if A is finite set and suppose you take any proper sub set of A then there can be no bijection between those two sets. So, why this will happen? Because if A is a if you take any set any sub set A of B then B also will be numerically for one of these J m's.

And then or one of this J m's and then again we use this then there is no bijection is m is different from n. So, we can say that finite set cannot be numerically equivalent to any proper sub set of itself all right. Of course, one can first show it here infinite follows from here if you take proper sub set of this it will be numerically connect to some J m, m where m is different from n. And hence there can be no bisection between those two sets, so let us just make this observation of finite set cannot be numerically equivalent to, a finite set cannot be numerically equivalent to its proper sub set. And once we defined what is meant by finite set, the definition of infinite set is clear right, whatever is not finite is infinite. So, will set A is of called the infinite set if it is not a finite set.

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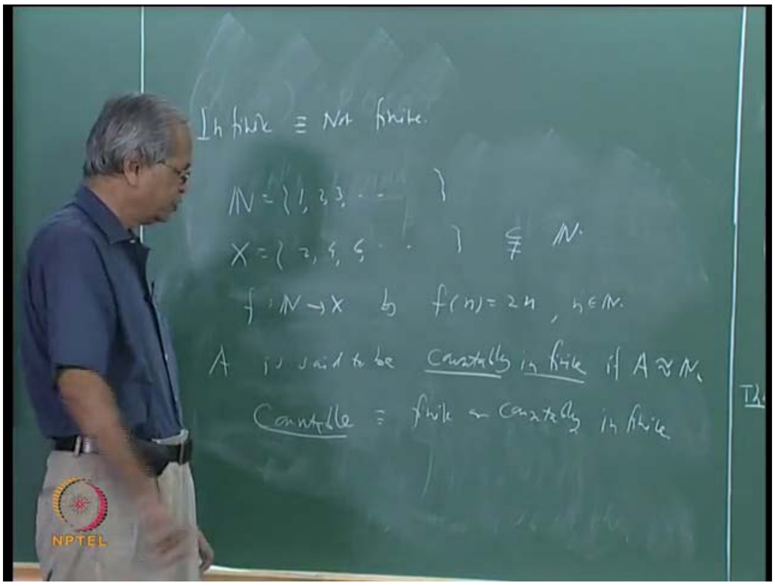

So, infinite this means not finite, what does it mean? First of all it means that it is not empty, so and it is not there is no 1 1 identity map between A and any of these segments, for no n A is numerically equivalent to J that is meaning of same it is not a finite set it is an infinite set. And when obvious example of an infinite set is this set of all natural numbers it is easy to show that this is not numerically equivalent to any of these because with the any sets segment, you will obviously find one element which is outside.

You will see that this property is not true, for infinite sets in case of infinite sets you can easily find sub sets which are numerically equivalent to the given sets and one can very easily find such subject. But just for the sake of completeness we will just take one example like this let that let us say that take the set N 1, 2, 3 etcetera and will x yeah fine we will take another sets suppose x is 2 4 6 etcetera, etcetera. So, this is the proper sub set of x right sorry proper sub set of N, and what is a ((Refer Time: 35:50)) map between an index. So, we just define f from let us say N to x by f of N is equal to 2 for, then f is 1 1 and 1 2.

Now, the next obvious question should such a thing happen for every infinite set, here we some example N is a an infinite set. And N has a proper sub set which is numerically equivalent to N. Now, the obvious question is suppose you are not necessarily N, but any infinite set will it always contain a proper sub set, which is numerically equivalent to itself. The answer is yes, it is not obvious it will require some work. So, let us now proceed to do that work.

And to do that will need one more definition and let one research, among the infinite sets we make some ((Refer Time: 37:03)) sub division, we say that say it is count ably infinite if it is numerically equivalent to N. So, we will say that this definition let us say A is said to be count ably infinite, if A is numerically equivalent to N. That means, there is a 1 1 and 1 2 map between A and the set of all natural numbers.

For example, this set x 2 4 6 etcetera this is the count ably infinite set and we will also use another term what is for countable, countable means finite or countable infinite. In fact there is a reason for this word countable as the word suggests it means, that elements in the set can be counted, what is the meaning of countable? Finite means its cumulatively equal if it is empty set we can forget about if it is non empty, it is either numerically equal to one of these N or to the whole of N, if it is equal to one of these N I can write element it has N elements I can write those elements as a 1, a 2 etcetera.

If it is numerically equivalent to N we can still write the elements as a 1 a 2 etcetera, but only thing is there will no last element. So, when the set is countable its elements can be arranged in certain order first, we can talk of what is first element, what is second element. And they can be counted in that order that is why it is called countable, so if a using language of a sequence if a set is countable you can give the elements of the set A as a sequence as a range of a sequence, if it is count ably infinite if it is finite then you can you can just write this elements as a 1, a 2, a n etcetera. Then before proceeding further let me also say something about this terminology. Unfortunately this terminology is not followed in all books certain author has used some other terms.

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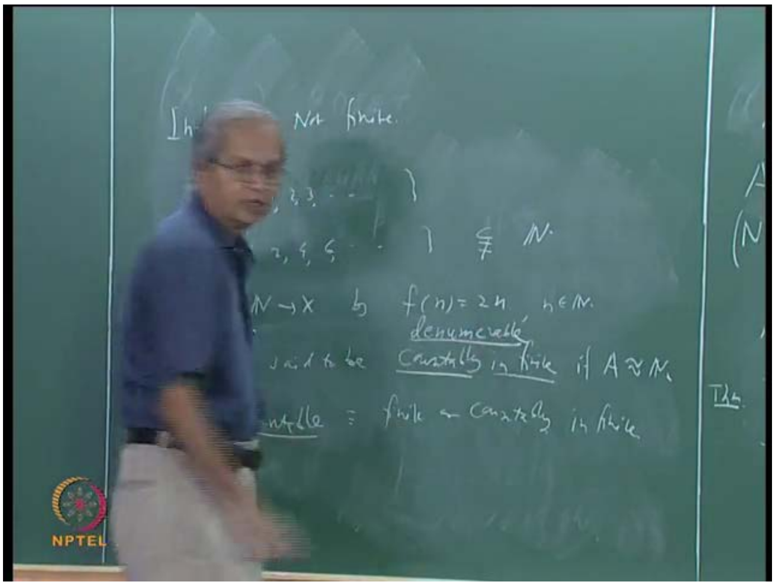

What we have called count ably infinite is also called denumerable, denumerable that is ruling rule in us the word is denumerable and what is worse is some books use countable for this denumerable, what is called what why do you, some books use countable for this and then what we have called countable they call at most countable. So, that there is something you should remember when you are looking at a book write in the beginning you see how the terms are defined, and then and then see subsequently, but we shall follow this notation for us countable denumerable is same as countable infinite. And countable means finite or countable finite. Now, let us go to what we were having in mind namely about showing that every infinite set as a proper sub set, which is numerically equivalent to each set, but even before going to that we need one more factor.

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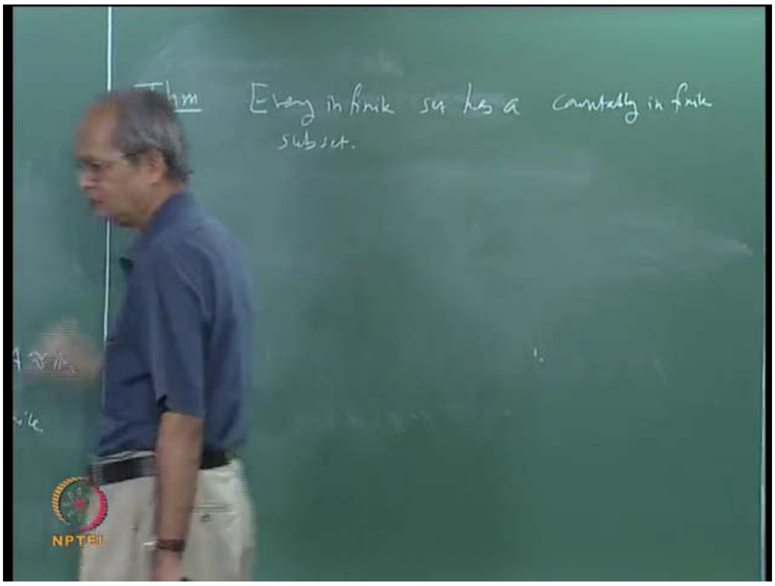

Let me write at the, it is the following every infinite set, has a count ably infinite sub set. See by the way before going to this, suppose I make the following statement that every count ably infinite set has a proper sub set which is numerically equivalent to it, is that clear to you right because if it is a contemporary sub set I can write the elements as a 1, a 2 etcetera. Then take the subset this a 2, a 4 etcetera that will be and you can easily establish 1, 1 correspondence between this. Now, is this clear to everybody.

Now, suppose I prove this theorem will it imply that every, every infinite set has a countable influent sub set and that countable influent sub set as I said which is numerically equivalent set. So, will it immediately mean that every infinite set has a proper sub set, for an example suppose let us say you start with the set A.

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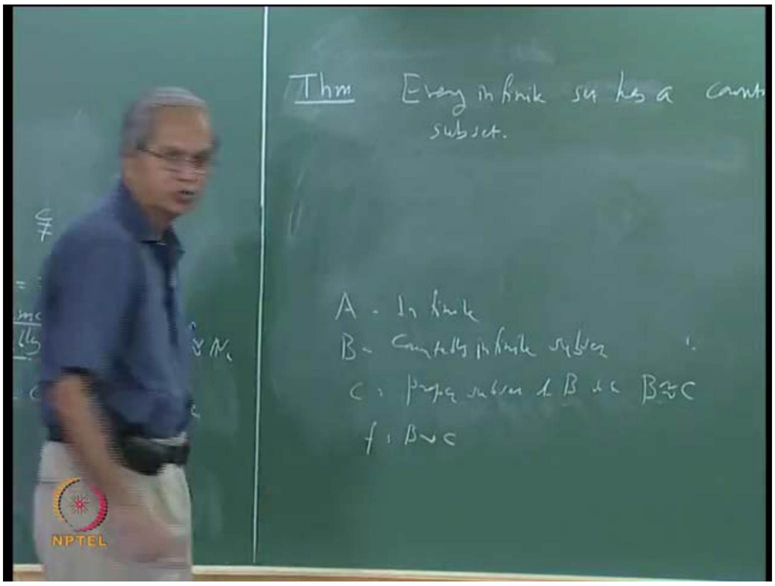

Suppose, this is our infinite set that you started with and suppose B is a count ably infinite sub set. Now, this B will have a proper sub set suppose I call that sub set as C. So, let us say as a proper sub set of B such that B is numerically equivalent to C right then from that can we say that immediately, can we say immediately A has a proper sub set which is numerically equivalent to A.

Student: ((Refer Time: 43:26)) yes, yes.

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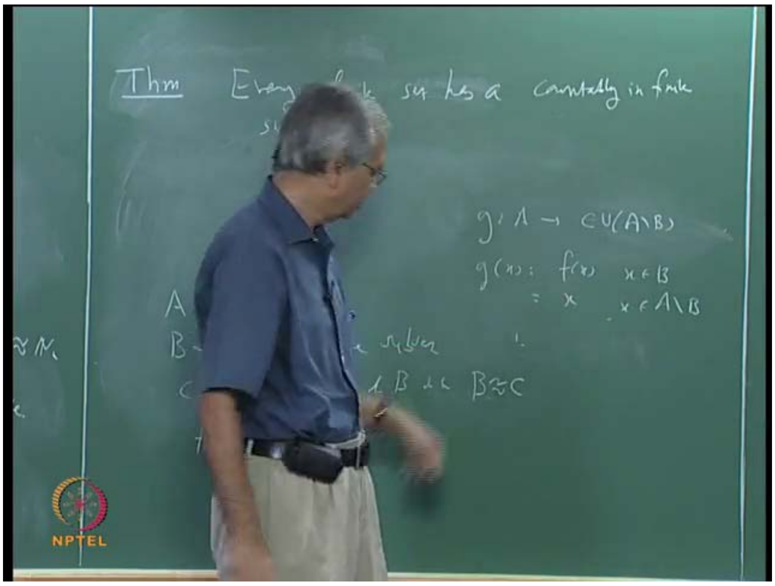

Tell me how, yes right what you do is see there is map 1 1 with B and C, then for those elements which are outside B you just identity that is take a map see there is a map, let us say f from B to C this is one words, and word 2. Now, whatever we do is that I will consider a map…

From let us say g from A to I will write this set later, the map is defined like this g of x is equal to f of x for x in B and equal to just x for x in A minus B, that is those elements which are outside B I will take g of x as simply x. And those element which are inside B I will take B of x is equal to x, is it obvious that this should 1 1 and on 2. Now what is the range of this map, see because the elements B goes to C right and the elements which are in A minus B that will go to A minus B. So, C union A minus B, right and is that a proper sub set of A yeah because we have moved the elements so this is not same as B right A band not yeah A, we have removed elements which are in B, but not in C. So, this is a proper sub set A and that is numerically equivalent to A. So, once we prove this theorem, let me write as a cardinality.

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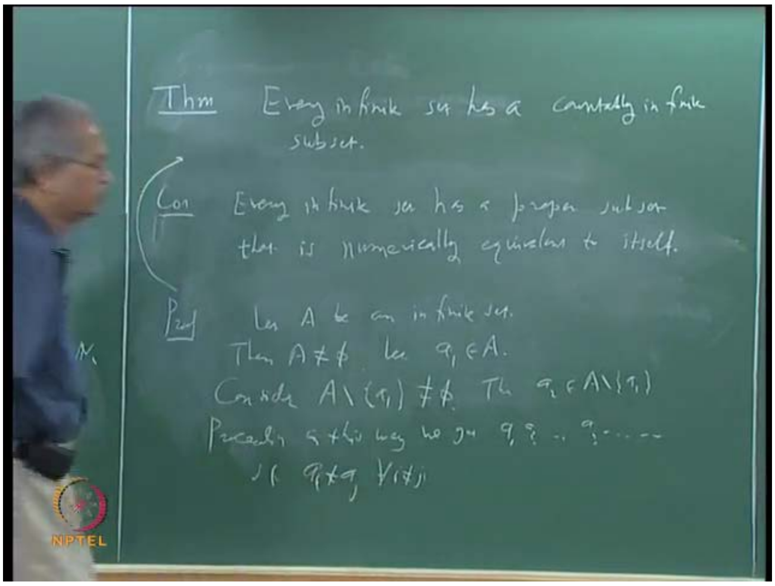

Every infinite set has a proper sub set that is numerically equivalent to itself. Of course, you have not yet proved the theorem, but I am sure that we have proved the theorem then this theorem will immediately follow from this theorem, this is the proof of that which I have discussed and what it means? Is that this is the fact which differentiates between finite and infinite sets. A finite set cannot have a proper sub set which is numerically equivalent to it and infinite set will always have a proper sub set which is numerically equivalent to itself.

And that is why in some books, this is taken as a definition of an infinite set this is taken as a definition of an infinite set, and then you define a finite set as the one which is not infinite etcetera, that is another approach of dealing going for the whole things. Let us now coming to the proof of this, let us say that the same is let A be the given set, let A be the infinite set, we want to show that a contains the count ably infinite sub set. First of all can A be empty it cannot be empty right. So, since let A be infinite then A is not empty what is the meaning that A is not empty, that there exists at least one element. So, A then A is not empty, someone they call that a 1 belong to A. Now, consider A minus a 1 let us remove that element a 1 from the set A, can this set finite, why?

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If it is finite it will be that there will exist some n since that A minus a 1 is numerically colon to J n right. That means there is 1 1 and identity map between A minus a 1 and this set 1 to N, then what you can do is that you can map this a 1 to n plus 1 then A will numerically equivalent to B segment J n plus 1 and that will make A to be finite right which is not the case. So, this is the infinite set if at all infinite set it has to be non empty.

So, A minus a 1 this must be non empty, this must be non empty then what follows from this again, if it is a non empty there must be some element in that I will call that element a 2. So, then a 2 belongs to A minus a 1 then one has to be done after this is clear. Now, consider A minus a 1 a 2 a 1 and a 2 are different now because a 2 is not same as a 1 right a 2 is taken from A minus a 1. So, it is different from a 1 A minus a 1 a 2 is also not finite by this same argument again. And then presenting in this way at the end stage are suppose, you follow this procedure for N steps, you will get the elements a 1 a 2 a n then consider a minus a 1 a 2 a n, that is also not empty.

So, what it means is that you can find one more element there that element is a suppose you call that element a n plus 1. Remember a n plus 1 is different from all of this a 1 a 2 a n, and this is un ending process you can continue in this way, continuing in this way what you will get? You will get a sequence. So, you can fill up the details will as simplessive proceeding in this way, we get a 1 a 2 etcetera a n. And all this elements are distinct such that a i not equal to a j for all i not equal to j. Now, what is the next thing to do you just collect all this element a 1 a 2 n that is a countable infinite sub set of a.

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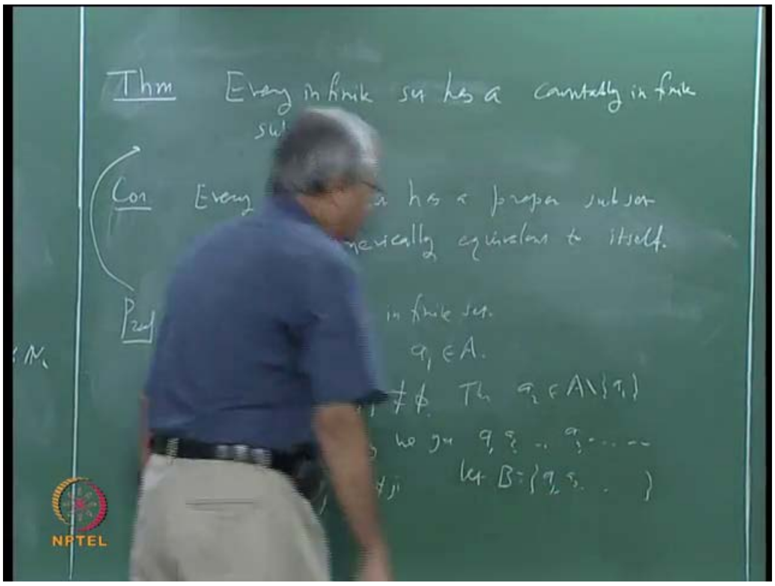

So, let B be equal to a 1 a 2 etcetera, etcetera then B is a countable infinite sub set of A and that completes the whole thing. We will stop with that we will consider some more property that is countable sets etcetera in the next class.