Real Analysis
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Lecture - 25
Uniform Continuity

So, we were discussing uniform continuity and related concepts and what we have seen is that uniform continuity is a stronger form of continuity. If a function is uniformly continuous it is continuous, but the converse is false. What we want to prove next is that, if a function is uniformly continuous on a dense subspace then, it can be extended to the whole metric space. Now, in order to prove that we shall need a couple of things, which we I will give as an exercise.

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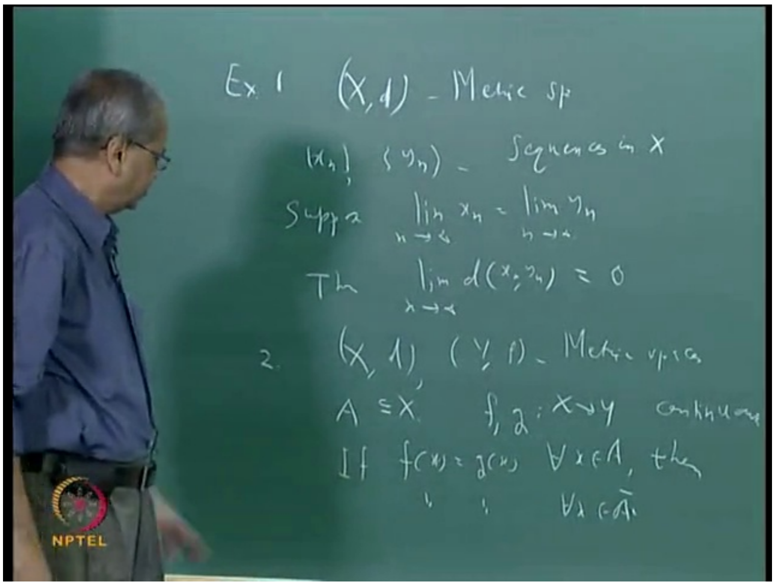

So, let us say first exercise is the following suppose X d is a matrix space and suppose we take two sequences x n and y n. Both are sequences in x and suppose both of them converge to the same limit. So, suppose limit of x n as n tends to infinity x, n is limit of y n as n tends to infinity. Then limit of d x n y n as n tends to infinity that is 0. This we will follow from an elementary application of triangle inequality. Then this second exercise is the following, suppose there are two matrices spaces x d, y rho those are two metric spaces.

And suppose A is a subset of x and we take two functions f and g, let me say functions from x to y, both continuous f and g from x to y continuous. Then what this exercise says that if n g coincides on A then, they coincide on a closure. So, if f x is equal to g x for every x in a then f x is equal to g x for every x in a closure. As far as the second one is concerned, you have seen that if you take any point in a closure then you can find sequence of elements in A.

Suppose, x is a point in a closure then there is a sequence x n of points in a such that x n tends to x. If x n tends to x then f is continuous if x n tends to f x. Similarly g is continuous so g x n tends to g x and f x n and g x n coincide so, that is the idea in the second exercise alright. Now, what we want to do is this, what I mentioned just now, suppose that is every function which is uniformly continuous on a dense subset can be extended to the whole matrix space as a uniformly continuous function.

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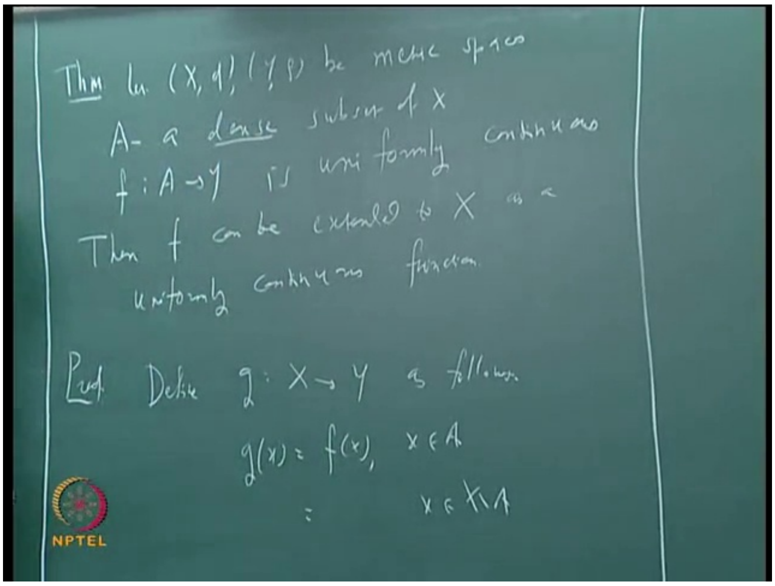

So, let us say that X d and Y rho are matrix spaces and a dense subset of x, f from A to y is uniformly continuous. Then, f can be extended to X as a uniformly continuous function. What is the meaning of saying that f can be extended it means, you can define a function g. Let us say function g, which is defined for all values of x such that g is uniformly continuous and g x is equal to f x for x in A, that is the meaning of saying that f can be extended to the whole of x. Now, let us look at the proof first let us see how the extension can be defined.

As far as it is has to be an extension for the points in A, its value has to be f x. So, let us say define we will consider let us say define g from x to y as follows. So, what is that, that is g of x is equal to f x for x in A. And we want to say how it is defined for x in x minus A and for that we use that A is a dense subset of x. There should be one more thing that is required here in the hypothesis, this space y must be complete these are matrix spaces and y rho is complete. And so there are two important subsets that is one is that y rho is complete and f is uniformly continuous.

Suppose, f is just simply continuous then also you cannot extend it in general then if y is not complete then also this extension is not possible and that you will see as we go along with the proof. When we take x in this x minus A since, A is dense in x every such x will be a limit of sequence of points in x. So, let us first see how this can be defined.

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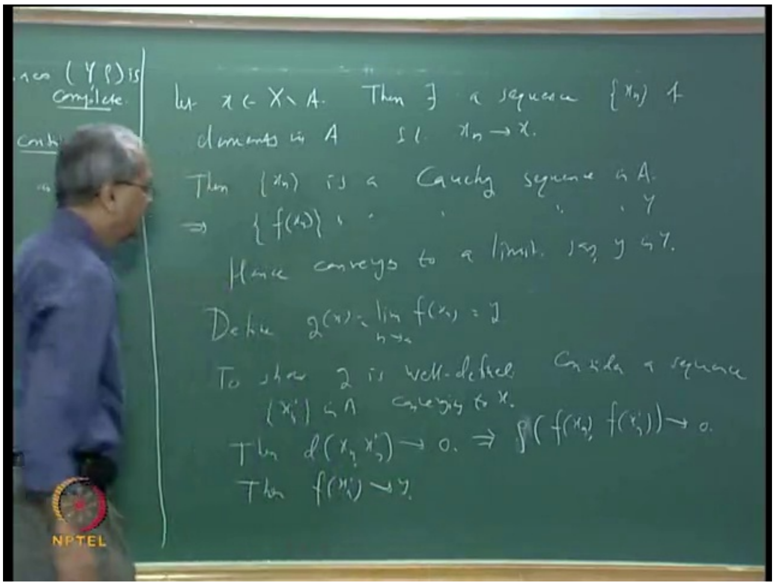

Let us take x in x minus A then we can say x is the sequence of elements in A such that that sequence converges to x. Then there exists a sequence suppose I call that sequence x n of elements in A such that, x n converges to x. If x n converges to x in particular x n is a Cauchy's sequence, so then because we know that every convergent sequence is Cauchy then x n is a Cauchy's sequence in a and f is uniformly continuous. So, we have see that is x n is Cauchy and if f is uniformly continuous, f x n should also be Cauchy. So, that implies that f x n is a Cauchy's sequence in y, but y is complete that is where we have used this. So, since y is complete this f x n should converge to some limit complete in y and hence converges to a limit suppose I call that limit as y to A.

Then we will define g of x as this number y so define g of x as limit of f x y, in this case we call that limit as y. Now, there is a problem with this definition and what is the problem? It is that sequence, which converges to x is not unique, it can happen that some other sequence that is given a point in A closure there may exist several sequences in A which converge to that point x. So, suppose you take some other sequence, suppose that sequence is x n prime then x n prime also converging to x.

Then if you follow this procedure then, g x should be defined a limit of f x n prime, but then unless we know that these two limits are the same, we cannot say that this is a well defined element. So, to show that the function g defined in this fashion is a well defined, we must show that this g of x does not depend on the particular choice of the sequence x n, that is the important thing. So, to show that g is well defined consider, another sequence which also converges to x.

Now, this is where I will use this first exercise, if there are two sequences and if both of them are converging to the same point then the distance between x n and y n that should converge to 0. So, then d x n x n prime this tends to 0. Now, the distance between x n and x n prime tends to 0 and f is uniformly continuous. So, we can show that distance between f x n and f x n prime that should also converge to 0. This implies that distance between f x n and f x n prime that also should tends to 0, but if that is the case remember f x n prime is also a Cauchy sequence by the same argument.

Just as f x n is the Cauchy's sequence f x n prime is also Cauchy's, so that also converges to certain limit. Suppose that limit is y prime then x n converges to y f x n prime converges to y prime and actually this distance is rho because that is in y and rho f x n f x n prime converges to 0. These three things along with primary equality will imply y must be same as y prime. So, using primary equality, you can show that f x n prime should also converge to the same limit namely y so then, f x n prime also converges to y.

In other words this limit of f x n does not depend on the choice of the particular sequence x n. So, let us again recall what we have done given any point x outside A. We find sequence x n in A, which converges to x that sequence is a Cauchy's sequence and hence f x n is a Cauchy's sequence, y is complete. So, f x n converges to that limit moreover that limit does not depend on which particular sequence you are choosing, whichever sequence you choose the limit of that f of that sequence will be the same.

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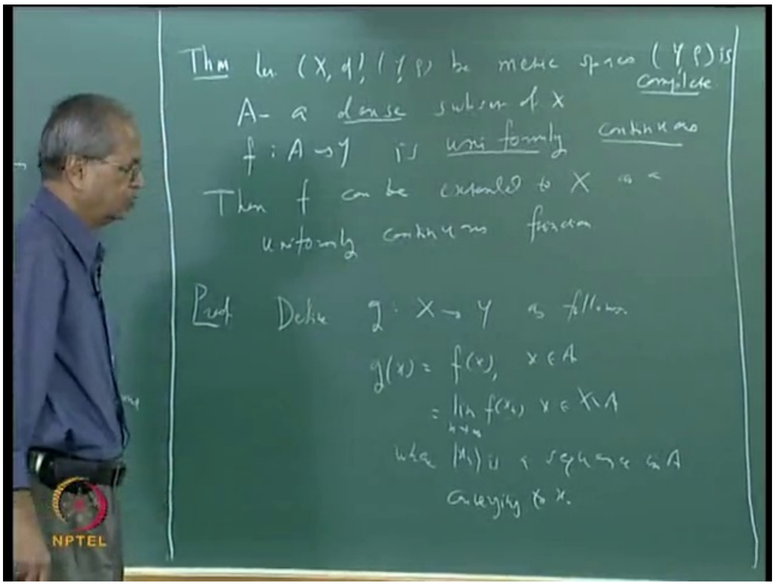

And that common value is what we take the definition of this g of x for x in x. So, let us say that g of x is limit of f of x n as n tends to infinity where, x n is a sequence in a converging to x. And we have seen that this limit does not depend on the particular choice of sequence now g is well defined.

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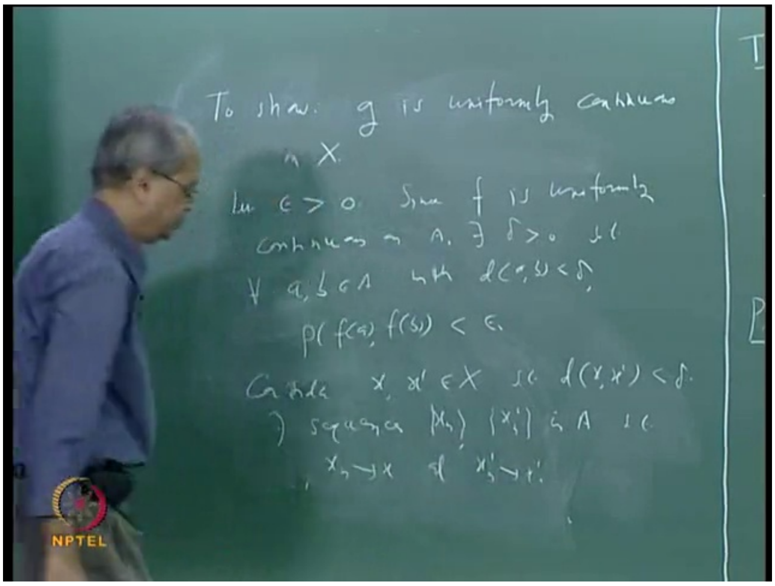

Now, what remains to be shown is that this g defined in this fashion is uniformly continuous on x that is to show g is uniformly continuous on x. Now, to do that we know that as far as the set A is concerned on that set g x is same as f x and that f is uniformly continuous. So, we shall use that property so, let us say that let epsilon bigger than 0 be given and then since f which is same as g on A is uniformly continuous on A there exists A.

Since f is uniformly continuous on A there exists a delta bigger than 0 such that if you take two points in A with a distance less than delta, the distance between f of those two points will be less than x. So, such that for all let us say for all a b in A with distance less than delta distance between f a and f b is less than epsilon. We have to show that the same thing happens for any two points in x. So, consider let us say x and let me say x prime in x such that, distance between x and x prime is less than delta.

Now, since x and x prime are in x there will exist sequences of elements let us say x n converging to x and x n prime converging to x prime x and x n prime lying in x. So, there exists a sequence x n and x n prime in A such that x n converges to x and x n prime converges to x prime. Then we know that f of x n in particular x n is a Cauchy's sequence, x n prime is a Cauchy's sequence, f of x n is a Cauchy's sequence, f of x n prime is a Cauchy's sequence and they converge.

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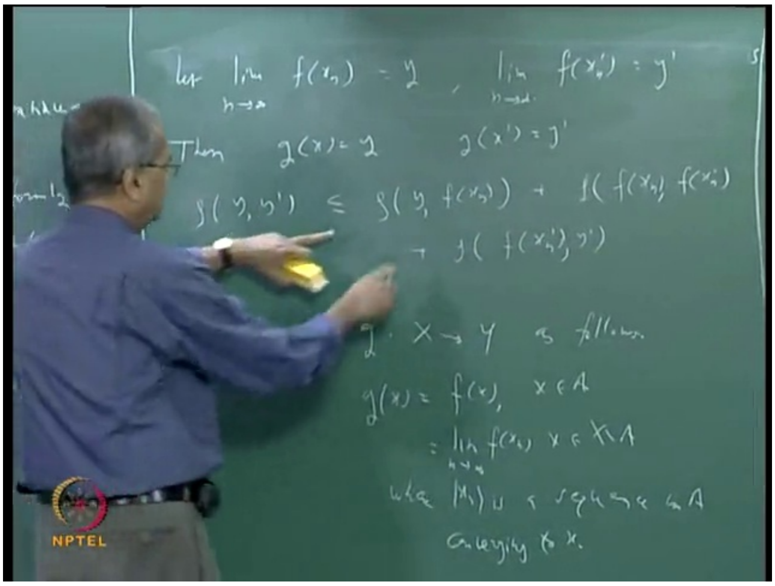

So, let us say that let limit of f of x n be equal to y and limit of f of x n prime, let us say that is y prime. Then what we do now is that by our definition g of x is y and g of x prime is y prime and what we want now is that, distance between y and y prime should be less than epsilon. Now consider rho of y prime. Now, basically again we again use the triangle inequality again and again. So, what we know is that f of x n prime converges so f of x n converges to y and f of x n prime converges to y prime. So, let us write that this so this is less than equal to distance between y and f of x n plus distance between f of x n and f of x n prime and finally, distance between f of x n prime and y prime. Now out of these three entries on the right hand side, we know that f of x n converges to y, so this term goes to 0 as n tends to infinity.

Similarly, f of x n prime converges to y prime, so this goes to 0 as n goes to infinity. So, only the remains to be discuss is what happens to this distance f of x n and f of x n prime. But x n and x n prime those are sequences in A, x n converges to x and x n prime converges to x prime so, we can say that distance between these two things x n converges to x and x n prime converges to distance between x and x prime.

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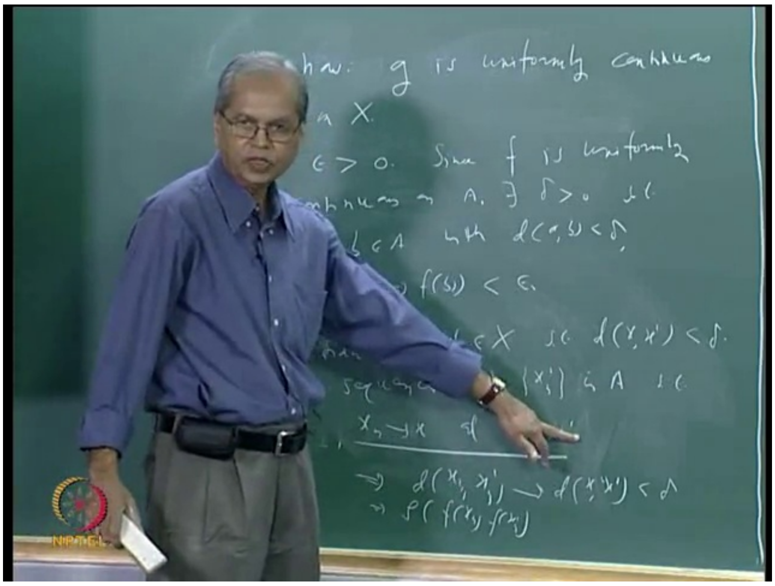

But distance between x and x prime is less than delta distance between x and x prime is less than delta. So, we can say that distance between f x n and f x n prime that should converge. Remember, this f is a uniformly continuous function, so if distance between x n and x n prime converges to distance between x n and x prime, what we must have is that this should imply that rho between f x n and f x n prime. That should converge to some number which is less than x 1 because what we know is that, for each a b the distance between a, b is less than delta.

See this distance between x and x prime is less than delta and this is something that converges to this number, which is less than delta. So, what we can say is that for large values of n, we can say there will exist some n 0, so that whenever n is bigger than or equal to n 0, distance between x n and x n prime will be less than delta. So, for those n this rho of x n and f x n prime must be less than epsilon, that is the argument that we seen. For large values of n we know that rho of f x n and f x n prime is less than epsilon, this happens for all large values of n. We can make argument more precise.

So, we can say this happens for all n bigger than or equal to that n 0, which n 0, that n 0 for which the whenever n is bigger than or equal to n 0 distance between x n and x n prime is less than delta. For those n we will also have rho of f x n and f x n prime is less than epsilon in other words, but this is independent of n.

So, we always choose n which is large enough such that, this goes to 0, this goes to 0 and this becomes less than epsilon. So, for large values of f you can say that rho y by y prime is less than epsilon and this is what we wanted to prove that is whenever x and x prime is less than delta rho, but what is y, y is g of x and y prime is g of x prime. So, that proves that g is uniformly continuous. Now, there is one more thing that is dependent on the completeness and uniform continuity, let us go to the proof of this also. And this is well known application of completeness you know that in practice several times we need.

To solve an equation like this f x is equal to y and for now itself there is a solving this equation is that we convert this equation in the following form g x is equal to x. This can be done in several ways and this is something you have learnt in numerical analysis that, one of the ways of solving this equation f x is equal to y is convert this into an equation of this form and find a solution of this. And as you know solution of such an equation is called fixed point of g.

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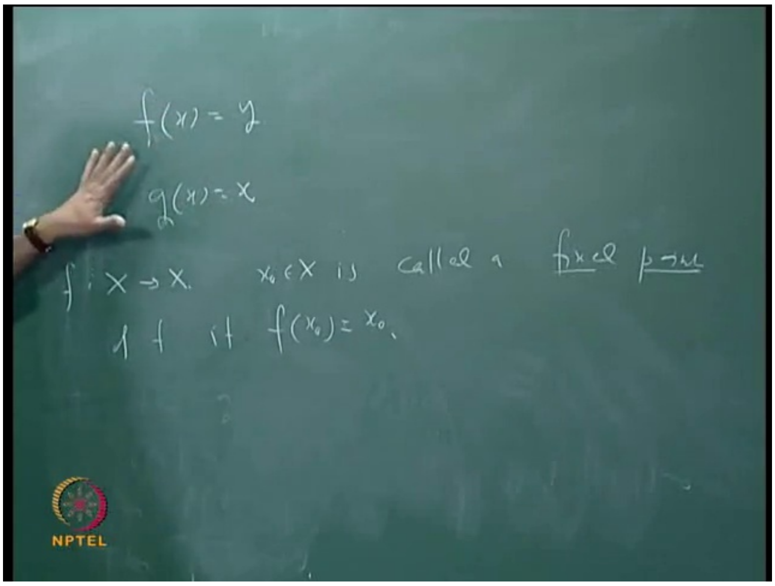

So, let us make a general definition, so suppose we take a map x from x to x then, x naught in x is called a fixed point of f, if f of x naught is equal to x naught. And given a map what we ask is whether such a map has a fixed point or not. And if we can show that a map has a fixed point then, it is equivalent to say that equation like this has a solution. If this map g has a fixed point, it is same as saying that the equation f x equal to y has a solution provided.

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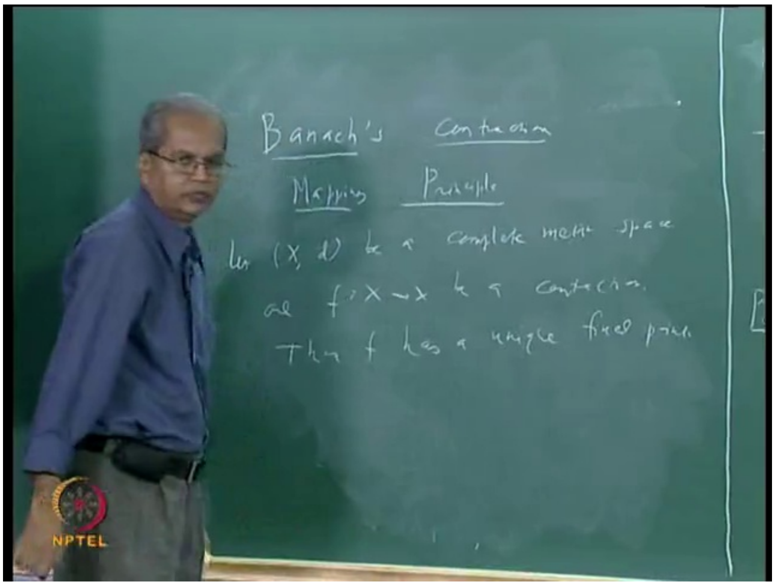

You have provided g in such a way that f x equal to y in fact only gives x is equal to x, this can be done. Now, what has completeness to do with this, one of the most famous theorems about the existence of fixed point is known as, let me read it is known as Banach’s contraction mapping principle and that is what we shall prove now.

It is known as Banach’s contraction mapping principle and the statement of this principle is very simple. By the way as I have mentioned earlier Banach’s was a very famous polish mathematician, who has done several things in real analysis and functional analysis. So, this name you will come across again and again in the course of functional analysis.

So, what does this statement say or what does this Banach’s contraction mapping principle say, it says simply this that is if you take a complete metric space and if f is a contraction mapping on that complete metric space then, it has a unique fixed point. So, the statement is very simple, so x be a complete metric space and f from x to x be a contraction or contraction metric then, f has a unique fixed point.

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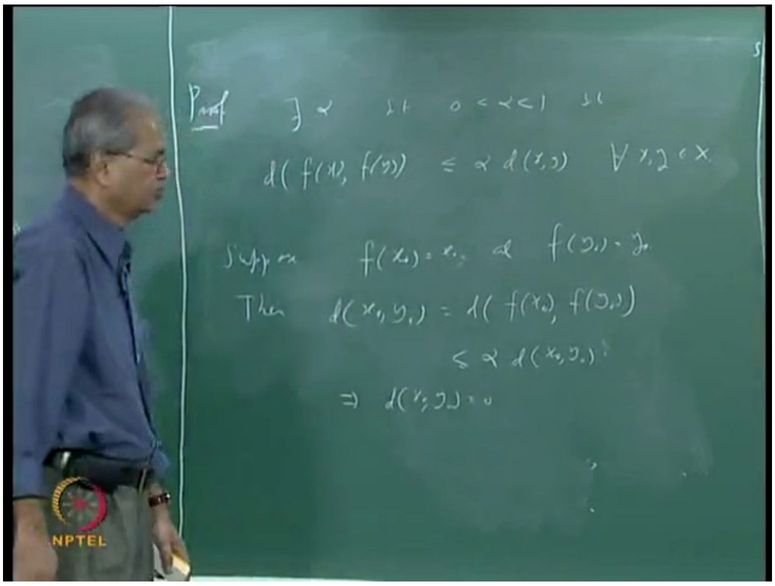

Now, let us look at the proof of this, as you can see from the statement, there are two parts that there exists a fixed point and that such a fixed point is unique. So, we have to prove the existence and uniqueness of the fixed point of this contraction map f. First of all let us recall, what is the meaning of saying that f is a contraction map, let us recall what does mean?

This means that there exists some alpha, such that 0 less than alpha less than 1 that is the meaning of contraction 0 less than alpha less than 1, such that distance between since f is going from x to y distance, between f x and f y is less than or equal to alpha times distance between x and y for all x y in x. That is the meaning of saying that it is a contraction and in particular it means that x is uniformly continuous. We have seen that contraction is a special case of uniformly continuous.

We can see first of all that from this it follows in a more or less trivial manner that if at all a fixed point exists, it must be uniform, there cannot be more than one fixed point. How does that follow? Let us say in anyway how does that follow let us say in a way how does one prove uniqueness? We can just say that show me that there are two fixed points, we should show that they are the same. So, suppose there are two fixed points suppose I call them x x naught and y naught.

Suppose f of x naught is equal to x naught and f of y naught is equal to y naught then, we should show that x naught and y naught must be same. So, consider distance between x naught and y naught then, distance between x naught and y naught, this must be same as distance between f x naught and f x naught because x naught is f x naught and y naught is f y naught. So, would be same as distance between f x naught and f y naught, but f is a contraction.

Look at this so, this right hand side this must be less than or equal to alpha times distance between x naught and y naught. Now, if this is any number bigger than 0, remember alpha is strictly less than 1 over what is distance between x naught and y naught is strictly less than distance between x naught and y naught. So, that cannot happen unless d x naught y naught is 0. So, this implies d x naught y naught, which is same as saying that x naught is equal to y naught. So, there cannot exist more than one fixed point that is clear, uniqueness is clear so remains to be proved now is the existence. Now to prove the existence let us proceed like this, we will take start from any point let us say I will call that point x naught.

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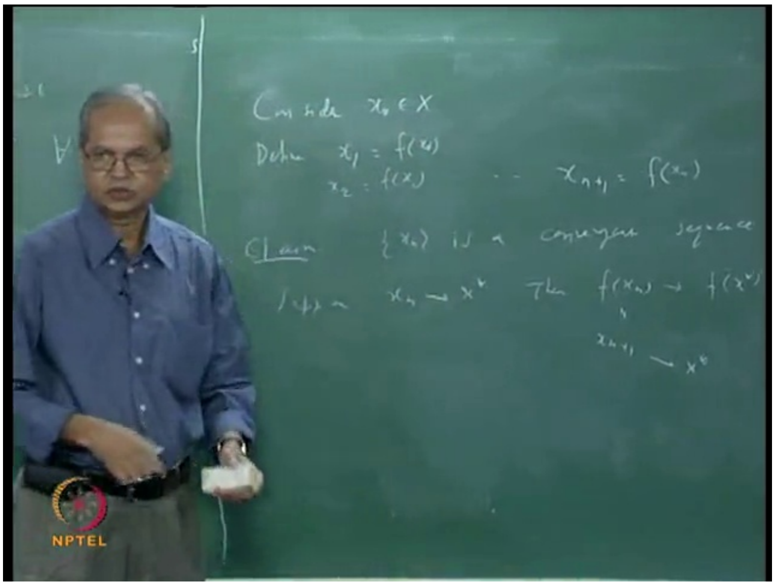

Let us consider a point x naught, consider point x naught in x and then define x 1 is equal to f x naught then, x two is equal to f x 1 actually. Define start from any point x naught, define x 1 is equal to f x naught, x 2 is equal to f x 1 then, x 3 is equal to f x 2 etcetera. In general proceeding like this defines x n plus 1 is equal to f x, this gives a sequence, this is a sequence. Our idea of the proof is to show that this sequence is convergent, we will show that this sequence x n is convergent.

So, we shall claim x n is a convergent sequence, once we show that x n is a convergent sequence then existence of x can be proved. Idea is that once we show it is a convergent sequence, it has some limit, suppose that limit is x. Let us say that limit is x star then, we will show that x start is required fixed point. So, we shall prove this claim afterwards suppose, this claim is proved then how does one complete the proof of the theorem then, x n is a convergent sequence, suppose x n converges to x star.

Then f is a continuous function so, f x n should converge to f x star then f of x n converges to f of x star, but f of x n is nothing but x n plus 1, that is how we have constructed f of x n, this is nothing but x n plus 1. Now, if x n converges to x star what about x n plus 1 x n plus 1 should also converge to x star. So, x n plus 1 should also converge to x star that means f of x n also converges to x star and f of x n also converges to f of x star.

So, what does that mean? It means that f of x star must be same as x star. So, whatever is the limit of this sequence that is a fixed point of the map f. So, the only thing that remains to be shown, the proof of the claim. We must show that the sequence x n is a convergent sequence, but for that we use the fact that this is a complete metric space. So, what is the way of showing that a sequence converges in a complete metric space it is a Cauchy's sequence. So, we shall show that the sequence is a Cauchy's sequence. Let us see how that can be shown and to do that we shall again use the fact that this is a contraction mapping.

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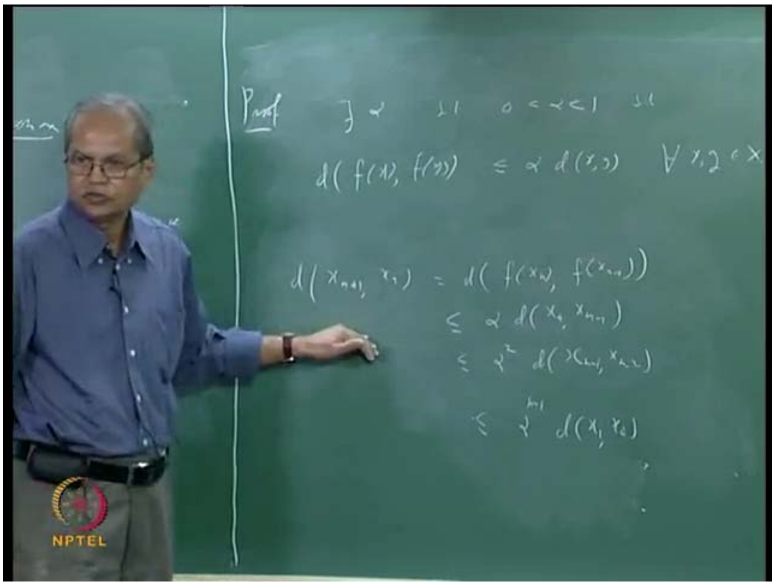

To do that let us consider the following, consider the distance between x n plus 1 and x n. Now, x n plus 1 it is same as f of x n and similarly, x n is same as f of x n minus 1. So, this is same as this distance between f of x n and f of x n minus 1, but what about that? Look at this distance between f of x n and f of x n minus 1, this should be less than or equal to alpha times distance between x n and x n minus 1. So, this must be less than or equal to alpha times distance between x n and x n minus 1.

So, what do we prove distance between x n plus 1 and x n must be less than or equal to alpha times between x n and x n minus 1. I can use it once again distance between x n and x n minus 1 must be less than or equal to alpha times distance between x n minus 1 and x n minus 2. So, I can say that this is less than two equal to Alpha Square times distance between x n minus 1 and x n minus 2.

We can go on like that and finally, this will be x less than or equal to alpha cube times distance between x n minus 2 and x n minus 3 etcetera and going on like this. Lastly we shall reach distance between x 1 and x naught, only question is what is the power of alpha? Here we will get alphas to the power n minus 1. So, given any n what we are shown is that distance between x n plus 1 and x n should be less than or equal to alpha to the power n minus 1 into distance between x one and x naught. Of course, that strictly does not prove that x n is a Cauchy's sequence, what this will certainly prove is this will go to 0 as n goes to infinity.

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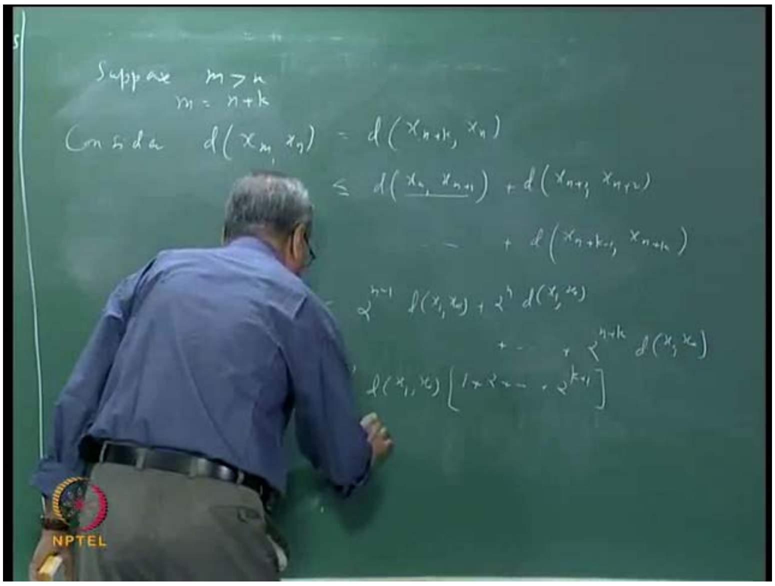

Because alpha is strictly less than 1, but that is not sufficient to show that it is a Cauchy's sequence. To show that it is a Cauchy's sequence we must estimate distance between x m and x n and we should show that that goes to 0 as m or that can be arbitrarily small when m and n are large. So, let us do it that way. Now instead of considering x n plus 1 and x n, consider distance between x m and x n. Now of course, if n is same as m this is 0 there is nothing, so consider m and n to be different.

If m and n are different one of them is bigger than the other. So, suppose m is bigger than n, bigger than n means what? It should be n plus something n plus 1, n plus 2. So, if m is bigger than n I can write m is equal to something let us say n plus k. So, this is same as distance between x n plus k and x n. Otherwise say that this is less than or equal to see it is distance between x n and x n plus k, this will be less than distance between x n and x n plus 1 and x n plus 2 etcetera.

Finally, x n plus k minus 1 and x n plus k so, we can say this is less than or equal to distance between that is used. So, this is less than or equal to distance between x n and x n plus 1 plus distance between x n plus 1 and x n plus 2 etcetera. Final number should be distance between x n plus k minus 1 and x n plus k. Now, for each of these numbers on the right hand side, we have estimates here.

So, this number distance between x n and x n plus 1, we have shown that that is less than or equal to alpha to the power n minus 1 into d x, x naught. So, this number is less than or equal to alpha to the power n minus 1 into d x 1 x naught. What about this, x n plus 1 and x n plus 2 again by the same argument that should be less than or equal to instead on n minus 1 it will be n. So, this will be less than less than or equal to alpha to the power n, d x 1 x naught. The next will be less than or equal to alpha to the power n plus 1 into d x 1 x naught and why not like this? This last will be less than or equal to what about this alpha to the power n plus k into d x 1 x naught. So, this d x 1 x naught is a common factor.

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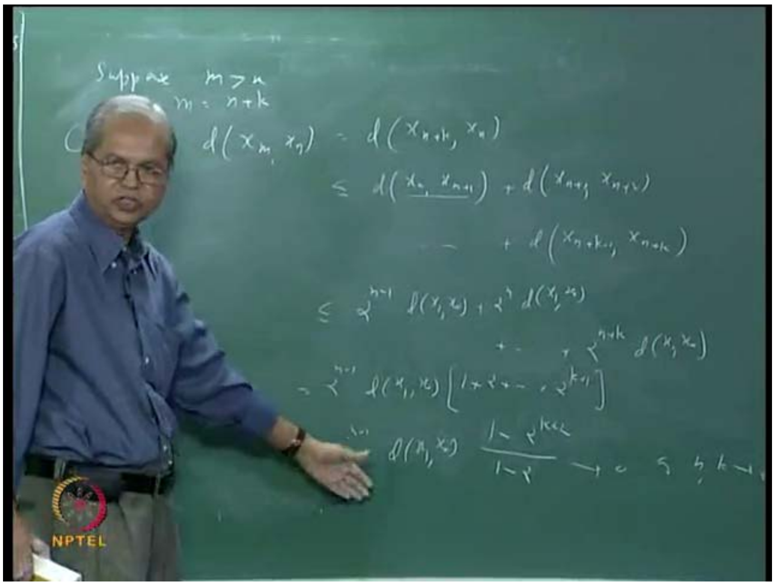

So, d x 1 x naught into alpha to the power n minus 1, in fact alpha to the power n minus 1 is also common factor. So, this equal to alpha to the power n minus 1 d x 1 x naught into what remains inside, it is 1 plus alpha etcetera, etcetera, up to what will be the last thing alpha to the power k plus 1. Now, what can we say about this 1 plus alpha x plus alpha to the power k plus 1, it is a geometric progression with common ratio alpha so, this is same as 1 minus alpha to the power k plus 2 divided by 1 minus k.

So, let us say this is the terminal, this term alpha to the power n minus 1 into d x 1 x naught multiplied by 1 minus alpha to the power k, k plus 2 divided 1 minus alpha. Now, the whole idea is what we want to show is that, this distance between x m and x n can be made arbitrary small for large values of m and n that is the meaning of showing that x m is a Cauchy's sequence.

If m and n both are large it is same as saying that m and k, m is n plus k, it is if I say that m and k both are large. So, if k is large this number alpha to the power k plus 2 goes to 0 alpha to the power n minus 1 also goes to 0. See saying that m and n both are large same as n and k are large. So, in other words I can say that this goes to 0 as n and k tends to infinity or which is same as saying that n and k tends to infinity is same as saying that m and n k tends to 0.

In other words we can make this argument quite precise and say that given epsilon bigger than 0, we can find some n 0 such that whenever m and n is bigger than that m 0. This distance between x m and x n will be less than this term, but which will be less than epsilon.

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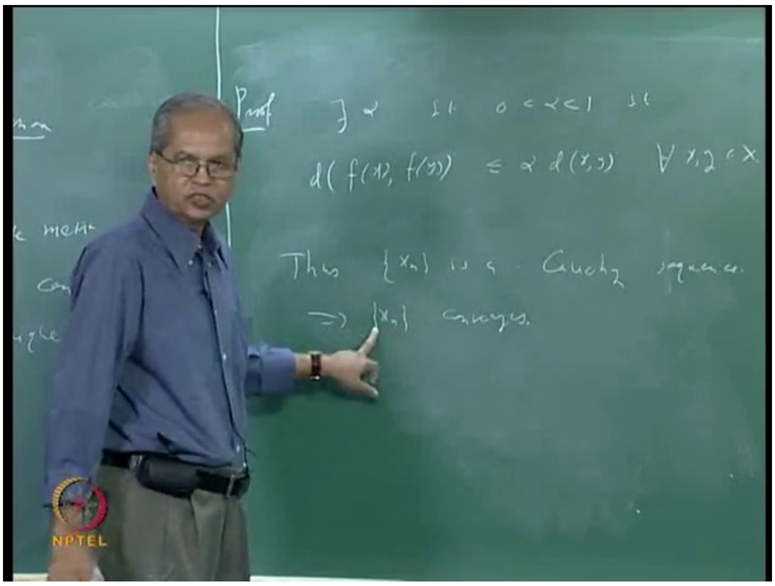

So, that shows that x 1, I will come back here that shows that x n is a Cauchy's sequence. Thus x n is a Cauchy's sequence and that was it, it is same as Cauchy's sequence and x is complete. So, this implies that x n converges that completes the proof of the claim that we wanted to show that x n is a convergent sequence. And we already shown that once we show that x n is a Cauchy's sequence x n converges and once we show x n converges. Suppose x n converges to x star, we have already shown that that x star is a fixed point of this map f and that completes the proof of this theorem. We will stop with this here.