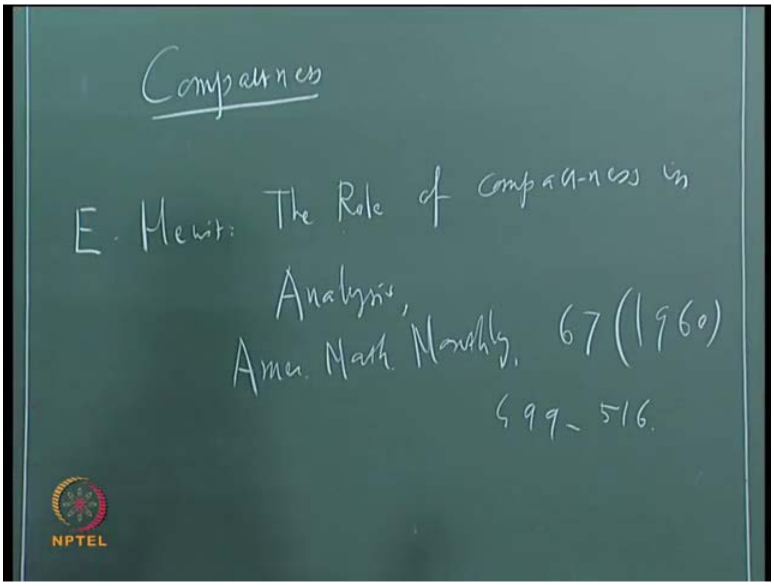
Real Analysis
Prof. S.H. Kulkarni
Department of Mathematics
Indian Institute of Technology Madras

Lecture - 28
Compactness

We will today discuss a property of metric spaces or certain subset of metric spaces. It is a very important property known as compactness.

(Refer Slide Time: 00:26)

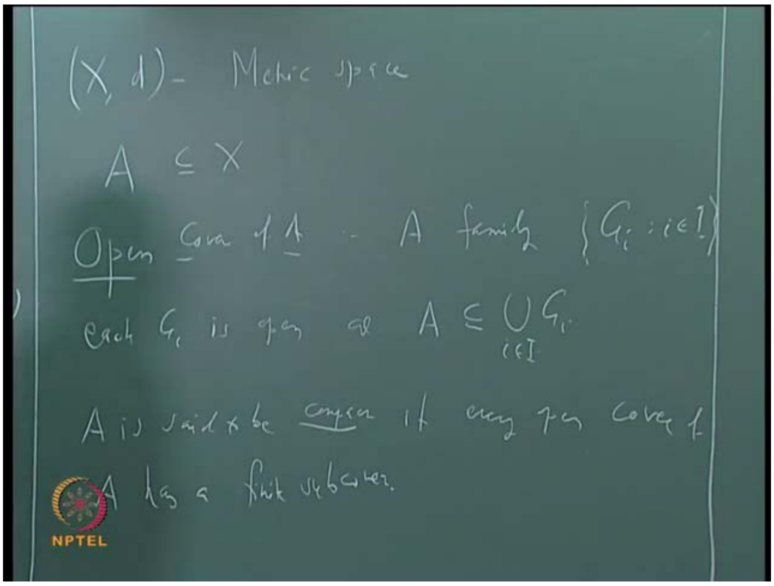


As usual, we shall define, what is meant by saying that a set is compact? And see certain example, of compact sets and some properties of compact set. Also, properties of continuous functions, defined on compact sets that is the idea. Compactness is supposed to be next base thing to be finite set. So, whatever we expect about a finite set with small modifications, those things are also true in case of compact sets. In this case I would like to point out one very interesting article.

This reference to that article it is E Hewit, role of compactness in analysis. Compactness in analysis, it appeared in a journal called American mathematical monthly. So, volume number 67, year is 1960. Pages 499 to 516. I do not remember, whether I have mentioned this journal earlier or not. American Mathematical monthly, that a very good journal. It contains articles, which are based on the BSC or MSC level mathematics. So, anybody with that is, you can understand most of the articles.

And another interesting part of that is, it contains a section called problem section. It gives some problems, with solutions are not known or some solutions are known. It invites the readers to submit the solutions, they have given some address and you can send a solution to that. After receiving solutions from the readers, they publish the best solution and also publish the names of the other people who have solved it. So, that is quite an interesting thing.

(Refer Slide Time: 02:55)



Coming back to our this idea of compactness. Let us again start with some metric space X, d. A metric space and we take a subset A of X and we will define what is meant by saying that this subset A is compact. Now, to do that we did some ideas, first of all, we shall define, what is meant by saying what is an open cover? Open cover of a in general a cover simply means a family of subsets whose union contains A.

Any cover of A is nothing but a family of subsets of X, whose union contains A. Open cover means a family of open subsets, that is open cover of A is nothing but a family let us say, suppose I call each element in that family as G suffix I. Say small i belongs to some big I, where big I is some indexing set. A family G i. So, what is the property of this family? First of all each G i is open. And A is contained in union G i.

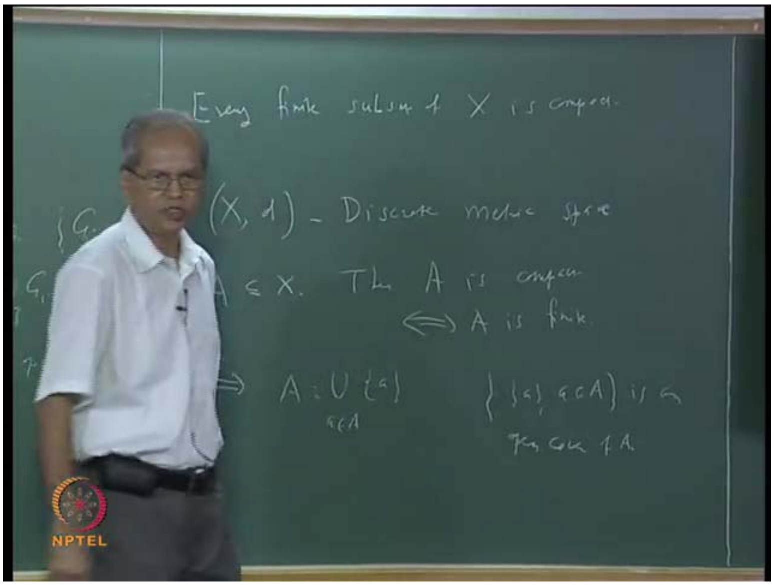
A is contained in union G i. So, again to repeat, what is an open cover? It is a family of open sets, whose union contains A. Such a thing is called as open cover of A. Alright, now when do we say that A is compact? Of course, there are many equivalent definitions, one which we shall use I am giving it now. It is that every open, cover should have a, every open cover of A, should have a finite sub cover. Sub cover means sub family.

Sub cover means sub family. So, given any such open cover, we should be able to find a finite number of sub sets here, such that that finite union should contain A. That finite union should contain A. So, that is the definition. So, A is said to be compact, if every open cover of A, every open cover of A has a finite sub cover. That means, if you can cover A by some family of open sets then that family of open set, should contains a finite sub family. Such that that finite sub family should contain should cover A.

Of course, remember we are not saying that A should be covered by some finite number of open sets, that will be always be true, because we can always take X and X contains A. So, this is a just one set which contains A. So, that is not the idea. What the idea is that whenever any family of open set covers A, that family should have a finite subfamily which should cover A. Alright now what are the obvious examples of set satisfying this property?

Let us take start form the very small things. If A is an empty set, this will always happen right? You take any one of those sets, that will be that will cover A. If A contains just one point then, if A is contained with individual, that one point should contain at least one of those. So, you just take that set, that will be cover for A. Proceeding like that, we can say if A is finite? Then suppose A is let us say A 1 A 2 An, then A 1 is contained in some G 1, A 2 belong to some G 2 etcetera. You just take those G 1, G 2, G n there union will cover.

(Refer Slide Time: 07:26)



So, what is obvious is that every finite set is compact. Every finite set is compact. That is the first observation that we can make. Every finite subset of X is compact. So that is why I said that compactness is the next thing to finiteness. Even that point is made a very clearly in this article. You can you can read this article, some time whenever you are free. Of course, if those were the only examples of compact set, then the this idea would not have been worth studying.

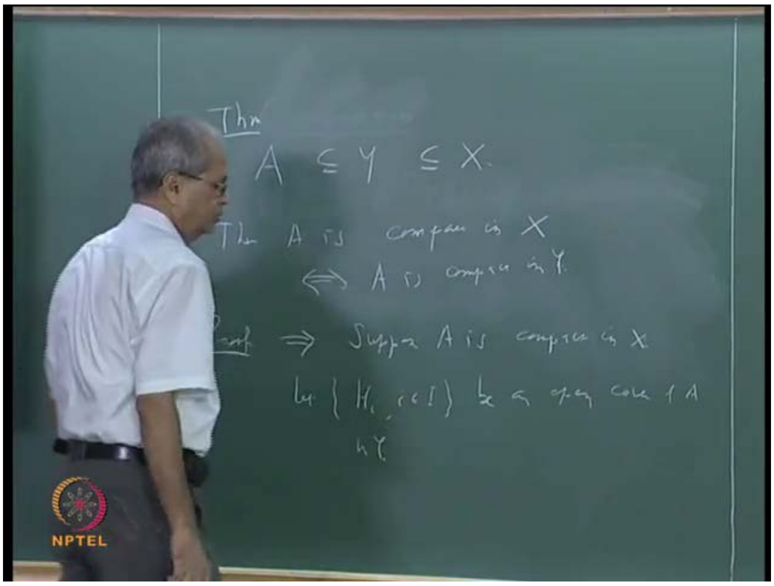
There are many examples of infinite sets, which are also compact and we shall see those examples in due course. But before proceeding that we can also mention one thing, if X d is a discrete metric space, then those are the only example of compact sets. As I said discrete metric space its main use is to understand the concepts. Sometimes whenever we define something, we intuitively feel something about that definition and whether that intuitive feeling is correct or not can be checked by looking at the discrete metric space.

So, let us say suppose now X d is a discrete metric space. Suppose now X d is a discrete metric space, then suppose we take subset A of X. Suppose I take subset A of X, then what I want to say is the following. Then A is compact if and only if A is finite. Of course, this way is true in any metric space. If A is finite, then A is compact. That we have that already solved that part is true in any metric space. What we want to say further is that in a discrete metric space, those are the only possible example of compact sets.

In other words in a discrete metric space A is compact implies A is finite. Alright how does one prove this? That is how will we prove this part A in A discrete metric space A is compact implies A is finite? Remember we have, we have already absorbed in a discrete metric space, every set is open. Every subset is open. So, in particular I can say that every single term it is open set. So, I can we can always say that, A is I can always write A as union of this single term sets. You know that single term set. Where you take or in other words, suppose we take this family A belonging to A.

This is an open cover for A, right? This is an open cover for. This open cover must have a finite sub cover, if A is compact. This open cover must have a finite sub cover that means what? That means there must exist a finite number of these sets, whose union covers A, right? But this set has single term, so that means a finite number of single term sets should cover A, right? That means A must be finite. Alright and where will this prove fails in an arbitrary metric space or in arbitrary metric space? When this is single cover of A, but not open cover, because this sets may not be open in arbitrary metric space.

(Refer Slide Time: 12:32)



Alright then, of course, that again leads to a question. Fine then what are the examples of compact sets in other metric spaces? We shall we shall answer that question it due course. But before proceeding to other things, let us also observe one more thing.

Compactness is one concept, which in a certain sense does not depend on, which metric space you are thinking about. For example, suppose I take a situation like this.

With say A is contained in y and y is contained in X. Then we have, we have seen that y can be itself regarded as a metric space. Now when I talk of A as a subset of y and A as a subset of X, right of course, if you take A as a subset of y and A as a subset of X. The meaning of open cover will change. All right for example, when you say some G i is an open cover of A, that if G, if you regard A as a subset of y, then each of this G i are open subset of y. On the other hand, if you regard A as a subset of X, then each of those G i are open sunset of X.

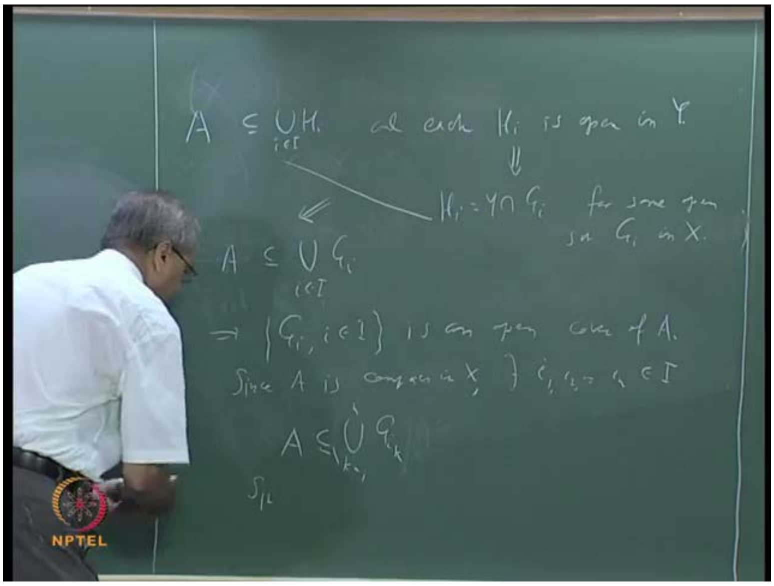
It can happen that a set is open in y, but not open in x etcetera. We have seen that these two things are different. We have also seen, what is the relationship between these two things? For example, y itself is always open in y, but it could not be opened in x. But what we will show is that, in case of compactness that difference does not exist. Whether you regard A is compact regards subset of y, if and only if its compact regarded as a subset of x.

So, in that sense this space, X really does not matter, when you are talking about the compactness of A. So, if we just write this as a theorem. Then A is compact, we shall simply say, A is compact in x. Compact in X means, compact regarded as a subset of X, if and only if A is compact in y. After proving this theorem, now simple we can simply talk of compact set, without bothering about where exactly that set lies. So, we can we can then sort of assume that, we can just talk of metric space itself as compact.

You can say that for example, saying that a metric space is open or metric space is closed does not make any sense, because X is always open in X or X is always closed in X. But X may not be compact in X. That is that is totally different notation. Alright, let us first see that, how we can prove this? To do this, we basically remember one old characterisation of open sets.

What we have said is that, if you take a set G of y then G is open in y, if and only if G in y is intersection some open set in X. G y intersection some open set in X. In fact basically, that is the characterisation that we shall use. So, let us say suppose A is compact. Let us say we want to prove this, suppose A is compact in X, then we want to prove that A is compact in y. A is compact in y. So, to show that A is compact in y, we should consider some open cover of A in y. So, let us say that, let us I call that open cover, H I, H I, i belonging to I, be an open cover of A in y.

(Refer Slide Time: 16:44)



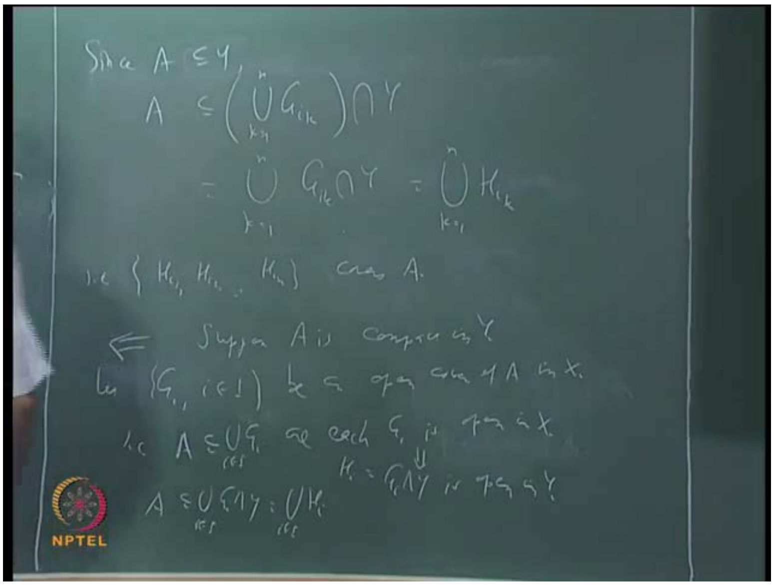
Now, what is the meaning of that? It means that A is contained in the union of H i and each H I, is open in y. That means the two things. First thing is that A is contained in union of H i and each H i is open in y. Alright, now if each H i is open in y, what does this mean? H i is open in y means, H i is y intersection some open set in X. That is what we have seen already, every open set in y is an intersection of y with some open sets in some open set in G.

So that means, since H i is open in y this implies that H i is equal to y intersection G i. H i is equal to y intersection G I, for some open set G i in x. Alright, now what we can observe from here is that, since A is contained in H I, that means A is contained in y intersection G i. A is contained in y intersection G i. But A is already contained in y. So, this two things will imply that A is contained in union of G I, i belonging to I. So, this part will imply, that is these two things will imply, that is A is contained in union H i and each H i is y intersection G I, that will imply that a is contained in union of G i. Is that clear?

But, each G i is open in X, that means, this is an open cover of A. That is that is this means G I, i belonging to I is an open cover of A. But we have to assume that A is compact in X. We have assumed that A is compact in X. So, every open cover of a must have a finite sub cover. So, since A is compact in X, this open cover has finite sub cover. This open cover has finite sub cover. What does this mean? That there exist a finitely many subsets here. Let us say G i one G i two etcetera. Such that A is contained in union of those finite numbers of subsets. So, I can say that suppose A is compact in X, there exist some indices suppose I call say i 1, i 2, suppose there are n of them. i n, in I.

Such that A is contained in union of sets corresponding to this particular indices A i 1, A i 2, etcetera. So, let say that A is contained in union. Let us say G i k. k going from 1 to n. Again, we use the fact that G i k, is nothing but y intersection ((Refer Time: 20:28)) sorry G is a is a, is fine we will come to that. A is contained in G i k, A is contained in union of G i k, but A is also contained in y. A is a subset of y. Fine since A is contained in union of G i k, I can say it is also contained in this intersection. I will write it later, this will confuse. Since A is contained in y. Continued there, since A is also contained in y and A is also contained in this. A is contained in the intersection of this and y, right?

(Refer Slide Time: 21:07)



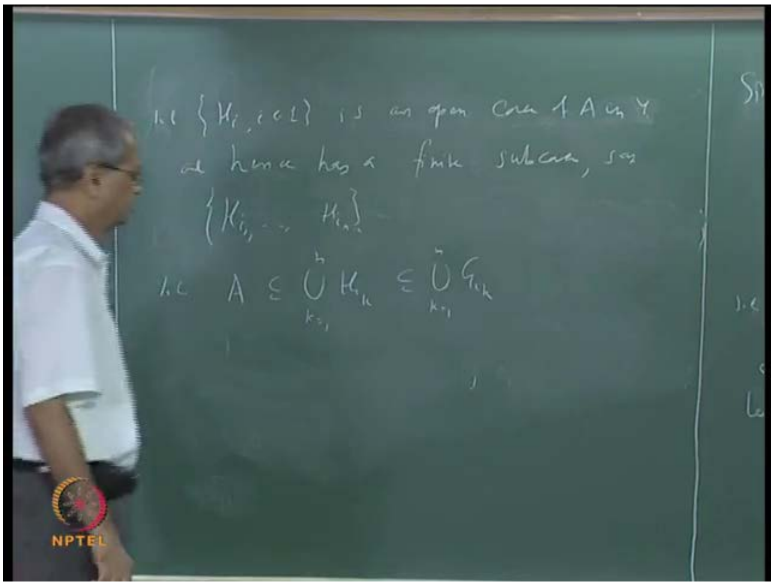
So, A is contained in union of G i k. k going from 1to n, intersection y. Now it is an elementary set theory. This is nothing but union of G i k, intersection y. That is this right hand side is nothing but union k going from 1 to n, G i k intersection y. What we have seen here for each index I, y intersection G i is nothing but H i. So, G i k, intersection y is nothing but H i k. So, this is nothing but union k going from 1 to n, H i k. In other words, this last, what we have put is, A is contained in union of H i k, k going from 1 to n. So, that means that is this, H i 1, H i 2 etcetera H i n covers A.

So, again remember starting from this open cover of A, we have produced a finite sub cover. Remember that is important this H i 1, H i 2, those are sets from the same cover. Which cover A. So that shows that every open cover of A in y has a finite sub cover. So, that shows that A is compact in y that shows that A is compact in y. Okay, is it clear now? How to go other way? Other way you start with assuming that A is compact in y, that is even easy, suppose A is compact in y. That we have understood A is compact in x, to start to show that A is compact means you would start from some open cover of A in X.

So, let us say that let us say G i, i belonging to I be an open cover of A in x. Again let me remind, this means that A is contained in union of G i and let us take me that is A is contained in union of G i, i belonging to I and each G I, is open in X, but if G i is open in X, but if we take its intersection with y, that will be open in y. So, each G i, is open in X, this will imply that G i, intersection y is open in y. Basically in a certain sense, we are reversing the steps not exactly.

Each so G i intersection y is open in y. So, suppose I call that set as H i. H i as G i, intersection y, then what we can say is that, since A is contained in union G i, and A is also contained in y, that will give that A is contained in union G i intersection y. Which is same as saying that A is contained in union H i that is union h i, i belonging to I. Alright, we continue the proof here. So, this last thing is same as saying that this H i is an open cover of A. This H i is an open cover of A in y.

(Refer Slide Time: 25:56)



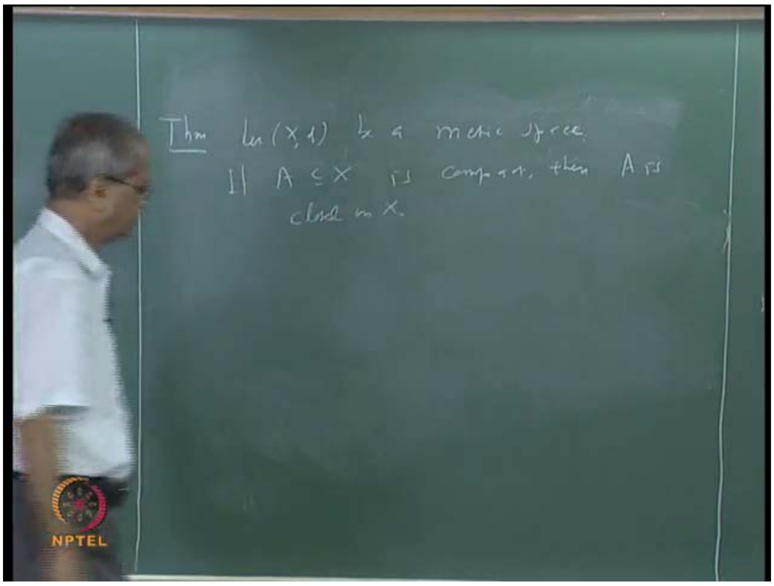
That is this H i, i belonging to I, is this H i is an open cover of A in y. And we have assumed that A is compact in y. We have assumed that A is compact in y. So, hence it has a finite sub cover. Hence, it has a finite sub cover. Suppose I call that sub cover H i 1,h i 2 etcetera H i n. Then that means, that is A is contained in union H i k. k going from 1 to n, but now we can say that each H i is G i intersection y. Each H i is G i intersection y. So, we can say that each H i is contained in G i. Each H i is contained in G i.

So, I can say that this is contained in union G i k. k going from 1 to n, but this last thing means that G i 1, G i 2, G i n covers A. That is a finite sub cover of this G i, i belong to I. So, again we started with an open cover of A in X. We produced a finite sub cover of that. So, that shows that A is compact in X. So, starting from the assumption that A is compact in y. We have now proved that A is compact in X. Is that clear? Now we have proved both things, so if A is a subset of y, saying that A is compact in y and A is compact in X, these two things are the same.

There is no difference in this. So, that is why now here afterwards, when we can when we talk of a subset A of X, without bothering about what X is, we can simply say that X is a compact set. Also, simply we can now talk about X as a compact metric space. Let me again remind X as a compact metric space makes sense, but X as an open metric space or X is a closed metric space that does not make any sense. Now let us also see some properties of compact metric space.

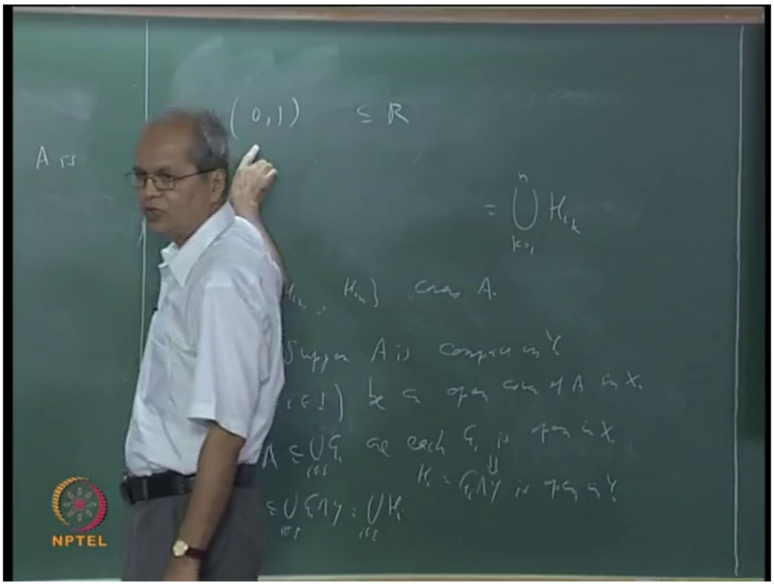
So, that that helps us in recognising which sets are compact in, for example, let us say. We want to know which sets which subsets of real line are compact. Of course, we already know till now, we know all finite subsets are compact. To know what are the other sets are compact, if we show that every compact set must produce some properties, then if we come across a set that does not have that property. We can certainly say that set is not compact.

(Refer Slide Time: 29:29)



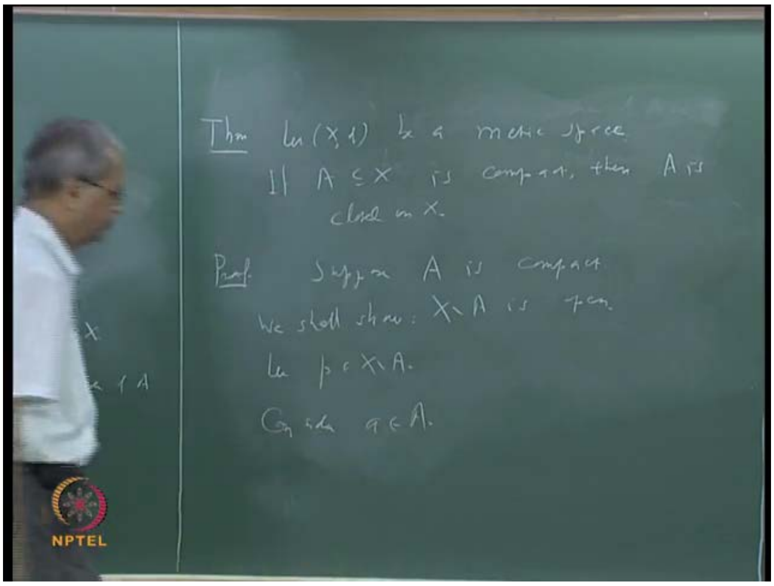
So, to that extent, let me just prove one following theorem. Let x,d be a metric space, what we shall prove is that, every compact set must be closed. Every compact set must be closed. So, if A is contained in x is compact, then A is closed in x. That is if A is compact in x then, A is closed in x. What is the use of this theorem? That suppose we come across a set which is not closed, we can certainly say that it is not compact.

(Refer Slide Time: 25:56)



For example, we can say that, for example, suppose I taker this interval 0 to 1 are a subset of R. Open interval 0 to 1 then, we then we know that this is not closed. Hence by this theorem, it is not compact. So, though this does not, though this does not still, which are the compact sets, it certainly says which are not compact set. So, if a set is not closed, we can we can immediately conclude it is not a compact set. That is after proving this theorem, but anyway that theorem we shall prove now.

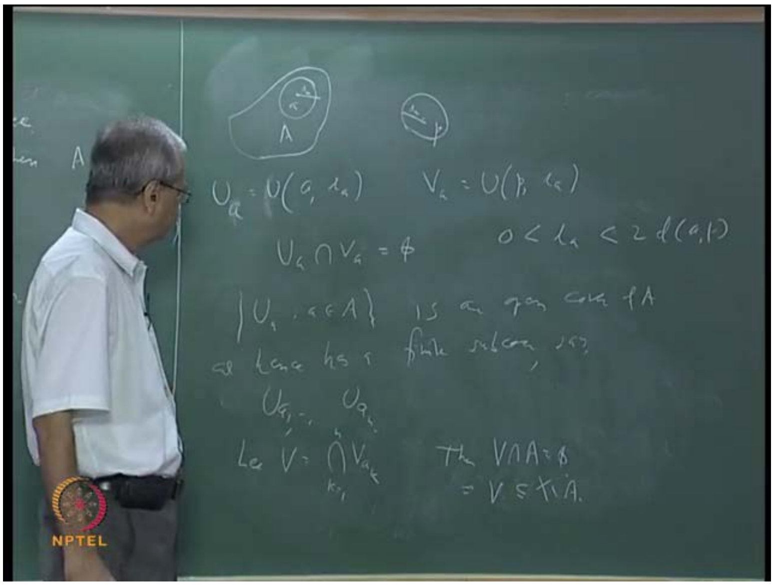
(Refer Slide Time: 31:09)



Suppose A is compact, alright we want to show that A is closed. We want to show that A is closed. Is it same as showing that x minus A is open. We have we have seen that, to say that a set is closed is same as its compliment is open. So, we shall show that, we shall show x minus A is open. So, how does one show that something is open? If you take any point in this, that is if you take any point in we shall show there exist an open ball with centre at that point if it is contained in that.

So, suppose I call that point as p. Let us say that p belong to x minus A. Then, we should find some positive number. Let us say r, such that open ball with centre at p and radius r is completely inside x minus A. If we can prove that that will be that x minus A is open. That will mean that A is closed. We proceed as follows. So, consider small a in a then, what we know is that, let me, I think draw some picture.

(Refer Slide Time: 33:07)



So, that may be clear. So, suppose this is the set A and this is the point p, which is outside A. Which is the outside A and a is a point in A small a is a point in A. What I can do is that, I can consider open balls with centre at a and centre at p, such that they are disjoint. That can be always done. What is the advantage? We can take the radius of each of this balls less than half of the distance between a and p. So, that actually does not matter, what radius I take. So, what I can say is I can always consider a subset to open balls with centre at a and centre at p, such that, they are disjoint.

They are disjoint. So, and let me call, give some notation, suppose I call, let us say U suffix a. Sorry, let us say let us say, U suffix a, as an open ball with centre at a and let us say radius r suffix a. Let us say radius r suffix a. Suppose v suffix a, as an open ball with radius p. Sorry, centre at p. U open ball with centre at p and radius r suffix p. Not r suffix p. Let me also, let me call that also as r suffix a.

What is r suffix a? r suffix a is some number that I want to choose in such a way, that U a intersection V a should be empty. I want to choose a number r suffix a, in such a way that the intersection U a and V a is empty. So, how should that number be chosen? It can be chosen that is, all that we need is that this is a wall, this is the radius r. Suppose a and that is the radius. This is also suppose r suffix a. So, if I choose r suffix a, in such way that two times r suffix a, is less then distance between a and p.

Then this will happen. That is you choose any number r say such that 0 less then r suffix a, less than 2 times distance between a and p. That is clear? So, let us again recall, what we have done. We have taken any point p in x minus a, and take any small a in A, then choose a number r a, which is strictly less than 2. Which is positive and strictly less than 2 distance between a and p. Then that U suffix a as U a r a. V suffix a as U p r a. Alright, now consider this family U suffix a, a belonging to A. That is an open cover of A. is an open cover of A.

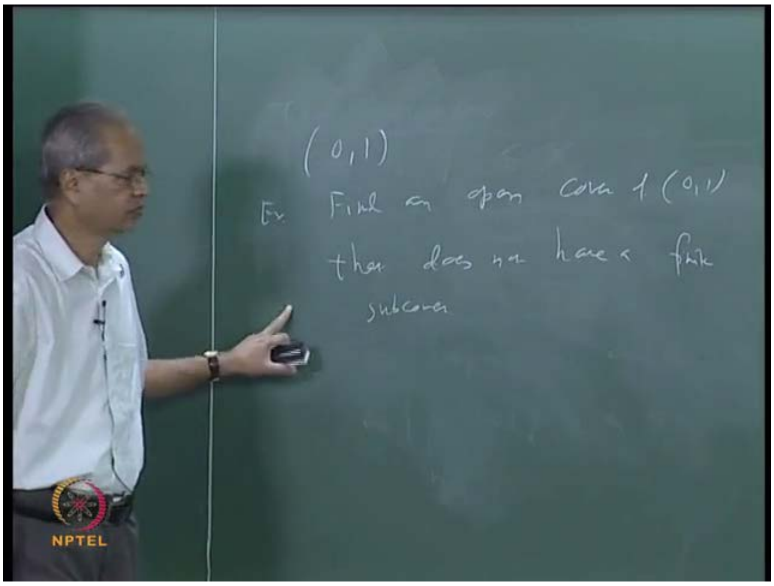
Obviously each of this is an open set, because this is open ball. Why it is an open cover of A? Because A belong to U a. a belong to U a and we are taking all a in A. So, this is obviously an open cover of a. Since a is compact, it should have a finite sub cover. Hence, has a finite sub cover has a finite sub cover. Suppose that finite sub cover, so say U a 1, U a 2 etcetera U a n. So, that means a is contained in U a 1 union U a 2 union U a n.

Now, consider the corresponding sets V a 1, V a 2. V a n and consider their intersection. So, let us say that, let V be intersection of v let us say V a k. k going from 1 to n. Then, since V a 1 intersection U a 1 is empty, v intersection u a 1 is empty etcetera. So, in other words, what I want to say is that V in fact what I will say is, that then V intersection a, is empty V intersection a is empty. Why V intersection a, is empty? Because V intersection U a 1, U a 2, U a n etcetera, everything is empty and a is contained in their union.

A is contained in their union. So, why V intersection a is empty, but why V intersection a is empty is same as saying that V is contained in x minus a. Why v intersection a, is empty? This means V is contained in x minus a. Alright, but is V an open ball? It is an intersection of finite number of open balls. So, if you take the radius is the minimum of r a 1, r a 2, r a n that is the thing in open ball with centre at p and radius that whatever is that number minimum. So, this is an open ball. Which is completely contained in x minus a with the centre at p with the centre at p.

So, that shows that x minus a, is open. that shows that x minus a, is open. So, what did we show? That every compact set is closed Every compact set is closed. So, in particular you can say, if you come across a set which is not closed, then that set cannot be compact. I shall just give you an exercise.

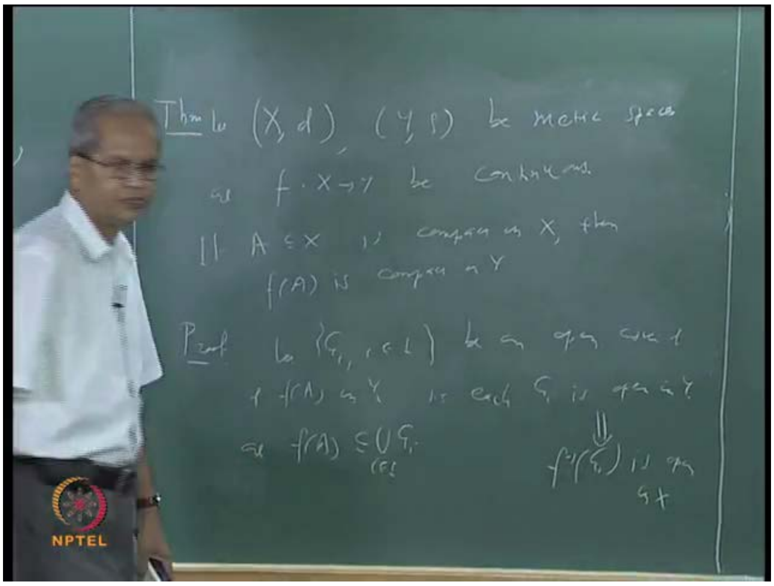
(Refer Slide Time: 39:58)



We have just now seen that, this set 0,1 is not closed. Hence, it is not compact. Alright now if a set is not compact by definition, what should happen? For a compact set, we mean that every open cover must have a finite sub cover. If a set is not compact, what should happen? There should exist, some open cover. Which does not have a finite sub cover. Now this set is not compact, so this should have some open cover. Which does not have a finite sub cover.

So, that is an exercise. Exercise is this, find and open cover of 0 to 1, that does not have a finite sub cover. That does not have a finite sub cover. Next we shall show that, the image of a compact set under a continuous function is always compact. We have done similar things, in case of connected sets. We had shown that image of connected set, under a continuous function is connected. Similarly, we will show that image of a compact set under a continuous function is compact.

(Refer Slide Time: 41:52)



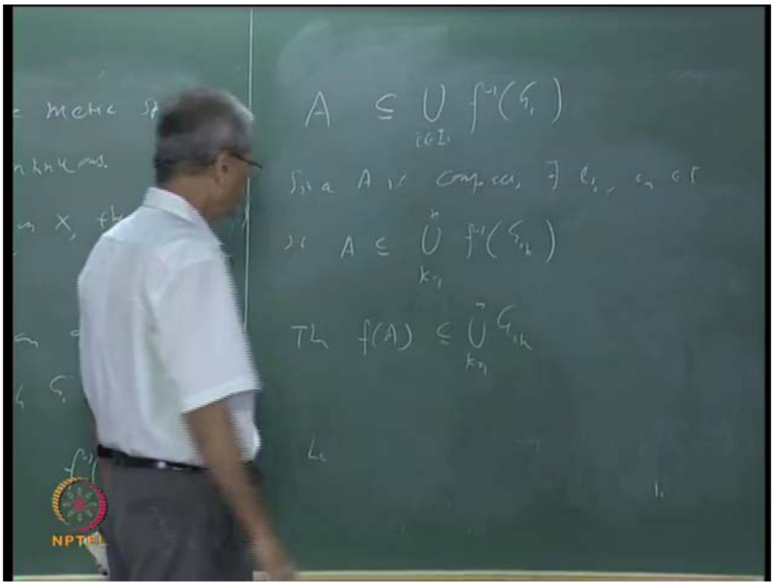
So, to do that, we have to discuss 2 metric spaces. Let x, d and y rho be metric spaces and f from x to y be continuous. That is in 2 metric spaces and a continuous function from one metric space to other metric space. What we want to say is that, if we take a subset a of x and if that subset is compact. Then f of a is compact in y. So, if a is a subset of x, is compact in x ,then f of a is compact in y. In words the image of a compact set under a continuous function is compact. Usually, it as saying that a continuous image of a compact set is compact.

We want to show that f of a is compact in y. So, by definition what does it mean? You, we have to show that, if you consider open cover f of a, then it has a finite sub cover. So, let us say that let G i, i belonging to I be an open cover. Be an open cover of f of a in y. What does it mean? It means two things, each G I, is open in y. That is, let me write that is each G i is open in y and f of a is contained in union G i Alright, but till now we have not used the fact that f is continuous.

But we have, we have proven earlier that if f is continuous, then inverse image of an open set is open. So, if each G i is open, it follows from this that f inverse G i is open in

x. Now f of A, is contained in union G i. So this means that, A is contained in f inverse G i. A is contained in union of, f inverse G i. So, this last thing will imply.

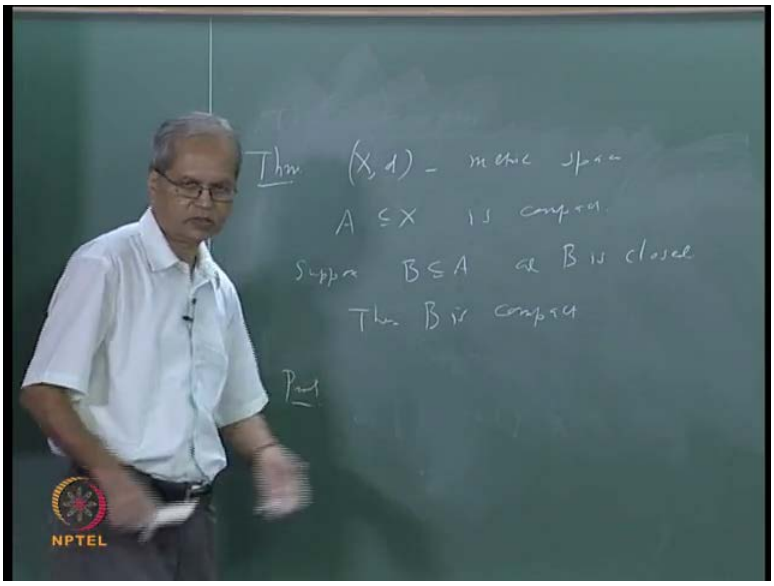
(Refer Slide Time: 45:36)



That A is contained in union f inverse G i. Each of the set f inverse G i is an open set in x. In other words, what you have here is an open cover of A in x. Alright, but we have assumed that A is compact in x. So, this open cover must have a finite sub cover. This open cover must have a finite sub cover. That means what? That means there should exist, some finite number of indices. Let us say 1,2 etcetera k. Such that A is contained in union of, f inverse G 1, f inverse G i 1, f inverse G i 2, etcetera. So, since A is compact, since A is compact, there exists, let us indices i1,i 2,i n, in I. Such that A is contained in union f inverse g.

Let us say G i k, k going from 1 to n. Alright, but then what can we say about f of A? f of A must contained in union of G i k, k going from 1to n. Then f of A must contained in union of G i k, k going from 1to n. Now does this prove whatever we wanted? We started with an open cover of, f of A and now we have produced a finite sub cover of that. So, this shows that every open cover of, f of A has a finite sub cover. That is same as saying f of A is compact. So, what we have proved, that a continuous image of a compact set is compact. Let us also prove one more thing. That suppose you take a compact set.

(Refer Slide Time: 48:33)



That suppose you take a compact set and if you take a closed subset of that then, that should also be compact. So, that a theorem suppose x, d is metric space. Let us say A is a subset of x, is compact. Suppose b is a subset of A, suppose B is a subset of A and B is closed. Of course, here this is slightly ((Refer Time:49:23)) by B is closed, what we mean? Whether b is closed in A or B is closed in x. We shall show that does not matter very much, in both the case.

We can show that B is compact, then B is compact. In other words closed subset, of a compact set is again compact. Now to prove this, alright since we do not have much time, I will just give you an idea, how we are going to prove this? To show that B is compact, what we need to show is that every open cover of B has a finite sub cover. We know that A has that property. So, the obvious idea should be to construct. If you start from an open cover, take any open cover of B. From that, if we can construct a cover of A. Then use the compact ness of A.

That is an idea. That is start form and open cover of A, sorry, start form an open cover of B and from an open cover of B find construct an open cover of A. Then using the compactness of A, that open cover of A will have a finite sub cover. Then come back to from that finite sub cover construct a finite sub cover of B. That is the idea, we shall, we shall see how exactly that idea is to be applied in the next class.