Real Analysis
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Lecture - 26
Connectedness

So, we have discussed the contraction mapping principle in the last class. Let us recall what it says once again. That is if X is a complete metric space and f is a contraction map on X then f has a unique fixed point. Now, this principle as I said has applications in several areas in particular differential equations one of the standard theorem, about the existence of differential equation. What is called Picard's theorem uses this contraction mapping principle.

One can also ask how does one come across this contraction mapping. One of the very standard ways of constructing these contraction maps is by using, what you have learnt in the undergraduate curricular as mean value theorem. That is if you remember what is the mean value theorem, that is what it say. Take suppose f is a function, which has derivatives. Suppose, you take say two points X and y. Suppose, f has derivatives in the interval containing X and y.

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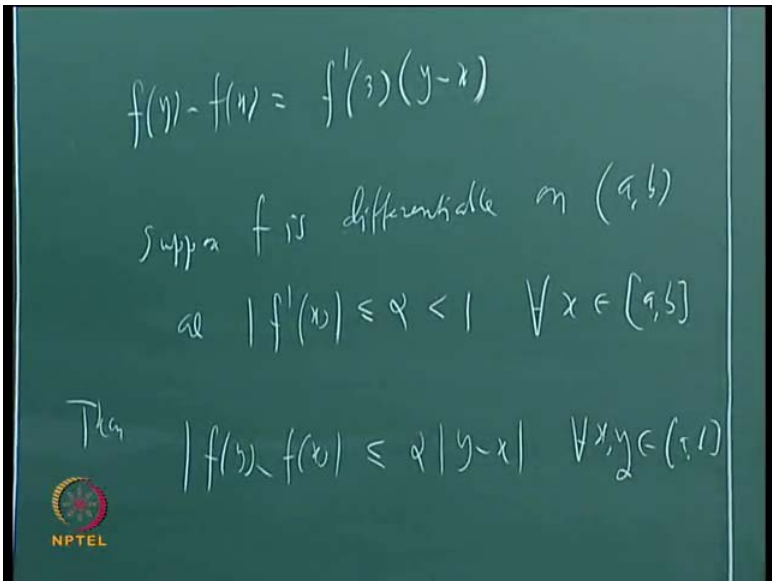

Then you know that f Y minus f x, this can be written as some f prime at z multiplied by, let us say Y minus x. Assuming that X is less than Y where z lies between X and X and y, that is some point, which lies between X and y. Now, suppose this f is such that suppose f is differentiable suppose f is differentiable. Let us say on some interval a b and suppose for all points in a b. Suppose, we know that absolute value of the derivative is less than or equal to some number alpha which is less than or equal to less than 1.

Then we can show using this that that map f is a contraction map that is, suppose. We now this that is f f prime at X is less than or equal on alpha. This alpha is strictly less than one and suppose, this happens for every X in a b. Then using that what we can say is that mod f Y minus f x. Then we can say that mode f Y minus f X less than or equal to alpha times mod Y minus x. This will happen for every for every X Y in a b and that means, which is same. As saying that f is a contraction map on this closed interval, a b f is a contraction map on this closed interval a b.

We have seen that the closed real line as a metric space is complete and closed subset of a metric space is again complete. So, any closed interval is a example of a complete metric space. So, if you take out map, which satisfies this property on any closed interval. Then that should have a that should have a should have a fixed point unique fixed point. Another thing that we discussed about this already started with the some certain special types of uniformly continuous functions. Another type that we discussed was what we call isometric. Look let us again recall what was an isometry. Suppose, we take say two metric spaces X let us say X d and Y rho.

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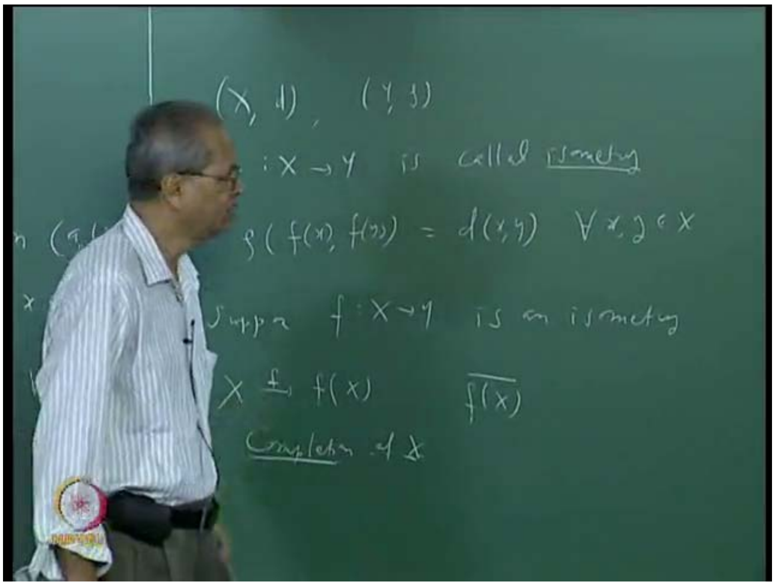

Let us recall we had said that f from X to Y is called isometry. If it preserves the distance isometry as the word means the preserves the distances that is what should happens is that. If you take say any two points X and Y here and look at the distance between f X and f Y f X and f y. Then that should be same as distance between X and y. So, that is for every X Y in X in other words isometry is a distance preserving method. What we have already observed is that isometry will always be one. If f is an isometry it will always be one. It may not be on to and if it is on to then we say that X and Y are isometric to each other.

If there exist an onto isometry from X to y, then we say that the two metric spaces X and Y are isometric to each other. We also seen that if the two metric spaces are isometric to each other. Then as metric spaces they are essentially the same. We can disregard the distance between the two, any two isometric space. Only thing that differs is that the names or labels of the elements change. Otherwise the distance between the two elements are exactly same as whatever happens in x. The same thing should happen in Y and whatever property X holds the same property will be derived for y. Also, in view of this what is usually done. Let us, now take one case.

Now, suppose f is an isomer suppose is an isometry. Suppose, f from X to Y is an isometry it may or may not be onto, but it that is it may or may not be onto y, but this it will always be onto f of X f of X will be a subset of subspace of y. So, f from X to f X will always be onto, so these two spaces X and f X. This is from X to f X this is a onto isometry, so we need. So, we can disregard the difference between X and f X and f X is contained in Y f X is contained in Y. This kind of thing we can say that we express this by saying, that X is isometrically embedded in y. This thing is called isometric embedding this space f X.

As I said that we did not regard that X and f X are different, we can regard X and f X are the same. So, by means of this isometry we can regard X as the sub-space of Y we can regard X as the sub-space of y. Now, let us see that Y is complete suppose Y is complete X may or may not be complete. Suppose, X is any arbitrary space and Y is complete then regarding this. So, f X as a subset of Y we can always say that we can always look at f X closure, f X closure will be complete. If Y is complete f X closure will be complete. So, even if X is not a complete metric space f X closure is a complete metric space. It contains X as a dense subspace, it contains X as a dense sub space.

So, such a metric space is called a completion of X it is called completion of x. What is meant by completion? Again, let me repeat it is a complete metric space, which contains the given metric space as dense subspace, that is called completion of a metric space. Now, obvious question is whether such a completion exists given any or can every metric space be completed and is such a completion unique. Then the answer is yes. We can always find a complete metric space, which contains the given metric space as a dense subspace. Then the completion is unique in certain sense it is unique, but is what is say is that it is unique up to isometry.

That is if you take any two completions of the given metric space. Then those are isometric to each other. Since, in view of our idea of not regarding isometric spaces as different. So, we can say that up to isometry the completion is unique. Now, how to go about constructing this completion? That problem I shall not discuss right now. That we will postpone for some time. Now, let us go to next property of a metric space that is next property. That a metric space may or may not have or subsets, of a metric space may or may not have that is a property, that is called Connectedness.

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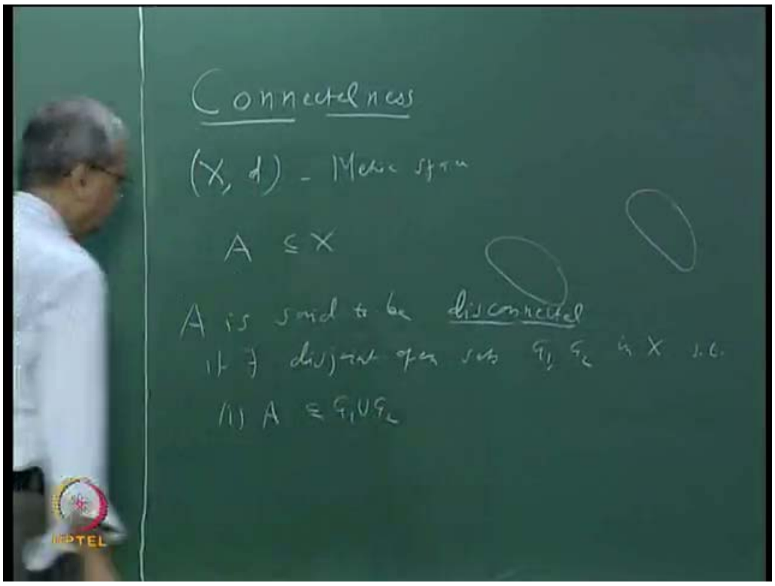

So, here let us say we are given a metric space X d and we are looking at a Subspace subset A of this. We want to say what is meant by saying that A is connected or a is not connected? Now, roughly speaking a connected set should mean that it is a single piece. For example, suppose you have suppose the set is like this say some points here and some points here. Then obviously we will regard this as not connected or disconnected. Only thing is that this what we think intuitively about connectedness. We have to make little more precise by giving the correct definition in the context of any metric space. So, the way in which it is defined is as follows, now it tells that it is simpler to define what is meant by not connected.

So, we shall define that first and then we shall come to connectedness. So, we say A is said to be disconnected. Disconnected means not connected is said to disconnected. You can keep this picture in mind for defining what is meant by disconnected. What should happen is that we should be able to find two open sets. Such that A is contained in those two open union of the two open sets. That those two open sets are disjoint should be disjoint and intersection of each of those sets with a should be non-empty. So, let me again repeat A is said to be disconnected. If there exists disjoint open sets open sets G 1 G 2 in X. Such that, what are the requirements? First thing is that et me write spell out completely. Such that A is contained in G 1 union G 2 second requirement is that G 1 and G 2 are disjoint.

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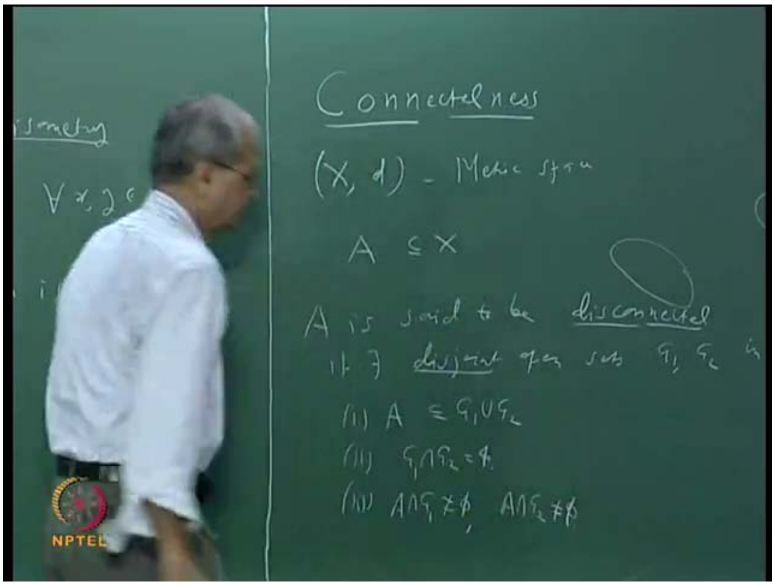

We need not write again, but let us emphasize G 1 intersection G 2 is empty and A intersection G 1 and A intersection G 2. Both sets are non empty A intersection G 1 is non-empty and A intersection G 2 is also non-empty. For example, suppose we take a set consisting of these two regions. As is in a complex plane in r 2 then you will obviously find some open space containing this G 1 some open space G 2 containing the disjoint sets. So, such a set is disconnected. So, if A is once having defined what is meant by disconnected set. Now, we can define a connected set is the one, which is not disconnected. So, we will say set A is connected if it is not disconnected.

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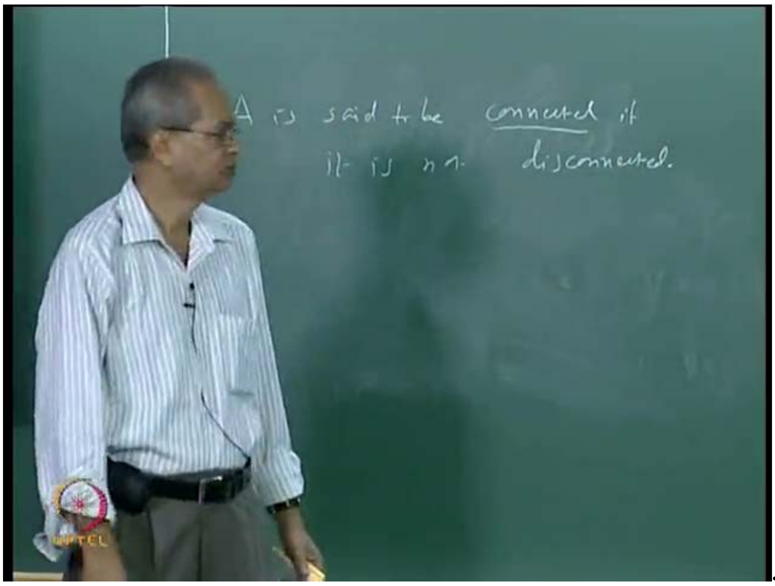

Of course, you may not find it in this way in all books in a book you may find what is first. It is defined what is meant by connected set, but then what will be the definition such two open sets do not exist, that is what is. Suppose, I wanted to without going to this procedure. Suppose, I wanted to find what is meant by connected set then what i would have to say is that A is connected. If they do not exist two disjoint two open sets satisfying all this property.

What is the meaning of that? If you take any two open sets. At least, one of the requirement is violated, that is the meaning of connected. Now, what are all these examples of connected sets or disconnected sets? Disconnected sets we have already seen. As far as connected set is concerned. Suppose, you just take a single point suppose a set consist of only one point. Then obviously this plus thing cannot happen G 1 and G 2 are disjoint and A intersection G 1 and A intersection G 2 is non-empty. That sort of thing cannot happen. So, a set consisting of one point is always connected.

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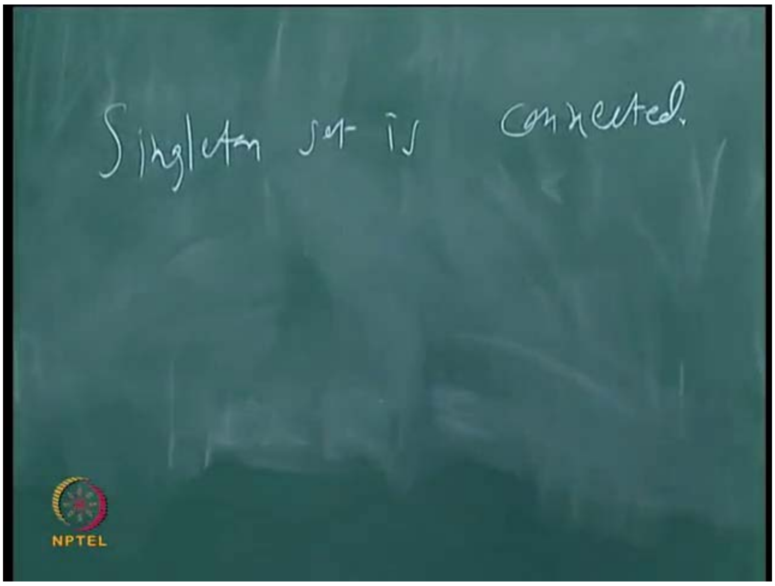

Whatever, be the metric space a singleton set is always connected. So, that is how standard example, so singleton set is connected in any metric. Of course, A set can have more than one points and still be connected, but we shall see those examples little later. On the other hand suppose you consider a discrete metric space. Suppose, X is a discrete metric space. Of course, if it contains only one point, it is obviously connected, but suppose X is a discrete metric space containing more than one points, is it clear? That X must be disconnected, because you can just take let us say one point. Then other as the complement, because in a discrete metric space you will know that any every subset is open.

So, you just write X as a union of two open sets that is for example, you can take say G 1 at just one point and G 2 as it is complement. Then that will satisfy all these properties. Of course, as a special case of this when A is the whole of x. When will such metric space X itself is disconnected instead of this X is contained in G 1 union G 2. It will be X equal to G 1, because since G 1 uinon G 2 will always be a subset of x. If X is also contained in gone union G 2. It simply means that X is equal to G 1 G 2. It means that if you can write X as a disjoint union of two non-empty open sets then X is a disconnected metric space.

Otherwise, it is connected if you cannot write X as a disjoint union of two non-empty open sets. Then X is a connected metric space and it is fairly straight forward to prove that this connectedness is a property, which does not depend on whether you regard A as a subset of X or a as a subset of itself. Because, we know that a itself is a metric space.

So, we can say one might think that there might be some difference between regarding a as a connected set as a metric as a subset of A. As a subset of X we have seen that kind of difference exists in open sets closed sets etcetera. Whether you regard A as a subset of A itself. That we know always open, but if you take a as a subset of X it may or may not be open, but in case of connectedness that is not the thing. A is connected or not connected does that does not depend on whether you regard A as a subset of X or any or A itself for any set lying between A and x. That can be proved, but we shall not go into prove of that right now.

Now, let us come back to the again to the most familiar metric space. Namely, the real line what are the connected sets in the real line. There we have the most satisfactory theorem or the most satisfactory answer intervals, are the connected sets in the real line. Those are the only sets. So, let us say, let us write that as a theorem.

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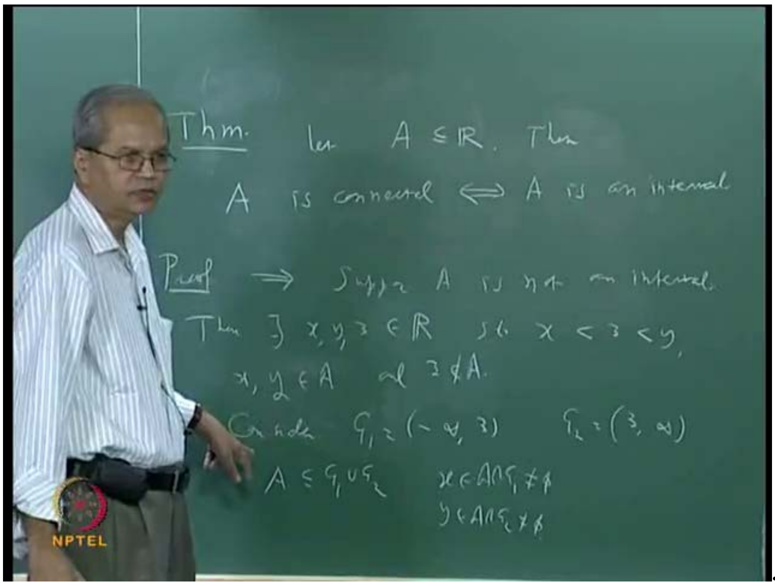

Let us say that let A be a subset of R by the way what about empty set. For example, can such a thing happen? If A is empty it is not possible. So, an empty set is connected set then that is virtually or trivially connected set. So, let A be a subset of R then what the theorem said? A is connected if and only if A is an interval. So, this gives a complete description of connected sets in the real line A is connected. If and only A is an interval.

Now, let me recall what is we have seen it how an interval is defined. You have seen that interval means, when we said A set is an interval. That if you take two points X and Y if X and Y belong to A then all the points lying between X and Y must also belong to A. That is the definition of interval.

Let us first prove it this way, suppose A is connected. We want to prove that A is an interval suppose A is connected. Now, if A is not an interval suppose A is not an interval, we should get a contradiction. Suppose, A is not an interval if a is not an interval what should happen? There should exist three points let us say x, Y and z. Let us say X less than z less than Y such that X and Y belong to A, but z does not belong to A. So, let us say then there exists x, y, z in R. There Exists x, y, z in R such that X less than z less than y x y belong to A and z does not belong to A.

Now, what we can do is that we can construct two open sets G 1 and G 2 satisfying all those properties listed there. So, let us say, so far we have considered G 1 is equal to consider G 1 is equal to , or let us say minus infinity to z open interval minus infinity to z and G 2 is equal to open interval z to infinity. G 1 and G 2 non-open sets they are open interval. So, open sets are they non-empty means they are infinite sets is this true that A is contained in G 1 union G2.

In fact G 1 union G 2 contains the whole of r except this point z. G 1 union G 2 is the whole of r except this point z and that point z is not in A. So, A is contained in G 1 union G 2 and what about A intersection G 1? A intersection G 1 contains X A intersection G 1 contains. So, X belongs to A intersection G 1. So, that is non-empty and A intersection G 2 that contains y. So, Y belongs to A intersection G 2, so that is non-empty. So, we have found sets G 1 and G 2 satisfying all these properties here. So, that means A is not connected A disconnected and that is A contradiction. We have started with assumption that A is connected.

So, every connected subset of real line is an interval. Let us, now see how we can prove it in other way. So, we shall we shall discuss this proof again tomorrow. So, for the time being we shall assume that whenever A is an interval A is also connected. So, whatever is the missing point part we shall again discuss this in tomorrow’s lecture. Let us proceed further, next what we want to say is that what happens to the images of the connected sets under the continuous functions.

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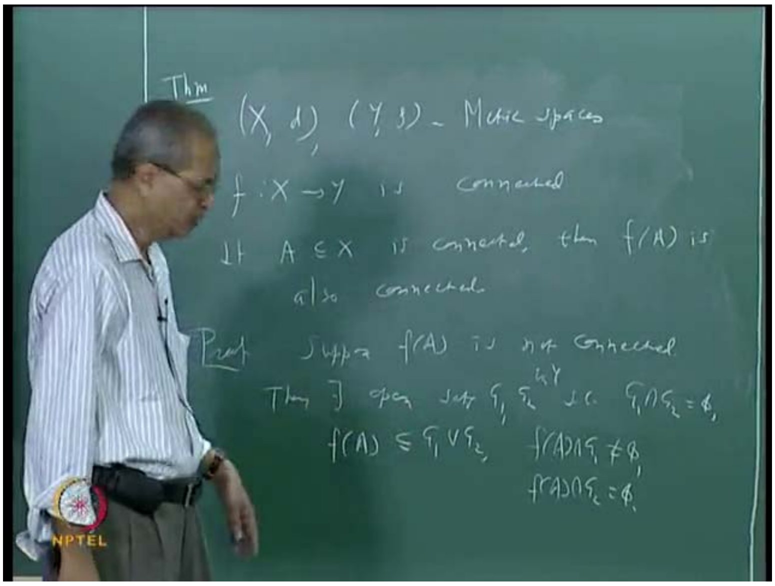

So, let us look at this. So, what does theorems suppose X d and Y rho are metric spaces and f from X to Y is connected. Then suppose you take A connected set in A then it is image in Y is also connected. So, that is what we want to prove. So, if A is contained in X is connected, then f of a is also connected. Basically, the whole proof depend of course, this is important theorem. About connectedness the whole proof depends basically on the fact, that inverse images of open sets are opens under continuous function, which is what we have seen some time earlier. So, suppose f of A is not connected.

Then we should we shall find there should exist open set G 1 G 2 satisfying all these properties, which we have written. Then consider their inverse images, then we will do the same thing with respect to A. That will show that A is not connected and that will be A contradiction. So, let us go through that proof. So, suppose f of A is not connected f of A is not connected. Then there exists open sets G 1 G 2 such that whatever is required G 1 intersection G 2 is empty, f of A is contained in G 1 union G 2.

Then f of A intersection G 1 is non-empty and f of A intersection G 2 is also non-empty. What we do after this is that, of course these open sets G 1 and G 2. Those are in Y open sets G 1 and G 2 in Y, because f of A is a subset of y. So, what we shall do now is that consider the inverse images of G 1 and G 2 under f. Then just rewrite these equations in terms of those. So, then first look at f inverse G 1 is f inverse G 1 non-empty.

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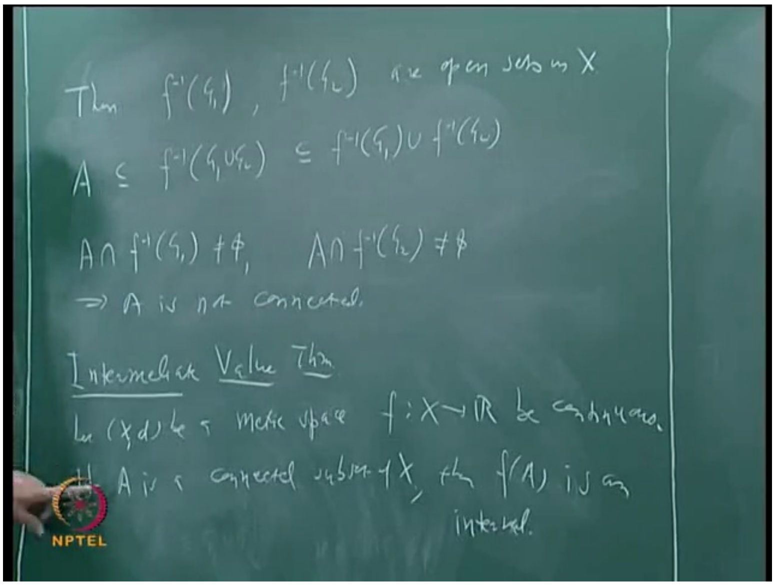

Why f inverse G 1 is non-empty? Because, this shows that f A intersection G 1 is non-empty. So, there should exist some X such that f X is in G 1 that, because suppose that f X f A is a set of all f X of the form X is in A. It is intersection with G 1 is non-empty, that means there exists some X in a such that f X intersection G 1 is non-empty. So, f inverse G 1 is non-empty. So, f inverse G 1 is first of all f inverse G 1 and f inverse G 2 are open sets In X where they are open because G 1 and G 2 are open in Y and f is continuous.

They we have seen that inverse image of an open set under a continuous function is open. Now, look at this equation f of A is contained in G 1 union G 2 f of A is contained in G 1 union G 2. So, that means that a is contained in f inverse G 1 union G 2. So, this means A is contained in f inverse of G 1 union G 2. We can always say that f inverse of G 1 union G 2 that is same as, in fact it is same as f inverse G 1 union G 2, but we can suddenly say it is contained in f inverse G 1 union f inverse G 2 actually this is equal.

So, A is contained f inverse G 1 union f inverse G 2 both these are open sets. Then what about A intersection f inverse G 1? That is where we come and look at this point again, if A intersection G 1 is non-empty. It means that there exists some X such that f X belongs to G 1, there is some X, such that f X belongs to G 1. So, that X should be in a as well as in f inverse G 1 see f A intersection. Let me repeat f A intersection G 1 is non-empty means there exists some X in A, such that f X is in G 1. So, that X is in a and also in f inverse G 1. So, we can say that A intersection f inverse G 1 is non-empty A intersection G 1 is non-empty.

In a similar way we can say that A intersection f inverse G 2 is also non-empty. A intersection f inverse G 2 is also non-empty, but that is the contradiction. That shows that A is not connected. So, this means A is not connected, now as a special case of this. If this second space Y is real line, then it means that if you take A connected set in x. Then it is image must be a connected set in R, which we have just proved that connected sets of R must be in intervals. So, that means image of a connected set under a continuous function will always be an interval image of A connected set, under a continuous real valued function will always be the interval.

This theorem of course, which follows immediately from here. The previous theorem is well known as an intermediate valued theorem. So, let us say that let X be a metric space and f from X to R be continuous. Of course, when I say nothing more about R it means with the usual metric. If we give some other metric that will be specifically said this is something I mentioned earlier also. If A is a connected subset of X then f of A is an interval.

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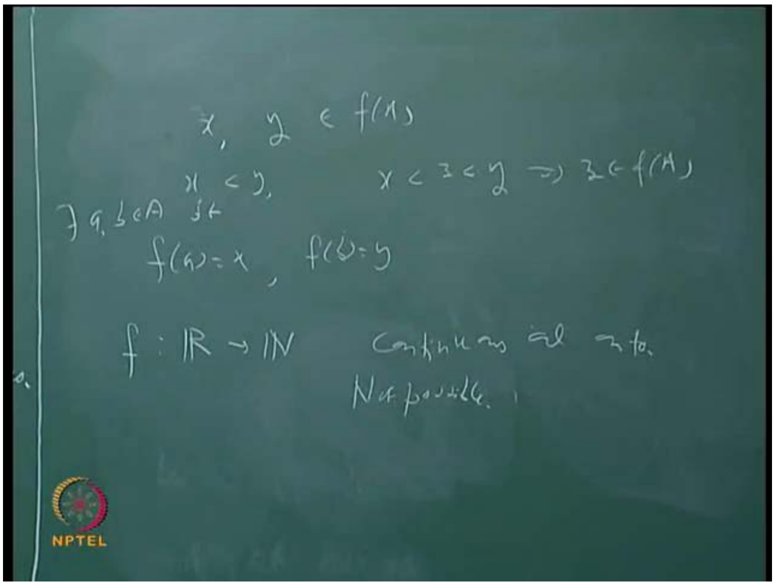

Of course, this follows immediately from this as such in a normal course. We should not simply call this as a corollary of this, because no new ideas are involved here. If A is connected f of A is connected. Since, f of A is a subset of R and every connected subset of R is an interval. So, f of A is an interval, why is it called intermediate value theorem?

Because, we can say that since f of A is an interval, what we can say is that. Suppose, there are any two points let us say X and Y belong to f of A and x. Let us say and X less than Y and suppose there exists some z X less than z less than y. Then this should imply that z also belongs to f of Z also belongs to f of A. So, in another words if f takes the value X and Y X Y belongs to f of A what there exists some A. Such that f of A equal to X and Y belongs to f of A means that what there some b such that f of b is equal to Y that is, let us they say f of A is equal to X that is there exists a b in A. Such that f of A is equal to X and f of b is equal to Y that means the function f, assumes the values X and y. If it does that it should assume all the values in between those two values or all the values, which are intermediate that is why it is called a intermediate value theorem. So, if f assumes any two real numbers it assumes all the values, which are in between those two values.

So, for example in particular suppose at some point f of A is less than 0. At some other point f of b is bigger than 0, then there should exist some point c such that f of c is equal to 0. So, this is something, which is used fairly often again in numerical analysis in finding the first approximation of any roots of the equation. So, intermediate value theorem is again quite useful in basically deciding, the starting points of many of the numerical algorithms. Usually, in applications this function f will go from some subset from r to r. That subset will usually be an interval that subset we usually, so image of an interval under a continuous function will always be interval.

This also answers for example, certain kinds of functions cannot exist. For example, can there exist a continuous function from r to n. Let us say continuous and onto is that possible. Let us say it is possible f ,let us say f from r to n f from r to n continuous and onto is that possible. Because, obviously that intermediate value property will be valid, right? n is not an interval. So, there cannot exist any continuous function going from r to r to n. So, such a thing is not possible.

Now, next we want to see how the operations on this connected sets take place. For example we would like to do what about the intersections. Unions of connected sets it is something similar to the interval. We have seen that intersection of any two intervals will again be an interval, but union need not be. In case of union what we know that union will be an interval if the intersection is non-empty, something similar is true for the connected sets also. That is if you take two connected sets and if their intersection is non-empty, then their union is again connected. Then this is true not only for two sets, but it is true for any family of sets. I will just state that theorem and then we shall see the proof tomorrow.

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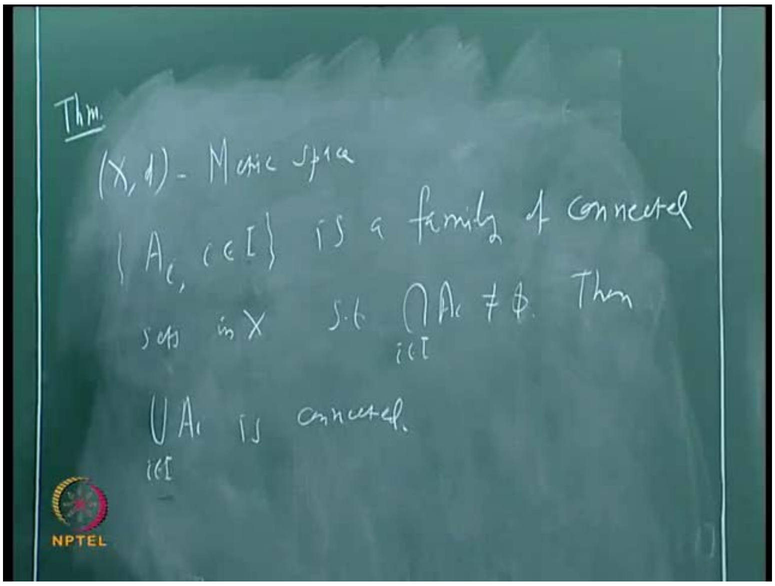

So, suppose X d is a metric space. Let us say That A i small i belonging to big I. Let us say I is some indexing symbol is a family of connected subsets is a family of connected sets in X of course. Such that their intersection is non-empty, such that intersection A I is non-empty then you want to say that union A I belonging to I is connected. We shall see the proof of this theorem in tomorrow’s class.