Real Analysis
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Lecture - 27
Connected Sets

So, we shall continue with our discussion on this property of connectedness of sets in metric spaces. We shall begin with the theorem about the characterization of connectedness in the subsets of real line. Namely, we shall show that a subset of real line is connected, if and only if it is an interval.

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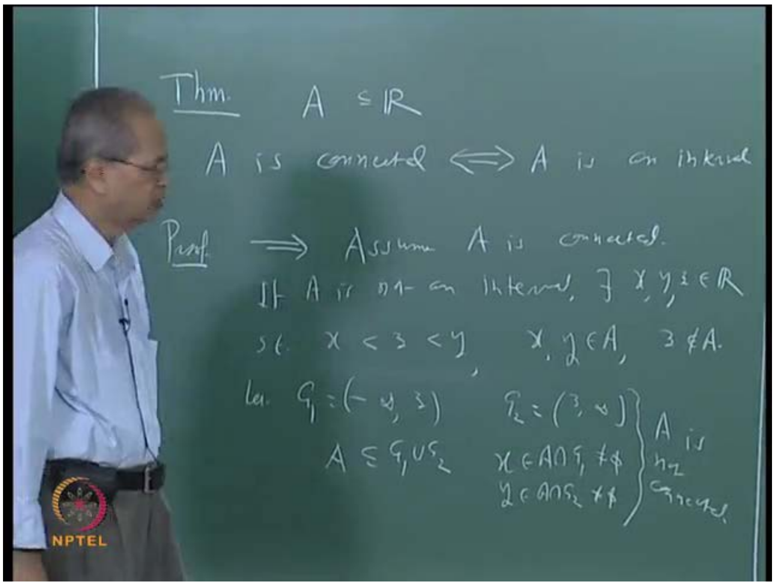

So, let us begin with that theorem. So, this is the theorem suppose A is a subset of R. Then what I want to show is that A is connected, if and only if A is an interval. As usual it is first assumed that A is connected. Now, as you have seen that if A is not an interval then there should exist three numbers x, y and z

Such that x less than z less than y and x and y belong to A, but z does not belong to A. So, if A is not an interval there exists a three real numbers x, y, z. Such that x less than z less than y x and y belong to a and z does not belong to A. That is the meaning of saying that A is not an interval. In this case we should show that A is not connected that will contradict this assume that A is connected. To shown that A is connected and to show that A is not connected, we should produce two open sets G 1 and G 2 satisfying those property. In this case it is easy let G 1 be equal to minus infinity to z and G 2 be equal to z to infinity then G 1 and G 2 are disjoint.

Their union is in fact r minus this z and since z does not belong to A, A is contained in union. So, A is contained in G 1 union G 2 and as far as a intersection G 1 is concerned. Since, x belongs to A and x also belongs to this interval G 1. So, x belongs to A intersection G 1, so x belongs to A intersection G 1. So, this is non-empty, similarly y belongs to A intersection G 2 that is non-empty. All these things show that A is not connected. That is A contradiction to the assumption that we started with.

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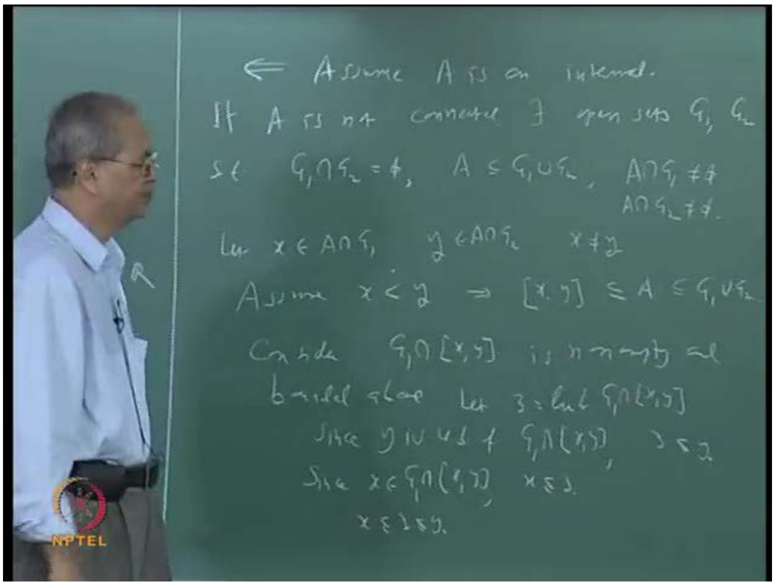

Now, let us look at this way assume that A is an interval and if A is not connected again, there should exist two open set G 1 and G 2 with the required property. If A is not connected there exists open sets G 1, G 2. Such that we have first of all they are disjoint G 1 intersection G 2 is empty and A intersection A. So, A is contained in G 1 union G 2. It has non-empty intersection with both the sets A intersection G 1 is non-empty and A intersection G 2 is also non-empty. Since, A intersection G 1 is non-empty we can consider some real number belonging to that. Let x belongs to A intersection G 1 and similarly y belongs to A intersection G 2.

Since, x belongs to G 1 and y belongs to G 2 they must be different x and y cannot be the same. So, x, x naught equal to y and as you have seen if x is not equal to y. Since, they are real numbers either x is less than y or x is bigger than y. We can assume any one of those things. So, assume x is less than y. Now, x and y both belong to A x is less than y. We have assume that A is an interval that means all the points lying between x and y must be in A. It is same as saying that this closed interval with the end points x and y. That must be completely contained in A that is, because of our assumption that A is A is an interval A is contained in G 1 union G 2

So, in particular this interval is contained in G 1 union G 2. Now, what we do is that we consider the intersection of this interval with this first set G 1. So, consider G 1 intersection x y then we observe two things, this is non-empty. Because, x belongs this interval also x belongs to G 1. So, this non-empty and it is bounded above, because this set is bounded above. So, for every element in that intersection belongs to this also, y is an upper bound. So, this is non-empty and bounded above.

If it is non-empty and bounded above, then we know that by l u B action every set, which is non-empty and bounded above, must have A least upper bound. So, let us call that least upper bound to be z. So, let z be l u B of this set G 1 intersection x, y. Now, first of all let us see where this x should lie. In fact crux of the proof is on that first of all z is the least upper bound and y is an upper bound. So, z must be less than or equal to y. So, z is l u B of x, since y is upper bound of this G 1 intersection x y. Every upper bound must be bigger than or equal to least upper bound, so z is less than or equal to y. On the other hand, x belongs to this set and z is an upper bound.

So, x must be less than or equal to z. So, since x belongs to this since x belongs to G 1 intersection x y and z is least upper bound. We have x less than or equal to z that means what x less than or equal to z less than or equal to y. So, x less than or equal to z less than or equal to y, let us just try to visualize this. So, this is an interval x, y, so z is somewhere here z lies somewhere in the interval.

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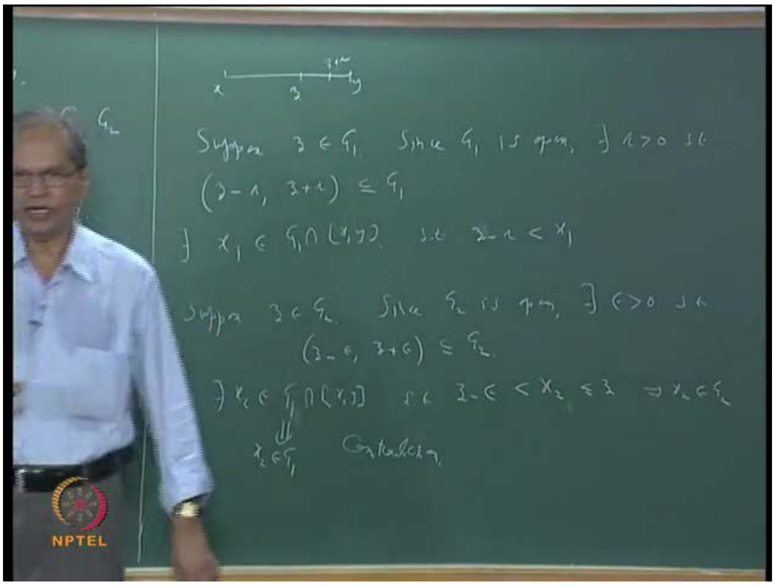

This whole interval lies in G 1 union G 2 this whole interval lies in G 1 union G 2. So, z must belong to G 1 or z must belong to G 2. Let us see the consequences of both of these. Suppose, z belongs to G 1 suppose z belongs to G 1 then G 1 is an open set G 1 is an open set. So, it must contain an open ball with center at z and in the in the real line open balls are nothing but open intervals. So, we can say that, so since G 1 is open we can say that since G 1 is open there exists r bigger than zero. Such that this interval z minus r to z plus r that is completely contained in G 1. Now, z minus see z is the least upper bound z is the least upper bound and z minus r is a number, which is strictly less than that.

So, that means what there should exist some number, which is strictly bigger than this. Since, z is the least upper bound and z minus r is strictly less than that. So, we can say there exists, let us say some number. Suppose, I call that number x 1 in this interval G 1 intersection x, y. Such that, so you can say that there exists x 1 belong to this G 1 intersection x y. Such that z minus r is less than x such that z minus r is less than x 1. So, let me leave this argument here. Let us see what happens if z belongs to G 2. So, suppose now z belongs to G 2, if z belongs to G 2 again we can say that there will exist some positive number.

Such that open interval with the center at z is completely contained in G 2 then, since G 2 is open there exists epsilon instead of r we call it epsilon. Such that z minus epsilon to z plus epsilon is completely contained in G 2, z minus epsilon to z plus epsilon is completely contained in G 2. Now, if z minus epsilon to z plus epsilon is completely contained in G 2 what we can say is that again. Let me let us make this same argument here, since z minus epsilon is not an upper bound there exist a number. Suppose, I call that number x 2, such that that number x 2 belongs to this interval G 1 intersection this set G 1 intersection x y.

It must be strictly bigger than z minus epsilon. So, we can say there exists x 2 in G 2 intersection x y G 1 intersection. Because, remember z is the least upper bound of this set G 1. So, any number smaller than that is not an upper bound. So, if you take any number smaller than that there should exist some element here, which is strictly bigger than that number, that is the argument we are using. So, there exists x 2 in this side such that z minus epsilon is less than x 2. Of course, x 2 is less than or equal to z x 2 is less than or equal to z, because x 2 is in this and z is the least upper bound. In other words this x 2 belongs to this interval, which is completely inside G 2, which is completely inside G 2.

So, and also x 2 belongs to G 1, because of this. So, see this, this implies in particular that x 2 belongs to G 1. That part implies that x 2 belongs to G 2. Now, this cannot happen, because G 1 intersection G 2 is empty. So, z belongs to G 2 is not possible z belongs to G 2. So, this is a contradiction. Now, in a similar way is this clear? This part is clear? In a similar way we can get A contradiction here.

Actually, now in this case we should have considered this part z to z plus r. Suppose, we take some number lying between z to z plus r and also, which is in this interval, also which is in this interval. Let us say that see, it is clear that is since this whole interval is contained in G 1. See, r cannot be strictly bigger than y, because z is the least upper bound z is a least upper bound. So, if you take any number, which is let see this z plus r can be somewhere here z plus r can be somewhere. So, suppose we take some number lying between z and y and z and z plus r. That is what I want is you add some number to it which should be less than both y as well as z plus r. Suppose, you do that suppose that number is what I call x 1, let us say consider x 1.

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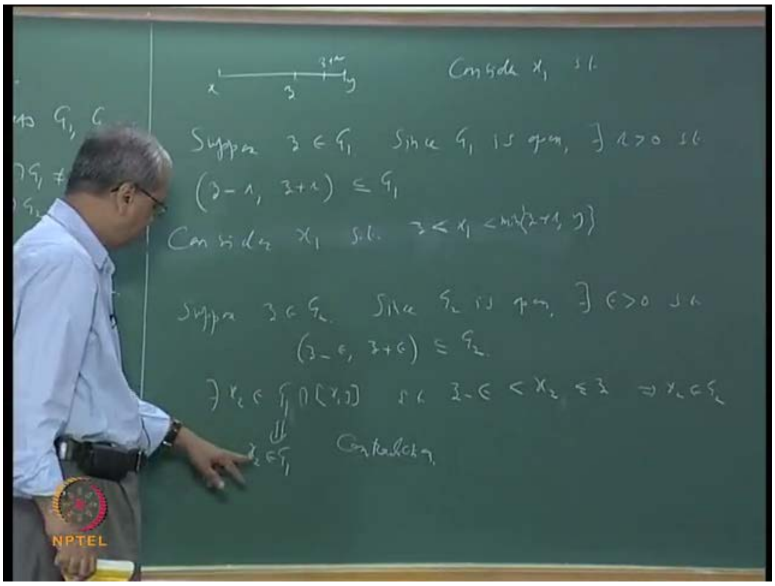

Such that I think let us let me remove this part. So, consider x 1 consider x 1, such that a x 1 is, let us say z is less than or equal to x 1 z is less than or equal to x 1. It is strictly less than both z plus r and y. Let us say minimum of z plus r and y. In fact I can take strict inequality here z is less than x 1. That is strictly less than z plus r and y, now this since this x 1 is less than z plus 1. It is in G 1 it is in G 1, but it is not in it is in G 1.

In a similar way we can show that this x 1, since it is strictly bigger than z remember this x 1 is strictly bigger than z and z is the least upper bound z is also. So, x 1 cannot be in this set G 1 intersection x 1, x 1 cannot be in this set that means it has, but this whole interval is contained in G 1 union G 2. So, x 2 must be in G 2. So, again A contradiction that is, we have shown that x 1 is both in G 1 as well as in G 2. So, that is again A contradiction, so contradiction to what contradiction to z belong to G 1. So, this contradiction shows that z does not belong to G 1. This argument shows that z cannot **b**elong to G 2, but z is in this interval x to y, which is containing G 1 union G 2.

Now, what is the basic source of all these contradictions? It is this assumption that A is not connected. We have assumed that A is not connected. So, we have got these sets G 1 and G 2 with these properties. So, this has led to all these, so this must be false. So, that means A must be connected then let us proceed to the other theorem, which I had mentioned yesterday.

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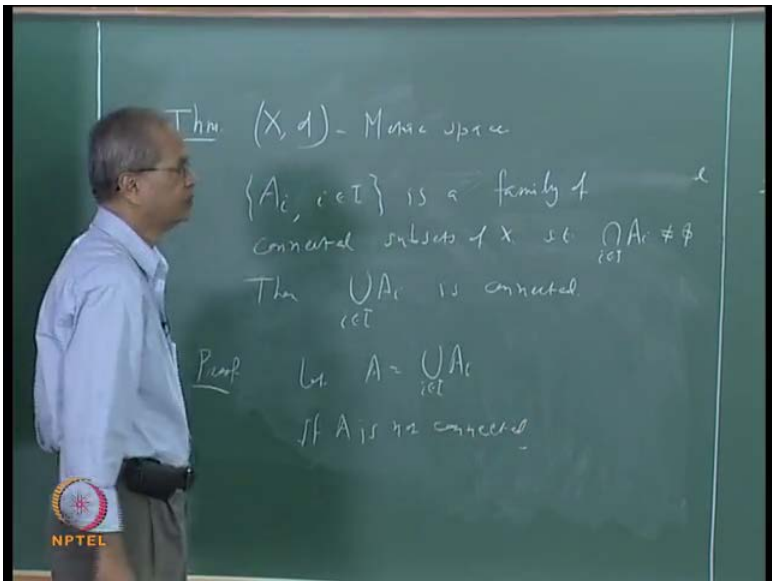

It is about the union of connected sets. So, suppose X d is a metric space. Let us say that we have taken a family of connected sets say A i. Let us say i belonging to big I is what we have called indexing set, it is an indexing set. So, let us say this is a family of connected subsets of x. What we want to show that is if intersection of these is non-empty. Then their union is also connected. So, let us say connected set such that intersection A i i belongs to I is non-empty. Then union A i I from is connected, let us look at the proof.

Now, this is fairly straight forward let us give some notation for this union. So, let A be equal to union A i i belonging to I. We shall just see what happens if A is not connected. Then get some conduction. So, if A is not connected I think I can just do this if A is not connected. Then there exists open sets G 1 G 2, such that G 1 and G 2 are disjoint A is contained in G 1 union G 2 A intersection G 1 and A intersection G 2. Both are non-empty, but A is a union of these sets. So, if A is contained in and let us see what follows from that. So, consider one A i.

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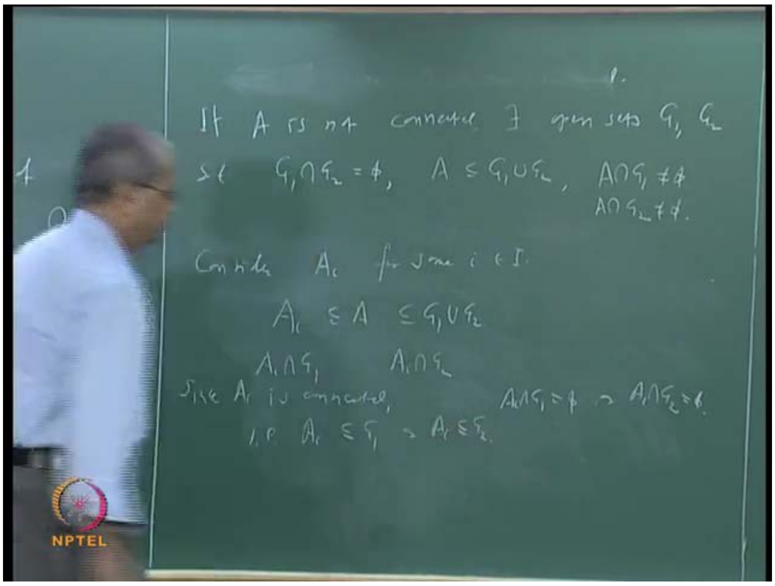

So, consider one such A i for some i belonging to I. Now, since A i is a subset of A, because A is a union of all such A i. This is contained in G 1 union G 2. So, A i is contained in G 1 union G 2 G 1 union G 2 are disjoint that is something that we already know. Let us look at, now A i intersection G 1 and A i intersection G 2. If both of these are non-empty if both of these are non-empty.

Then what is the meaning of that, see suppose you have two open sets G 1 and G 2, which are disjoint and A i is contained in their union. If A i intersection G 1 and if A i intersection G 2, if both of these are non-empty. That means that A i is not connected. That means that A i is not connected that is there, but our assumption is that each A i is a connected. Remember, here our assumption is each A i is connected. So, that means it is it cannot happen that both of these are non-empty, it cannot happen that both of these are non-empty.

So, at least one of them must be empty. So, since so what is the argument here? Now, since A i is connected, since A i is connected one of A i intersection G 1 or A i intersection. We can simply say A i intersection G 1 is empty or A i intersection G 2 is empty. Remember, none of these A i's can be an empty set. Because, in that case this intersection would have been empty. So, each A i is non empty, but what is the meaning of this? Suppose, A i intersection G 2 is empty, but A i is contained in G 1 union G 2. So, if it is intersection with G 1 is empty, it must be contained in G 2.

Similarly, if it is intersection with G 2 is empty then it must be contained in G 1. That means A i must be contained in G 1 or in G 2. So, that is this means that is A i is contained in G 1 or A i is contained in G 2. So, what did we show if we take any of these sets in the family each A i is either contained in G 1 or is contained in G 2. Now, let us see what are the possibilities it can of course, happen that some of the A i's are in G 1. Some of the A i's are in G 2, but if that happens is this possible see G 1 intersection G 1 and G 2 are disjoint. Suppose, some of the A i's are contained in G 1 and some of the A i's are contained in G 2, then their intersection will be empty.

So, since if this since this is non-empty all of them must be either contained in G 1 or all of them must be contained in G 2, but suppose all of them are contained in G 2. Let us say all of them are contained in G 2. That will mean all of them are contained in G 2 and that will contradict this that A intersection G 1 is non-empty. On the other hand if all of them are contained in G 1, then A is their median, then that will mean that A is contained in G 1. That will imply that A intersection G 2 is empty. So, whatever you do that will lead to a contradiction. Again, what is the source of this contradiction, our assumption that A is not connected.

So, that completes proof that if each A i is a connected set and if you take and if their intersection of all this family is non-empty. Then the union over that family again is A connected set. Then what we also want to know is how is the what is the relationship between. Suppose, A is connected then what can we say about A closure and it should be and one can expect that A closure should also be connected. In fact we will do something more any set, which lies between A and A closure is connected. If A is connected, so let us let me just state that.

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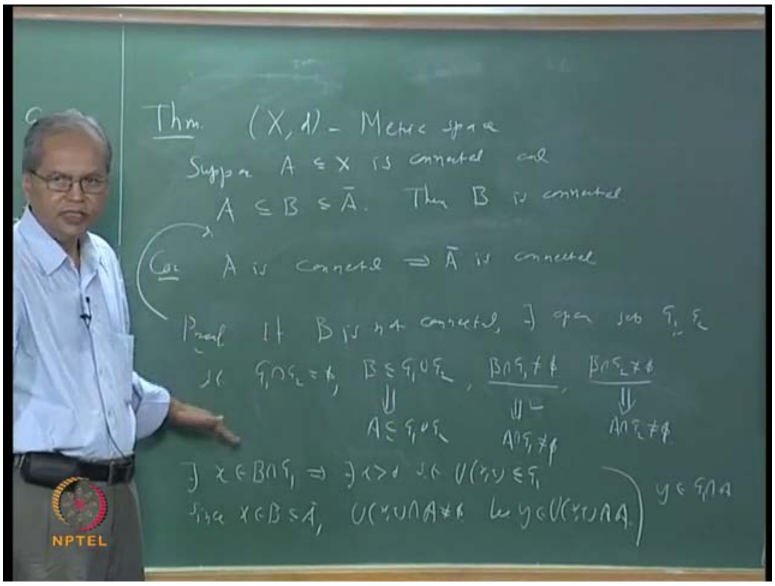

Again let us start from this is again this things are true in any arbitrary metric space. Remember, the earlier theorem we had proved for the sets in the real line. Whereas, this is true in any metric space. So, again let us say x p is a metric space then. Let us say suppose A contained in x is connected. There is another set B such that A is contained in A and B is contained in A closure, then B is connected. So, even before discussing the proof of this you can see one of the obvious consequences that is that if A is connected then A closure is connected. Because, in that case you can take just B as just equal to a closure. So, I can just write A quick corollary of this A is connected, this implies A closure is connected.

Now, let us look at the proof of this we need to show that B is connected. So, we start with the usual way if B is not connected. Then there should exist two open set G 1 and G 2 etcetera. So, if B is not connected there exists open sets G 1 G 2 such that G 1 intersection G 2 is empty, B is contained in G 1 union G 2. Then B intersection G 1 is non-empty and B intersection G 2 is also non-empty. We will show from here that if this is the case then A also cannot be connected.

That will be the contradiction to the assumption A is connected. To show that A is not connected we have to again consider the open set, construct open sets G 1 G 2 satisfying these properties just be replaced by A. Now, as far as this is concerned there is A is not G 1 and G 2 are disjoint anyway, but can you say that this implies that a is contained in G 1 union G 2. Because, A is contained in B and if B is contained in G 1 in union G 2. Then that will say that A is contained in G 1 union G 2. Next question is this from here B intersection G 1 is non-empty. Can I say that does this imply that A intersection G 1 is non-empty?

This is the question, what is the answer? See remember here, now in this we have used this part A is contained in B. We have not yet used this part B is contained in a closure, every point of B is A closure of point of A. Every point of B is A closure of point of A. So, let us start from a B intersection G 1 is non-empty, that means there exists some point in B intersection G 1. So, there exists let us say x in B intersection G 1 B intersection G 1. So, this x belongs to B and it also belongs to G 1. Now, since x belongs to G 1 and G 1 is an open set x belongs to B one and G 1. So, we can say that there exists an open ball with center at x which is contained in G 1.

So, this implies there exists r bigger than 0, such that u x r is completely contained in G 1. Can you see what is the next obvious thing to do x is in B, which is in a closure. So, that means x is A closure point of A. So, any open ball with center at x should have a non-empty intersection with A. Every open probable center at x should contain a point from A. So, since next thing is since x belongs to B, which is contained in a closure. What we must have is that u x r intersection A is non-empty. Now, what is the meaning of that u x r intersection A is non-empty consider some point in that.

So, let y belong to this let y belong to u x r intersection A, does it means that y belongs to A in particular, does it also means that y belongs to G 1? Because, u x r is completely contained in G 1, so and y is in u x r. So, it is contained in G 1. So, this these two things you imply that y belongs to G 1 intersection A. So, what we show starting from the fact that B intersection G 1 is non-empty, we showed that a intersection G 1 is non-empty. So, this is true here is the true right in a similar way from the fact that B intersection G 2 is non-empty. You will be able to show that a intersection G 2 is non-empty. So, similarly from here this will imply that A intersection G 2 is non-empty, but again so what is, let us again take the stock.

What does it mean? G 1 and G 2 are disjoint A is contained in G 1 union G 2 A intersection G 1 is non-empty a intersection G 2 is non-empty, which means that A is not connected and that is A contradiction. So, whenever A is connected its closure is also connected and not only that any set, which lies between A and A closure is also connected. Now, let us consider one important concept, which is associated with this idea of connectedness a given metric space may or may not be connected, but it will always have connected subsets.

For example, singleton set is also always connected, but we often seen examples of the sets, which are not singleton and still connected. So, one can ask the question can we think of something like maximum possible or biggest possible connected set. We have seen here for example, that if A is connected A closure is also connected. So, that is one way of enlarging connected sets, what is the other way? What you can do is that suppose you start with a connected set. Then you think of all those connected sets, which have non-empty intersection. For example, you take a family like this suppose you take all connected sets, which contain a given connected set or in particular you start with a point.

Consider all the connected sets, which contain that particular point then their union must also be connected. So, that way we can form new and new connected sets. So, the question is suppose you reach a stage beyond, which you cannot extend. Suppose, you reach a stage beyond, which you cannot extend. Then obviously such a set will be called a maximal connected set.

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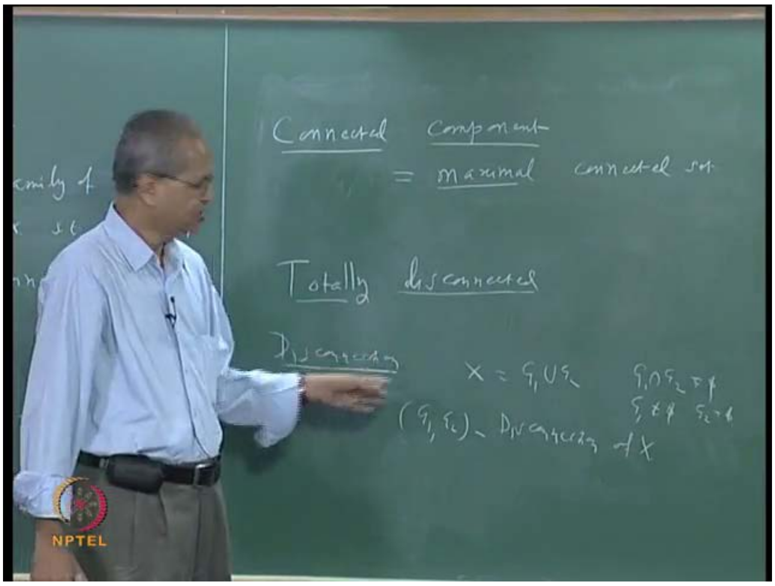

It is called connected component is the definition connected component is another way of saying it is a maximal connected set again recall our idea of the maximal elements.

Suppose, you take all the connected sets then there is this inclusion is already partial order in that. So, it is maximal in that sense or to make it. So, where do we say that a set is A connected component, it means that it is a maximal connected set it means it is connected. It is not properly contained in any other connected set. That is the meaning of maximal connected set. Let me repeat again a maximal connected set means, first of all it must be connected. It is not properly contained in any other connected set that is a component and what we can show is that even if a given metric space is not connected.

We can write that metric space always as a disjoint union of its connected components. Of course, the things can be very bad, for example if you look at a discrete metric space. If you look at a discrete metric space then, it is clear? That a singleton set is connected, but if you take any set containing two points. It is disconnected, which means in the case of discrete metric space. Every singleton is a connected component of you cannot do anything better than that. So, that is a space we can say or that is a kind of a thing that is as much opposite to being connected. So, things like that are called totally disconnected metric spaces like that are called totally disconnected.

Of course, this is not exactly what the definition of that, but that is the idea totally disconnected means, given any two distinct points you can find what is called disconnection. I have not defined this word disconnection, but let me just say again, what is meant by disconnection? Suppose, A is not connected then what should happen? This whole space is x and if that is not connected. Then it should be contained in two open sets G 1 and G 2 that is x is x is equal to G 1 union G 2. Then G 1 intersection G 2 is empty and G 1 and G 2 both are non-empty. So, normally we would have said G 1 intersection x is non-empty by G 1 intersection x is just G 1.

Similarly, G 2 intersection x is simply and x is we would have said x is contained in G 1 union G 2, but that is same as x is equal to gone union G 2. So, the metric space itself is not connected then you can find two open sets that is like this. So, this G 1 G 2 that is called a disconnection of x that is called a disconnection of x if such sets exist. Of course, given a metric space there could several disconnection given a metric space, there could be several disconnections. Like, for example if you take a discrete metric space. You can take G 1 as one singleton set and G 2 as its Complement that will be a disconnection.

You will do it for each single points, there will be several disconnection. So, what is meant by a totally disconnected metric space. It is that given any two points of that space you can find a disconnection. Such that G 1 contains one of those points and G 2 contains the other points that is in other words any two distinct points, can be separated by a disconnection. That is what is called totally disconnected metric space. The example of that, which we have seen till now is the discrete metric space. There are many other examples also, for example if you take the set of all rational numbers. That is also set of all rational numbers as a metric space not as a sub-space of r, but as metric space itself.

Then there also you will see that is also an example of a totally disconnected space. Then let me just say how does it follow that every metric space can retain as a disjoint union of its components. What we can see is that let us say first of all is it near that let me just say suppose X d is a metric space.

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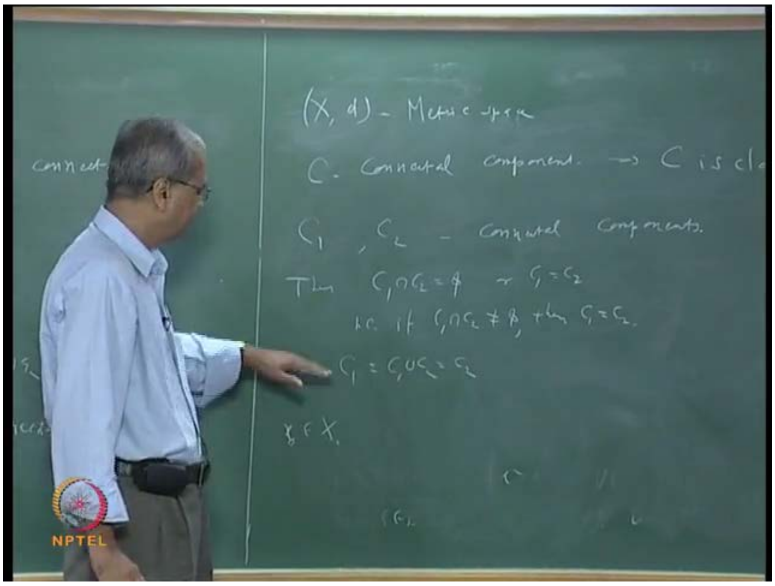

Suppose, C is a connected component we can we can see some of these properties of C is it is it obvious that C must be closed, is it clear? Remember, we have already shown that if C is connected then C closure is connected.

So, if C is maximal that means C is not properly connected in C closure, which is same at C and C closure must coincide. So, if C is a connected component then C is closed. Secondly, suppose you take two connected components C 1 and C 2 suppose C 1 and C 2 are connected components. Then either they should be disjoint or they should be same. You can see that is why then we then C 1 intersection C 2 is empty or C 1 is equal to C 2, that is if this is non-empty, which is say missing. That is if C 1 intersection C 2 is non-empty, then C 1 is equal to C 2 how does this last thing follow? Suppose, this is non-empty of then you look at C 1 union C 2.

If C 1 intersection C 2 is non-empty then C 1 union C 2 must be connected. If C 1 union C 2 must be connected C 1 and C 2. Both are components we know what C 1 is contained in C 1 union C 2 that is C 1 is contained in C 1 union C 2. C 1 intersection C 2 is non-empty this is a connected set, but if this is maximal, this must be coincide with C 1 union C 2. So, C 1 must be equal to C 1 union C 2 and similarly, C 2 must be equal to C 1 union C 2, because C 2 is also connected component C 2 is Also maximal connected set.

So, it cannot be properly contained in any other contained set and C 1 union C 2 contains C 2, that means these two must coincide. So, any two connected components either they are the same or they are disjoint then since a singleton x is a connected set. What we can do as I said just, now take all the take all the connected sets, which let us say suppose let us start with a point. Let us say x belongs to x or let me say x naught belongs to x then take all the connected sets, which contain x naught there. Then certainly one such set consists that is singleton x. So, take all the possible sets which contain x naught the consider their union.

Since, all of them contain x naught their intersection is non-empty. So, their union must be connected that union must be connected and that will be a connected component. Because, if any other connected set contains that union it also must contain x naught, which means already it is a part of the family. So, every singleton point x naught every point x naught is contained in some connected component. So, we just same as saying that x can be written as A union, if union of connected components. So, even though the metric space itself, let me again repeat. Even though the metric space itself is not connected we can always write it as union of its connected components. Any two such all these components are closed and any two distinct components are disjoint.

So, a metric space can be written as what is called disjoint of its connected components. Now, while discussing completeness we discussed, we also proved what is called Bannach's contraction mapping principle. As you said that is an example of what is called fixed point theorem. It is a theorem that shorter map has a fixed map. Now, connectedness also leads to some fixed point theorem. Let me just give an example we are using the fact that a set in a real line is also connected. If and only if it is an interval we have proved what is called intermediate value theorem.

What does the theorem say that is it if a function takes any two values. Suppose, earlier suppose a function goes from a connected set to a real line, if it takes any two values. Then it takes all the values in between those two values. That is an intermediate value theorem, using that we can say that.

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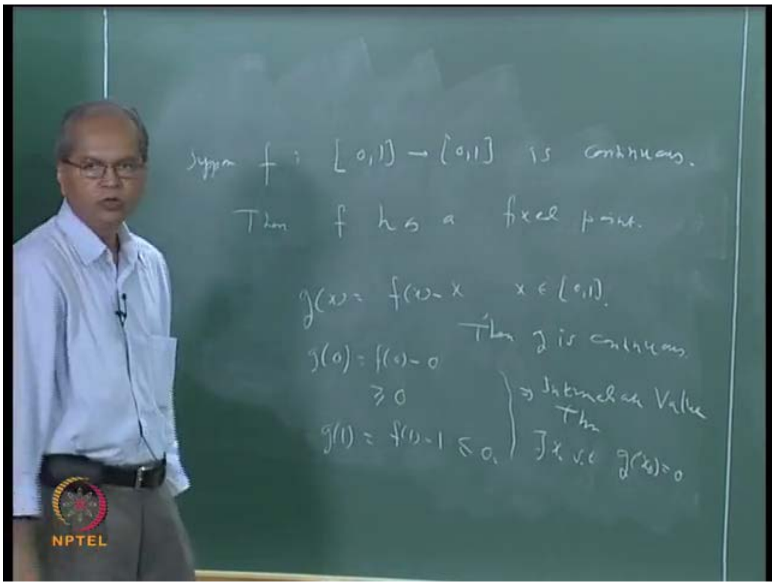

Suppose, let us say if is f function let me take to begin with the interval 0, 1 to 0, 1. It is not very important it can be any interval a, b to a, b. Suppose, f from 0, 1 to 0, 1 is continuous. Then f has a fixed point and this follows in a very straight forward manner by an immediate application of what is called intermediate value theorem.

So, how it follow we can just consider a function. Let us say g x is equal to f x minus x for x in 0 to 1. Then what is g 0 first of all then g is continuous then g is continuous g is continuous g 0 is f 0 minus 0. f goes from the interval 0 1 to 0 1. So, f 0 is some number lying between 0 and 1. So, it is clear that f 0 minus 0 must be bigger than or equal to 0, right f 0 minus. So, this is bigger than or equal to 0. Now, similarly I look at g of 1. So, g of one will be what will be f 1 minus 1.

What can you say about this again remember f goes from interval 0 to 1 to interval 0 to 1. So, f 1 lies between 0 and 1. So, what can you say about f 1 minus 1? It must be less than or equal to 0. It must be less than or equal to 0. So, g is a function, which is bigger than or equal to 0 at this at f 0 and it is less than or equal to 0 at 1. So, what does the intermediate value theorem say? It means it must take it means it must be, see suppose this is strictly bigger than 0 or if it is 0. Suppose, one of these values are 0, then that is what we want. So, we can say that these two things will imply by intermediate value theorem.

There exists we can say some x naught such that g of x naught is 0, but if g of x naught is 0. That is same as saying f x naught equal to x naught, g of x naught is zero is saying that f x naught is equal to x naught. So, it where will we use connectedness in applying this intermediate value theorem. Now, you can see a few things here there was the thing in particular about this interval 0 to 1.

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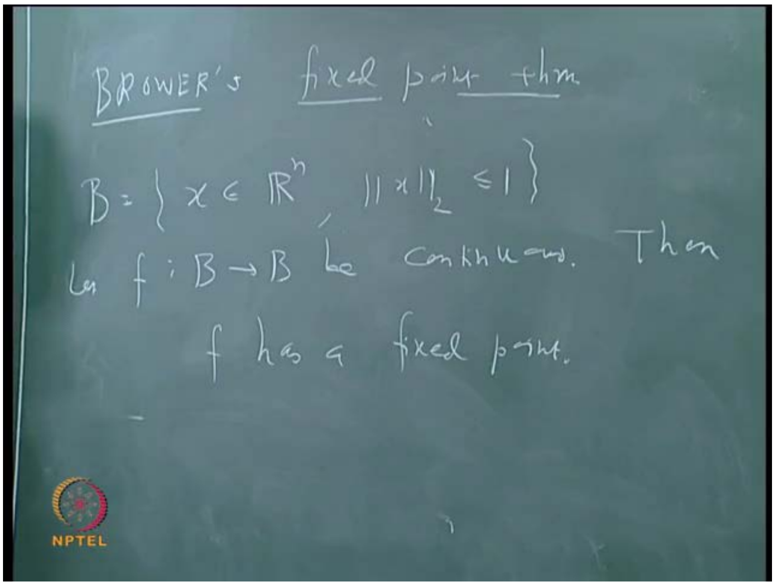

The same argument would have worked if you take and replace this interval by any other interval a b to a b. For example, I have taken minus 1 to 1. Instead of taking 1 to 1 I could have taken minus 1 to 1. That is what leads to another important theorem I just state that theorem. We shall not discuss the proof of this, because that is fairly complicated. It is it is very well known theorem, it is called Brewer’s fixed point theorem.

Brewer’s fixed point theorem, so instead of this interval 0 to 1 we take what is called a closed unit ball in R n closed unit ball in R n. Suppose, I call that ball as b let us say what is a closed unit ball in r n it is the ball with centre at 0 and radius 1. So, it is a set of all x in R n with norm of x whatever norm you take here norm of x plus x nor equal to 1. You can taken any 0 Norm here

So, for the time being let me just state the use of euclidian norm x 2 x nor equal to 0. Let f from b to b be continuous then f has a fixed point. As I said we proof of this involves lots of advanced techniques. So, we shall not go into this proof here in this course, but we can see that this is a special case of Brewer’s theorem, if I take the interval minus 1 to 1. Here this instead of 0 to 1 that is the thing, but the closed unit ball of the of R closed unit ball of R, but so in this case the proof the proof is easy, but in any other cases. The proof is complicated. So, we shall not discuss it here, you can compare this theorem with contraction mapping principle.

You can see that here, we have only as here as well as here only assumption is that f is continuous. Whereas, in the case of contraction mapping it was that it is contraction, but of course is also weak. We have only said that it has a fixed point whereas, in the case of contraction mapping. We had said that it has a unique fixed point nothing is said about uniqueness, here nothing is said about uniqueness here. So, hypothesis are weaker, so conclusion is also weaker. We will stop with this.