Real Analysis
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Lecture - 29
Compactness (Continued)

So, we were discussing the property of compactness, in the last class. Let us again recall, that we have defined, that set is compact. Means every open cover of that set should have a finite sub cover. We have seen some example of compact sets and then we have also seen a few properties. Namely, that every compact set is closed. Till now the definitions, of compact sets are in terms of open sets. So, we shall also consider some criteria about compactness in terms of closed sets.

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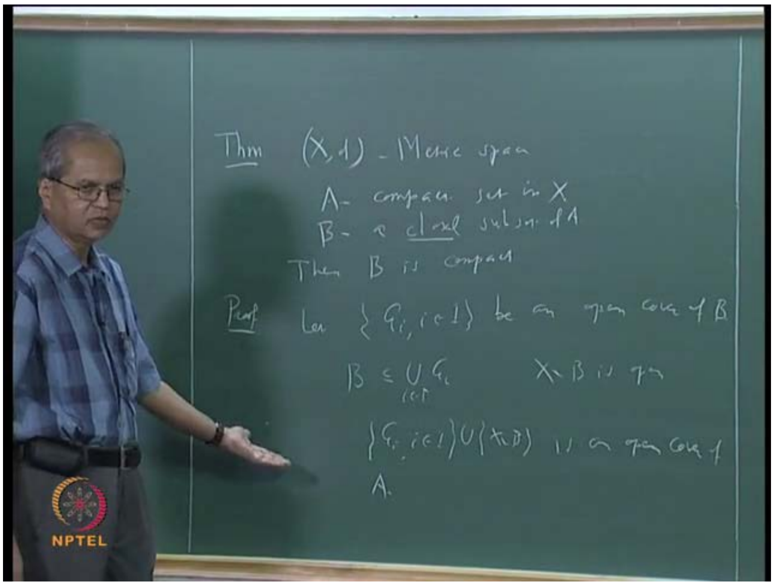

But before that, let us complete one theorem, we started yesterday. That was the following suppose x, d is metric space. And A is the compact set in x. Compact set in x. And B is a closed sub set of A, then B is compact. In other words, so closed subset of a compact set is always compact. So, let us look at the proof. As usual, to prove that any set is compact, we need to show that every open cover has a finite sub cover. So, start with an open cover of B.

So, let say that, let G i, i belonging to I, be an open cover of B Again let me repeat that, this means is each G i is open. Another thing is that B is contained in the union of G i.

Alright, now suppose, you take this G i, then this is a cover for B, but we are given that A is compact. That property has to be used somehow. That means, we have to construct a cover of A, then we can get it, square it. Of course, using the compactness of A, but this G i only covers B. So, what about the point outside B, well those are not covered by this. But what we can say, is that, we are given that B is closed, that means its complement is open, its compliment is open.

So B, is closed means, this x minus B is open. Alright now the idea is simple, this suppose I add this open set, to this family. Will that cover A, because those point which are in B, those already in one of the those already in G i. Those which are outside B, those will be in x minus B. So, what you can say is that, this G I, i belonging to I. Then union this single term x minus B. This is an open cover of A. This is an open cover of A. Open cover of A. Since, A is compact, this should have a finite sub cover. Since A is compact, it has a finite sub cover. But that finite sub cover is not A sub cover of this.

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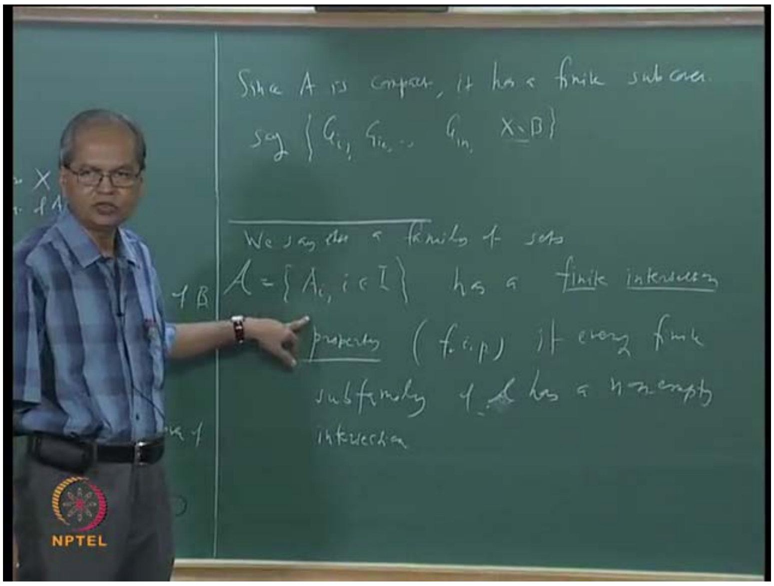

So, it is a sub cover of this new family. But what is the difference? Only difference is, this additional set. Only difference is, this additional set. So, that sub cover may or may not contain this. But does not matter, whichever is the case. We shall show that, sub, anyway it is a, it will be a sub cover of A. It will be sub cover of A. So, the finite sub cover, let us say, g i 1, g i 2, say g i n and perhaps this x minus B, right? This may or may not be there, this may or may not be there.

It does not, if it is not there, if x minus B is not there then this g i 1, g i 2, say g i n cover B. See whatever is, since B is contained in A. B is contained in A, any cover of A, is also cover of B, right? So, since, look at, since this is a cover of A, it is also a cover of A. Only problem will be that, if it if this finite sub cover contains this set. That is not the sub cover of this original family that we started with. That is the only problem. If this finite sub cover, does not contain this set, there is no problem. Then this, then this sub cover will contain only g i 1, g i 2, say g i n, that is already a sub cover of this family. That is cover of B.

So, we have proved that every open cover has a finite sub cover. Only question is what is to be done, if this finite sub cover also contains this set x minus B. What is, it is obvious, what is to be done? Just remove that set from this, just remove that set, because this is a cover of A. We are not interested in the cover of A. What we want is cover of B. Obviously, this is not going to contain any point of B. So, even if you remove this, open set from this, remaining sets will also cover B, is that clear? So, that proves that. Then this remaining set, are coming from this family. So, whichever is the case, we have proved that, every open cover has a finite sub cover. Is that clear?

Then, we shall also discuss one more property of the compact set. So, far we have, proved for, every compact set is closed. Alright before going to that property, let us also characterize this compactness in terms of closed sets. To do that, it is convenient to discuss the whole metric space, instead of subset. Alright, let me first give you some definitions. So, suppose you have a family of sets, let some family A i, i belong to I. We I is any, big I is any indexing set. We say that such a family has a finite intersection property. That is what we want to say, is that such a family has a finite intersection property.

We say that this family of sets, this A i, i belong to I, has a finite intersection property. Has a finite intersection property. There is a standard short form used for this. f, i, p finite intersection property. It is clear from the name, what should, it should mean. Finite intersection property means, if you take any finite sub family of this, then that has a non-empty intersection. That is the meaning of saying that, finite sub finite intersection property.

We said, it has a finite intersection property, if every finite sub family, let me give some notation for this family. Suppose, I call this script A. If every finite subfamily of script A has a non-empty intersection. Of course, we are not saying that the intersection over the whole family is non-empty. That may or may not be the case, but we said is that if you take any collection any finite collection of sets from here. Say A i 1, A i 2, A i n, etcetera. Then intersection over that finite sub family should be non-empty. That is called, if a family has that property.

If a family of sets, has that property. We say that the family has the finite intersection property. Is clear? What is an obvious example of this kind of property? We have seen earlier, what is what we had called decreasing sequence of sets. Suppose, it is a sequence of set. A 1contains A 2, A 2 contains A 3, etcetera and suppose all of them are non-empty, then if you take intersection over any finite sub family that intersection will be smallest among those sets. For example, say A 1 intersection A 2 will be A 2 or A 1 intersection A 2 intersection A 10, that will be A 10. Whatever, once it is a decreasing family of set.

So, decreasing family of sets in which, each set is non-empty. We have this, finite intersection property. But of course, such a family may not have the property that, intersection over the whole family is non-empty. That may or may not be true. Where does compactness come into picture? What we can show is that in a, in a compact suppose a metric place is compact. And if you take a family of closed sets, which has a finite intersection property, then intersection over the entire family, is also non empty. Intersection over the entire family is also non empty. This is equivalent to compactness, equivalent to compactness.

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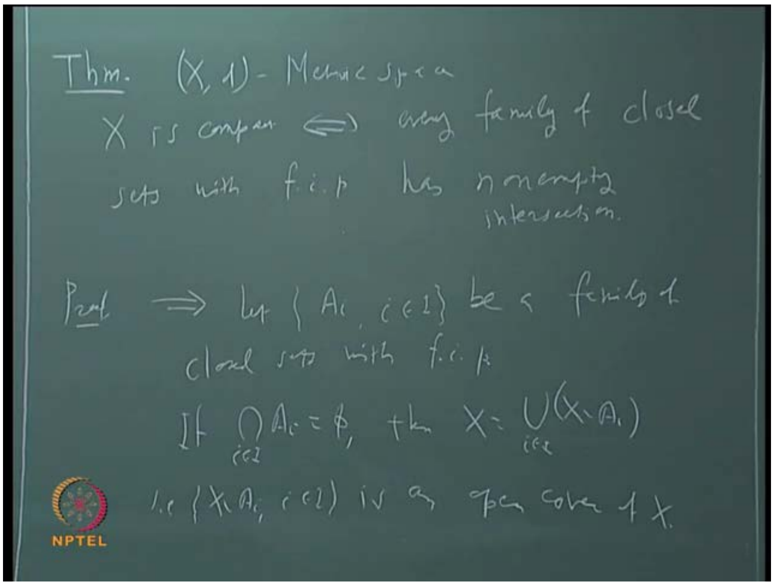

So, that is the theorem. As I said, it is convenient to write, in term of the whole metric space x instead of taking some subset of x. We have seen earlier, that there is, that makes no difference. Whether you take a compact set of metric space or talk of compactness of the whole space, that is more or less same. It makes no difference. So, let us say that x is a metric space, then x is compact, if and only if every family of closed sets, every family of closed sets with finite intersection family, with finite intersection property.

Suppose, you take a family of closed sets, with finite intersection property. Then, intersection over the whole family is non-empty. So, I have presented every family of closed set with finite intersection property has non empty intersection. Has non empty intersection. Alright and the proof involves, actually just one, only one idea. Namely a set is open if and only if its compliment is closed. And compliments of the unions are intersection. That is that, these are the only ideas used. So, let us take this, suppose x is compact and we want to prove that every family of closed sets with non-empty intersection has a, sorry, with finite intersection property has a non-empty intersection.

So, start with a family of closed sets with finite intersection property. So, let us say that let A i, i belonging to I, be a family of closed set. Closed sets with finite intersection property, then we want to show that intersection over the whole family is not empty. Intersection over the whole family is non-empty. Suppose, this is false, suppose this is false. So, if intersection over A i, i belonging to I, is empty. If intersection of A i, i belonging to I, is empty.

What should happen? What can we say about the compliments of A i? Suppose, this intersection is empty, its compliment should be whole of x. That should be nothing but union A i compliments, right? Then, what we can say is that, then x is equal to union where i belonging to I, x minus A I, right? But each of this x minus A i is open, each of this x minus A i is open and their union is x. That means this is an open cover for x. That is, what we can say is that is, x minus A i, i belonging to I, is an open cover of x. Now, what is? How to proceed is obvious, we have assumed that x is compact. So, every open cover must have a finite sub cover. So, I will continue with that.

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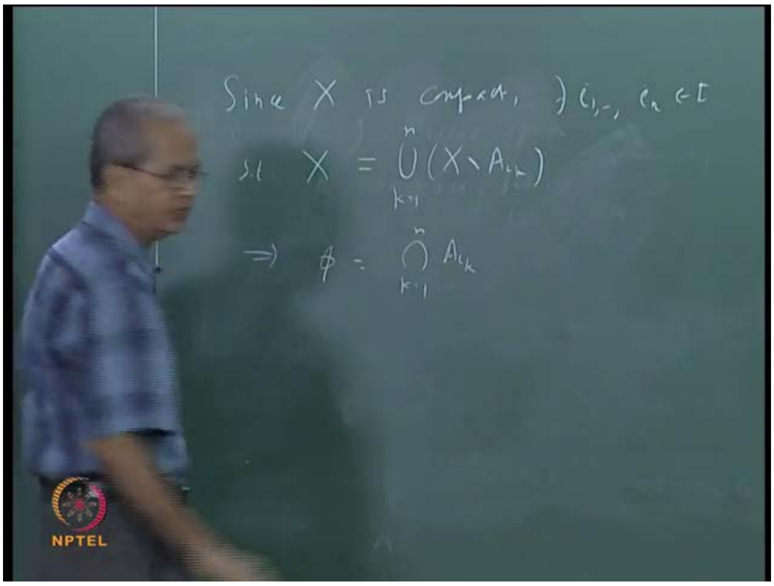

So, what is argument? Since, x is compact, this open cover consisting of compliments of A i must have a finite sub cover. Which is same as saying that, there should exist finite number of indices. Let us say i 1, i 2,i n such that x minus A i 1, x minus A i 2 etcetera that should cover x. That is, there exists, let us say i 1, i 2,i n in I, such that x is, x is contained in any. We just same as saying, because x is the whole space. x is equal to union x minus A i k, k going from 1to n. Got it?

Now what is the meaning of this, if x is the union of this x minus a i k. Suppose I take again the compliments of both sides. Compliment of this will be empty. What will be the compliment of this? It will be the intersection of A i k, k going from 1to n. So, this that is, this means, phi is intersection A i k, k going from 1to n. Alright, and that contradicts this, finite intersection. We have assumed that A i, is a family with a finite intersection property. Now we have produced a finite sub family, A i 1, A i 2, A I, which has empty intersects.

So, what this show? If x is compact, then every family of closed sets with finite intersection property must have non empty intersection. What is a converse? Converse is that suppose a metric space has this property, that every family of closed set with finite intersection property has non empty intersection. Then we should prove that, x is compact. What I will do is, I shall leave you that part of the proof, as exercise. You will understand how to proceed, to show that x is compact. You start with any open cover and then you have to show it has a finite sub cover. Then whatever open cover you take, you start to take the compliments and show that, that family of closed sets has finite intersection property.

Then using basically, again principles involved are same that is compliment of a closed set is open and vice versa. Then compliments of this union is intersection and that kind of things. So, base elementary set theory and this fact that, a set is open if and only its compliment is closed. Using this two things you will be able to prove this. Now, as I just said that, obvious examples of the families with the finite intersection property, are the decreasing sequences of sets. So, what this means is that, in a compact metric space, if you take a decreasing sequence of closed sets and if each of the closed set is non-empty.

Then, their intersection also must be non-empty. Something similar what you have seen in the case of Cantor’s intersection theorem. There also, we had taken the decreasing sequence of closed set. But there, we assumed something additional things. That the diameter goes to zero. And then, we had proved that the intersection consists of just one single point. In case of compact sets we need not show many things about the diameter the whole thing happens inside the compact metric space. And then, intersection also may contain more than one point. All that we can say it is non-empty.

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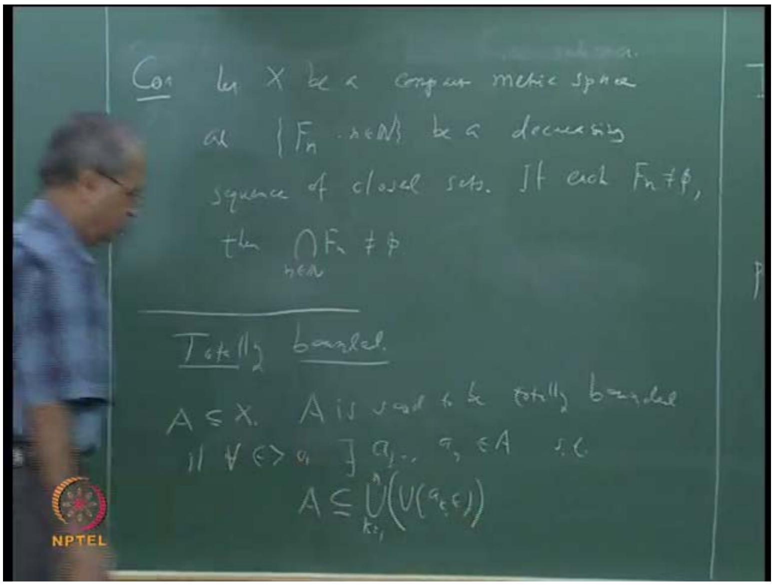

So, let us since this is an importance consequence. Let us write it as a corollary. So, let x be a compact metric space, compact metric space and let us say that f i, i belonging to I, be a decreasing family of closed sets. Then if each f i is non-empty, sorry, if each not decreasing family, let me say f n instead of f i, f n, n belonging to N, be a decreasing sequence of closed sets, because if we take i to be any arbitrary indexing set. Then decreasing sequence does not mean anything, because unless that i has an order, we cannot talk about decreasing.

So, that is why we take decreasing sequence. So, decreasing sequence of closed sets. If each f n, is non-empty then, intersection f n, n belonging to N, is also non empty. This follows from this theorem in a straight forward manner, because if each f n is non-empty then, this family has a finite intersection property, right? Hence, intersection over the whole family must also be non-empty. Alright, there are in some books, you will also find a statement. The same statement written in a slightly different manner.

What will be said is that, if the intersection over a whole family is empty, then intersection over at least, then there exists a finite sub family over which the intersection is empty. But you can see that, that is basically same as this, stated in a different language. Similarly, here statement might be, if the intersection over the f n is empty, then at least one f n is empty. Of course, if one f n is empty, all substituent f n will be empty, because it is a decreasing family. Now, coming back to the properties of compact sets.

We have already shown that every compact set is closed. We also want to show that every compact set is bounded. In fact, it is something more than bounded. It is just slightly stronger property and we shall discuss that property. It is what is called totally bounded. It is a new term that we shall define. So, what is meant by totally bounded, this means the following. Suppose, you take A as a subset of x. What is meant by bounded? We know that bounded in a metric space, we have defined that a set is bounded, if its diameter is finite. So, it is bounded if its diameter is finite.

Totally bounded means, something more. What we want is that, what should happen is positive. If we have given any positive number. Let us say epsilon, then you should be able to cover this epsilon, by a finite number of balls with radius epsilon. So, we will say that A is said to be totally bounded, totally bounded, it said to be totally bounded, if for every epsilon bigger than zero. What should happen, is that there should exist finite number of points in this A. And A should be contained in the union of open balls with centre at those points and radius epsilon.

This you should be able to do for every epsilon. So, for every epsilon bigger than zero, there exists A 1, A 2, A n in A, there exists A 1, A 2, A n in A, such that A is contained in union. Let us say k going from 1to n of open balls, with centre at A k and radius epsilon. That is you take open balls with centre at A 1, A 2, etcetera. Each with radius epsilon, then A must contained in the union. Remember it is a finite union. It is of so, that is important.

For every epsilon there exist a finite subset of A. So, that exist a finite subset of A. And A should be contained in the union of open balls with centre at those finite number of points. In some books this set A 1, A 2, A n is called an epsilon net. Epsilon net means, it is a set of, it is a finite set, such that, the given set is contained in the union of open balls with centre at that points. So, such a thing is called an epsilon net. So, the difference between total points will be, every epsilon has a finite epsilon net. But we shall not use that now. Is it clear? That every totally bounded set is, bounded is bounded.

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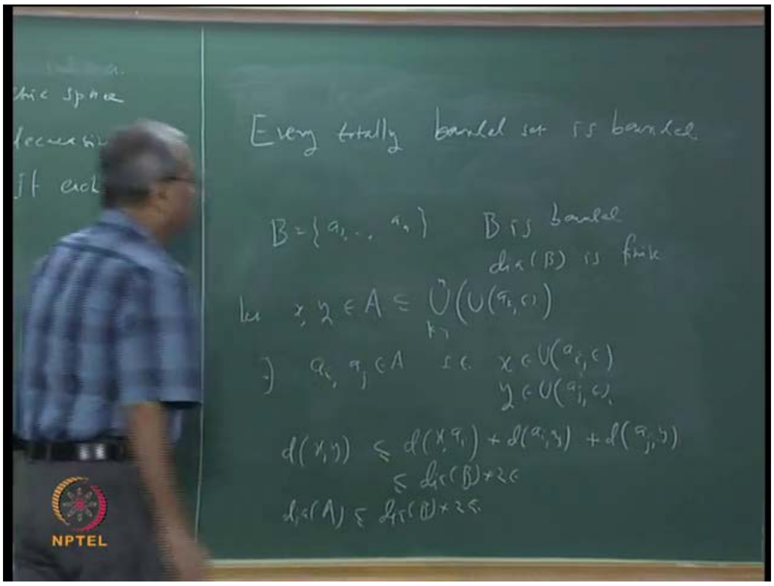

How does one prove that? Suppose A is totally bounded. Suppose A is totally bounded. Then, alright suppose, A is totally bounded, then what we know, that for every epsilon, there exists some finite subset A 1, A 2, A n. Suppose I call that subset B. Suppose I call this set B. Let us say that B is A 1, A 2, A n. Is this clear that B is bounded? It is a finite set. So, its diameter will be, you take distance between A 1, A 2. A 2, A 3 it is that distance between A i, A j for all pairs A i and A j that will be finite number of real numbers, finite subset of real numbers. Take their maximum, that will be the diameter of B. So, diameter of B is finite.

So, B is bounded. So, diameter of B is finite. What can we say about relationship between diameter of A and diameter of A? Less than equal to diameter of B plus epsilon. Let us say, how does one prove that? To look at the diameter of A. What we should do? We should take any two point x and y in A. And show that distance between x and y is less then equal to that number. Whatever is the diameter? So, suppose we take x and y in A, that x and y belong to A. Then, we know that A is contained in this union that k equal to 1to n, open balls with centre at a k and radius epsilon. So, what follows from this? That each of this x and y is contained in one of those balls.

Of course, they may not be contained in the same balls. They may be contained in different balls. So, we can say x is contained in some ball and y may be contained in some other ball. So, we can say that, this means, there exist let us say this a i and a j in A.

Such that, such that, x belong to open ball with centre at a j and sorry it is centre at a i and radius epsilon and y belongs to open ball with centre at a j and radius epsilon. Now, what can we say distance between x and y? We can say the distance between x and y is less then equal to distance between x and a i plus distance between a i and a j and plus distance between a j and y. And plus distance between a j and y. This is less then epsilon.

This is also less then epsilon. What about this? This is less then equal to diameter of B. So, we can say that, this is less than the whole thing is less then equal to diameter of B plus 2epsilon. So, what is that prove? That diameter of A is less then equal to diameter of B plus 2epsilon diameter of A. So, diameter of A is less then equal to diameter of B plus 2epsilon. So, that proves that, every totally bounded set is bounded. What is the obvious question to ask here? Whether the converse is also true. Can we say that every bounded set is totally bounded? Well obviously, if that were the case. We should not have defined the two things separately.

As I said total bounded set is a stronger concept than bounded set. Every totally bounded set, is bounded, but in general the converse is false. But as you know to show that the converse is false, we have to give a counter example. Giving a counter example is easy. Again, as I said earlier also several times. When we think of counter examples, the best metric space to start with is discrete metric space. You start with a discrete metric space. Take any infinite set and give a discrete metric on that. Let us say for example, we can take R.

We take R and d, where d is a discrete metric. Where d is a discrete metric on R. Now is this bounded, right? Discrete metric, distance between any two point is 0 or 1. So, then diameter of R is 1. Therefore, R, d is, so R is bounded. Is it totally bounded? If it is not totally bounded, how does one show that? For some epsilon, this should be false. There should exist some epsilon, such that this cannot be covered by finite number of balls with radius.

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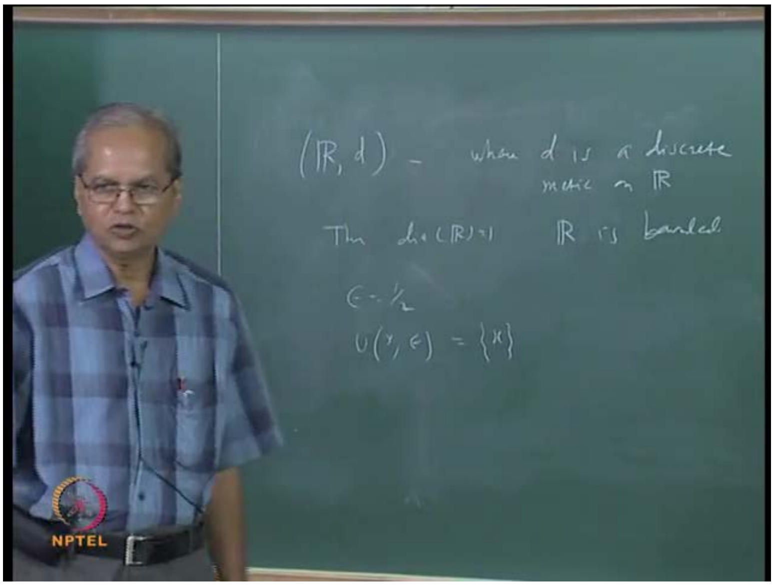

What is such epsilon? Yeah, you can take for example, take epsilon equal to half. For example, okay. If you take epsilon is equal to half, and if you take, let us say any point in the real line. Where if you take say, open ball with centre at x and radius this epsilon. What will be that? That will just single term x. That will be just single term x. Right? So, if you take any finite number of ball like that, there union will be just a finite subset of r. That can never be R. So, this is an example of set, which is bounded, but not totally bounded.

So, total bounded ness is a much stronger concept than bounded ness. So, let us again recall, totally bounded means this. Every totally bounded set is bounded, but the converse is false. Now, the question is, what is compactness to do with all these? What we want to say is that, every compact set is totally bounded. Every compact set is totally bounded.

Now, before proceeding with that, let me also mention one more thing. In case of real line or in case of R 2, R 3, R k, with any of those metrics. In those spaces it so happen, with the usual metric, not the discrete metric. Suppose if I use R with usual metric, then every bounded set is totally bounded. In R with usual metric, every bounded set is also totally bounded. This happens in case of some metric spaces, it does not happen in case of all metric spaces. Let me, let us just see, how? Why that should happen?

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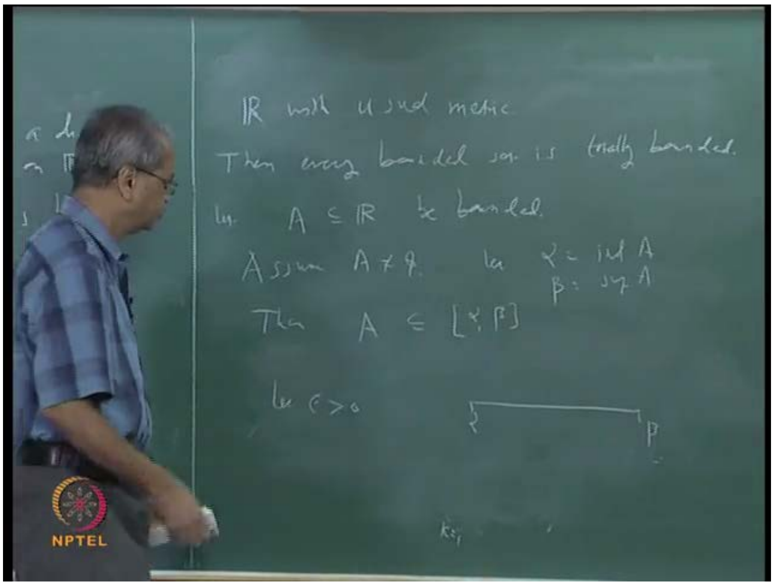

Now let us say R with usual metric. Let us say R with usual metric, then every bounded set is totally bounded. Now it is very sure that every bounded set is totally bounded. Let us let us say, we are talking about R, so let us take A is a, let us say A is a subset of R. Let A be bounded. If A is empty set, nothing to be, if A is empty set obviously it is totally bounded. We do not have to bother about that, So, assume A is non-empty and because of our l u b action.

We know that first set is non-empty and bounded about, it must have a list upper bound and similarly, if it is non-empty and bounded below it must have greatest lower bound. So, since it is bounded and non-empty, it means it is bounded above and below. So, let us say that, let alpha be increment of A, of A that is greatest lower bound. Beta be equal to ((Refer Time: 39:07)) of A. Then A is contained in, then A is contained in the closed interval of alpha to beta. So, if we cover this interval alpha to beta by finite number of open intervals. Then, it will mean A will be totally bounded and that is easy.

For example, suppose thought epsilon is given. Let epsilon be greater than zero and let us say this is alpha and that is beta. A is inside this. So, can we find a finite number of points, in this alpha to beta, such that they cover this interval, they cover this interval. In fact, let me let me also say one more thing here and will come back to this will come back to this.

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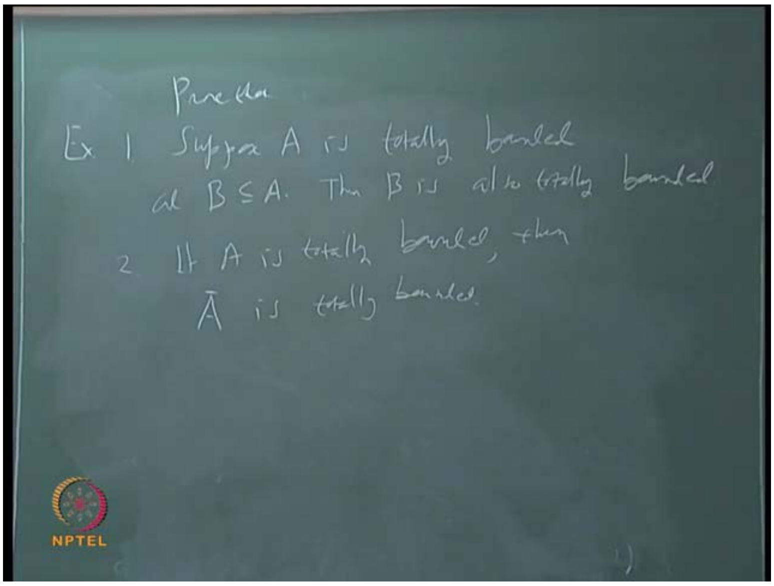

Those two things I will give it to you as exercise. Suppose A is totally bounded and B is a subset of A. Suppose A is totally bounded and B is a subset of A. What can you say about B? Then B also must be totally bounded. Then b is also totally bounded, in other words every subset of totally bounded set is again totally bounded. That is one thing, second thing is, if a set A is totally bounded, then its closer is also totally bounded. So, if A is totally bounded then, A closer, is also totally bounded. This is exercise for you, prove that.

What is the relevance of all this while discussing here? We will just prove that this interval, alpha to beta is totally bounded. That will mean that A is totally bounded. That is the idea. We shall just prove that this interval alpha to beta is totally bounded. And, because of this first part here, it will imply that A is totally bounded. How does one prove that this is bounded? You should, given any epsilon, bigger than zero. You should find point in this interval, such that, balls with centres at those points in this case, balls means open intervals. Suppose, there is any point x here, A 1 here, then it means interval A 1minus epsilon to A 1 plus epsilon. Intervals like this should cover this completely.

Now what is the obvious way of going about that? We can start arbitrary, for example, I can choose A 1 is equal to alpha. Let say I will choose A 1 is equal to alpha. Then of course, there is nothing like A 1minus alpha. So the first thing will be a this is the first interval between A one minus epsilon to A one plus epsilon.

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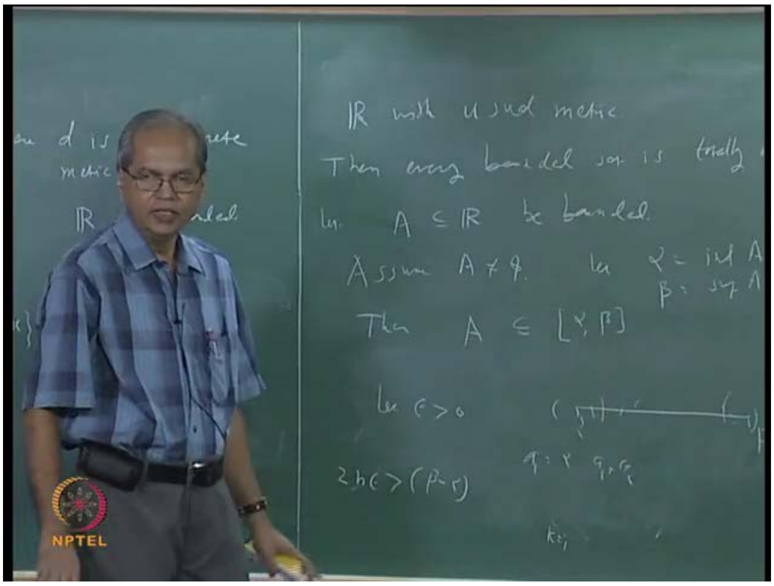

Let us say second point I choose as, let us say, A 1 plus epsilon by 2. Then there will be next interval etcetera. So, idea is that if all of the cover all of them should cover, what is the total length of all these intervals? Each interval is of length 2 epsilon. Each interval is of length 2 epsilon. So, suppose there are n such intervals, the total length will be 2n epsilon. So, suppose we choose n, 2n epsilon is bigger than this. Beta minus alpha, then their union will cover this.

Of course, you can choose the point appropriately. That is we can choose the points, that next point is within this. That is the union covers the whole interval A,B. In other words, what we can do is, choose n such that 2n epsilon is bigger than beta minus alpha. Can this be done always? Again use the Archimedean property. Whatever be this numbers, epsilon and beta minus alpha into, we can always choose n such that n is bigger than beta minus alpha divided by two epsilon.

So, choose any such n and then choose those n points and then show that their, this n intervals will cover this given interval alpha to beta. Is it clear? This can be done, whatever be alpha and beta or whatever is epsilon. So, given any epsilon, you can always find n points. Let us say A 1, A 2, A n, such that the interval, A j minus epsilon to A j plus epsilon their union will cover, their union will cover this interval alpha to beta.

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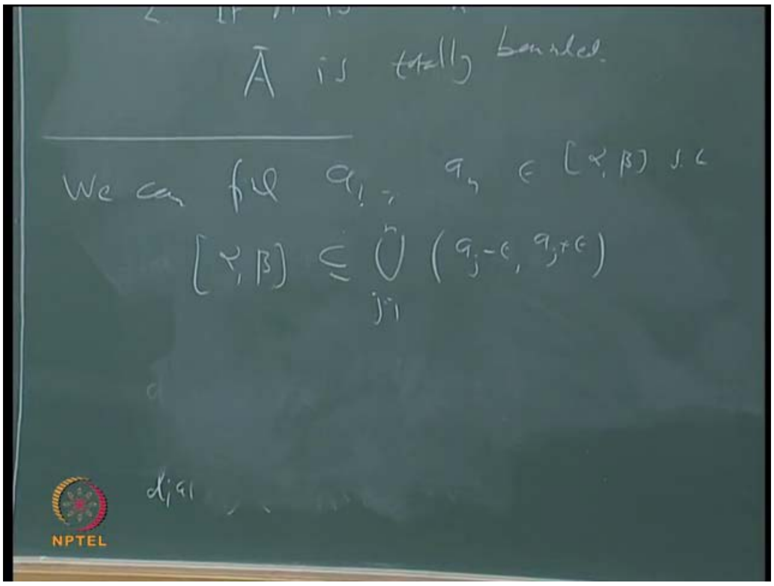

So, we can that, I can continue here, that you can find A 1, A 2, A n, of this interval alpha to beta. Such that alpha to beta, contained in union of say intervals A j minus epsilon to A j plus epsilon for j going from 1 to n. Alright, is this clear? So in R, every bounded set is totally bounded. Can we do a similar thing in, let us say in R 2? Can we do, because instead of the interval, instead of the interval, that will be some closed ball.

Close ball with some finite radius and some centre, because every bounded set will be contained in some closed ball. We can. So, if you can show that every closed ball is totally bounded, just as here. See, remember to show that every bounded set is totally bounded, it was enough to show that every closed interval like this is totally bounded. So, similarly, if you can show in R 2, that every closed ball, closed ball with finite radius is totally bounded. Then you will be able to show that every bounded set is totally bounded. Not only in R 2, but R 3, R k any of those spaces.

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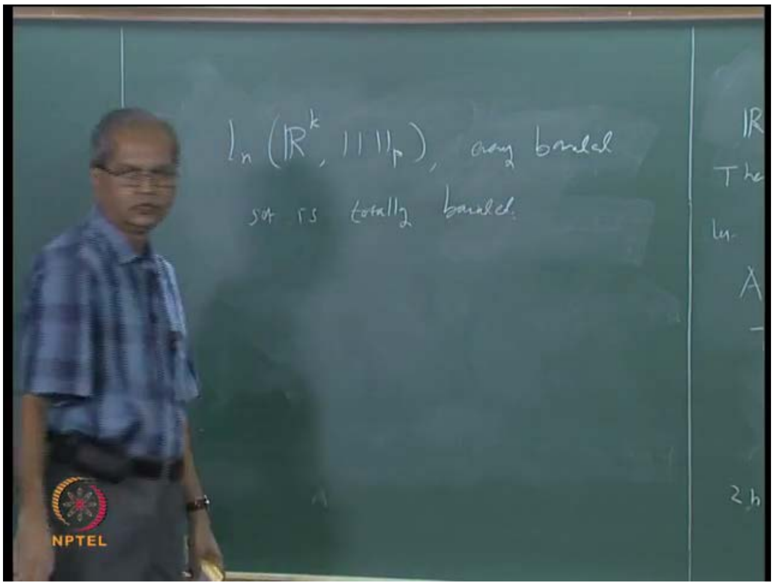

So, let me just say that in R k, in R k with any of those metric, take any of these metric. There will be small modifications in the argument. If you change the metric, but whatever be the metric you take in R k, any of these terms. So, that every bounded set is totally bounded. So, let us again take every, of whatever we have shown. About the compacts. We have shown that, every compact set is closed and every compact set is also totally bounded, and hence bounded.

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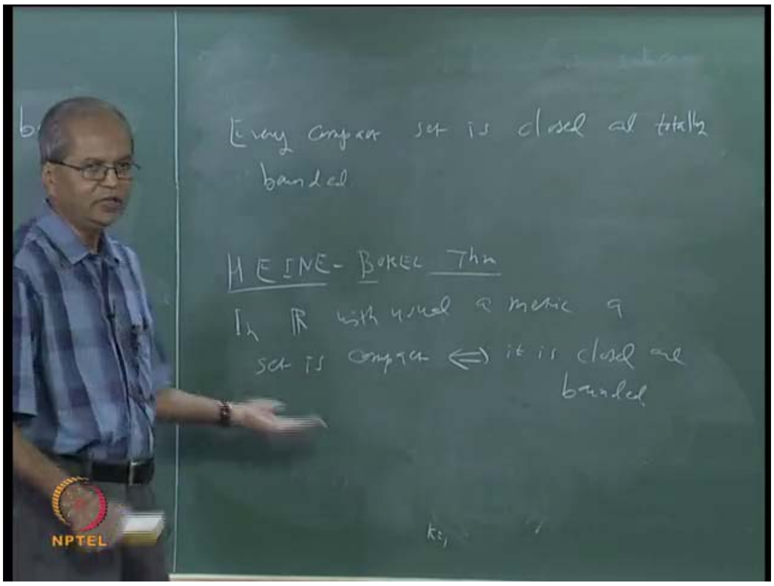

So, let us just write, every compact set is closed, and totally bounded. Of course, it also means that, every compact set is closed and bounded. Every compact set is closed and bounded. In real line, with the usual metric, you can just replace this totally bounded ness by bounded ness. What is interesting is that, in the in the real line, the converse is also true. What will be the converse? Converse is that every, in the, in since, we are talking about real line, we can say every bounded and closed set is also compact.

Every bounded and closed set. And that is a very well-known theorem, that is known as Heine-Borel theorem. In let us say, in R with usual metric, usual metric, a set is compact, a set is compact, if and only if, it is closed and bounded. Actually, if a set is compact, that it is closed and bounded, that part is true in every metric space. In fact, something more is true, not only bounded in its, but it is totally bounded. But what is particular about this real line is, that the converse is also true.

That is what is called as actually, Heine-Borel theorem .That in the real line every closed and bounded set is compact. We already know that, in every metric space, if a set is compact it must be closed and bounded. The importance of this theorem, is it gives us complete description of compact sets in the real line. We can now recognise all compact sets in the real line. Compact sets must be every, it should be closed and it should be bounded and that is all.

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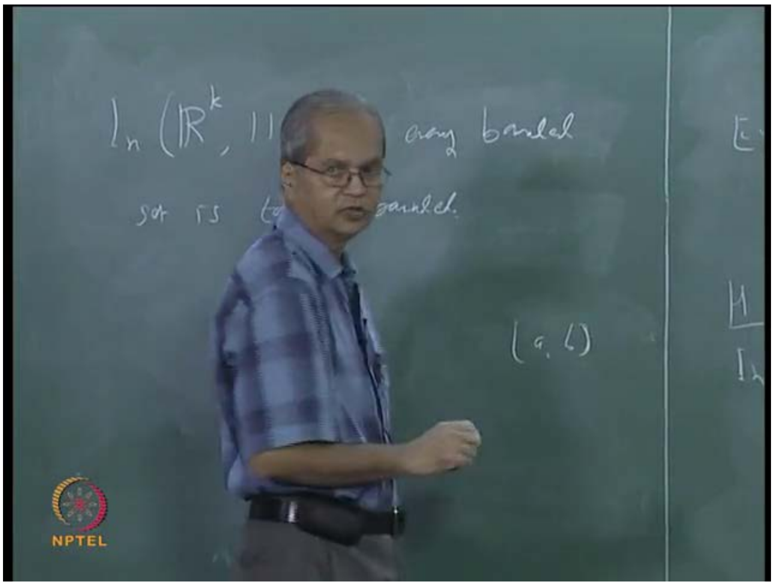

Those are the compact sets. Now we have several examples. For example, every closed interval, every interval line, this let us say, a, b that is an example of a compact set. You take any closed and bounded set, that must be, that must be compact. We shall discuss the proof of this in the next class. We stop at this.