Real Analysis
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Lecture - 24
Continuous Functions on a Metric Space

So, we were discussing the properties of continuous functions, let us recall that we had proved the following.

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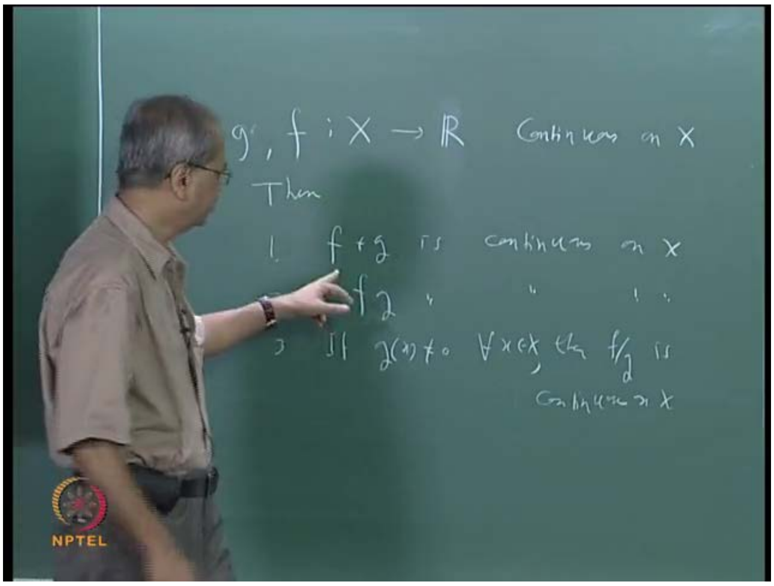

That if you take function f from x to y then f is continuous if and only if inverse image of any open set, inverse image of every open set in y is open set in x and that is equivalent. This is something quite useful in deciding about the continuity of a function and also proving several properties of the continuous function. Is it clear to you that from this it follows that if x is discrete metric space, then every function on it is continuous, is that clear because whatever open set you take, here its inverse image whatever be the set in x.

Since, in x every set is open, so any function defined on a discrete metric space is always continuous fine, now let us also record a few more things that will follow from this. Now, let us say suppose you have two function suppose g from x, let us say g and f let us say r continuous then now this for this let us take this metric space as R. Suppose f and g are real valued functions and suppose both of them are continuous then we want to say this let f plus g is continuous, secondly f g is continuous.

If g of x is not 0 everywhere in x then f by g is also continuous and here by continuous means continuous on x everywhere of course we can also write a theorem at a particular point. So, if g x is not equal to 0 for every x in x then f by g is continuous on x as I said, here has stated theorem for continuous on x, but you can as well state it for continuity at a point instead of say on x if I change this that suppose both are continuous at some point. Let us say a point x naught in x suppose f and g are continuous at a point x, naught in x then f plus g is continuous at that point x naught f g is continuous etcetera.

Last thing will be if g x naught is not 0 then f by g is continuous at that point x correct, now I just, as we have seen as far as proof of any of these things are concern it will follow from two things. You just make two guesses that to show that function is continuous at all points in x, just take any arbitrary point suppose that point is let us say x naught. If that is an isolated point then it is already continuous there is nothing to be proved, if it is a limit point then it then the required conversion will follow from the corresponding theorem over the limits. For example we already proved that limit of f x f plus g of x as x goes to x naught that is same as for example, that is for example, we have proved that.

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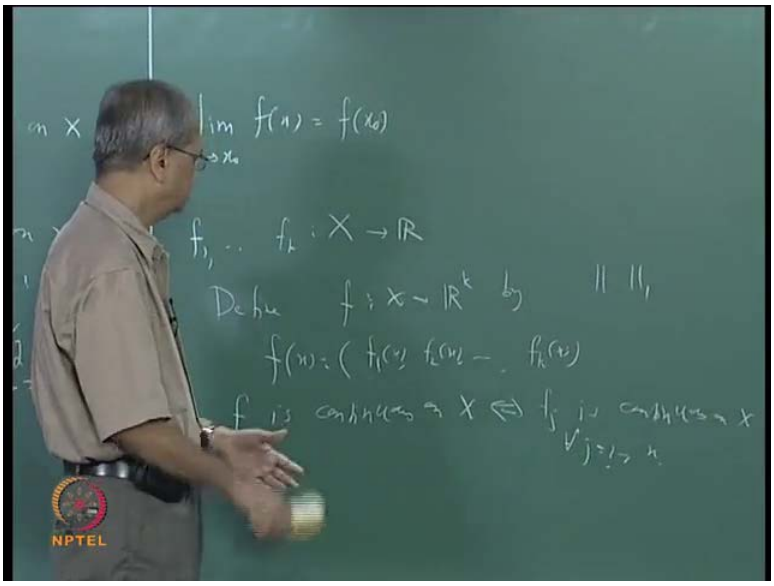

Limit of f x that is f is continuous at x naught f is continuous, see x naught is a limit point saying that f is continuous at x naught means this. Similarly, g is continuous at x naught will be limit of g x is extends to x naught, this g of x naught and then by the corresponding theorem of the limits, it will follow that limit of f plus g at x naught is same as f x naught plus g x naught. That will show that f plus g is continuous at that point x naught, since x naught was arbitrary on the whole of x and similarly other two things also can be.

So, the point is conceptually there is nothing new in this, all this assumptions will follow from the corresponding assumptions about the about the limits. Then we can also say one more thing, here suppose we take let us say k functions let us say f 1, f 2, f t. Suppose these are k functions from x to R suppose each, let us I will talk of continuity for the time being, just take these are real valued functions.

Then I can construct what is called a vector valued function that is the functions which goes from x to R k. Suppose I define that function as f, f is a function define f from x to R k by f of x is equal to f 1 x, f 2 x etcetera f k x. Then f will be a function from x to R k any if you take k real valued functions it will lead to a function f from x to R k. Similarly, other way also if you take any function x any function f from x to r k that give rise to k real valued functions. So, similar just similar things as we have seen in the case of sequences, now what we want say is the following if all of these functions are continuous then f is continuous.

Conversely if f from x to R k is continuous then each of this must be continuous, by the way these are sometimes called coordinate functions given by this function f. So, as I shall this f is continuous on x of course, here when I say R k, I should say what is the metric on R k, but actually it does not matter which ever metric you take this theorem is true. But, to just make the, just to make the things concrete let us take the metric given by let us say norm suffix 1.

But the same more or less the same proof will work for any other metric or any other metric given by any of those l p norms f is continuous on x if and all if f j is continuous on x for each say. To see this, all that we need to see the relationship between the absolute value in R and norm in R k. Since already use this small x for the points in x, here let me use something else for the point in, here R k.

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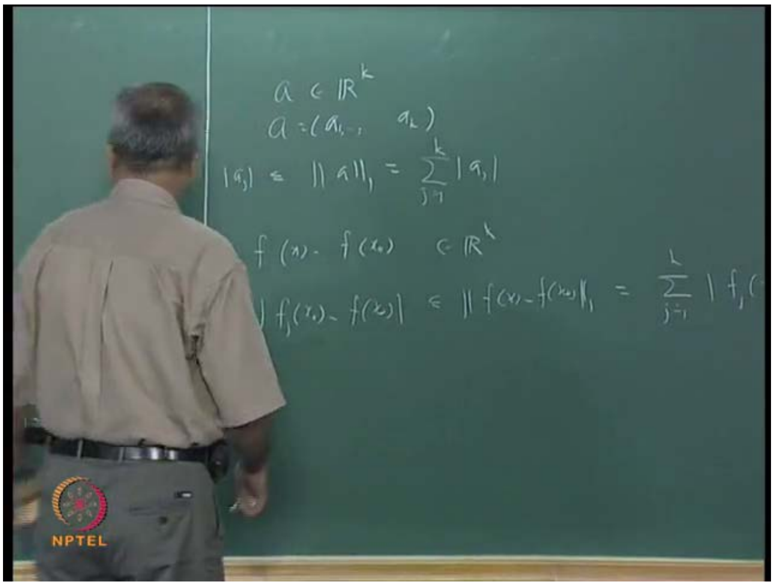

So, suppose here I take a point, let us say a in R k, so a is a 1, a 2, a k, so what we know is the following. That is norm of a suffix of 1, norm of a suffix 1 this is nothing but sigma mod a g, g going from 1 to k and so in particular each of this mod, so each of this mod is less than or equal to norm of a, each of this mod is less than or equal to norm of a. So, in particular see, so how does, how this proof will go, suppose I want to, suppose we assume that f is continuous on x.

So, f is continuous at sum x naught let us say, suppose f is continuous at then let us say we want to each of this are continuous at x naught. So, just look at say at x naught minus f say of x essentially what we will use is this, we shall say that. Suppose we look at f say of x minus f j of x naught or let us say, let us say f, f of x, f minus f of x naught this will be an element in R k, f of x minus f of x naught this will be an element in r k, take that as this point a.

Suppose I apply this then what will happen that is z the coordinate of this z-th coordinate of that will be f say of x naught minus f at x naught that should be less not equal to norm of f x minus f naught. Now, and let us also say what is and what is this norm of f x minus, f x naught this is nothing but sigma naught f x minus f of x naught. So, once you realise this inequalities the proof of this is immediate, for example suppose f is continuous on x that means let us say f is continuous at x naught for each x and we want to show that is continuous at x naught.

Now, what is the meaning of saying x is continuous at x naught, it means given any epsilon, given any epsilon you can find a delta. Such that whenever distance between x and x naught is less than delta, distance between f x and f x naught is less than epsilon that means whenever distance between x and x naught is less than delta this is less than epsilon. But, if this is less than epsilon this also should be less than epsilon because this quantity is less not equal. So, that shows f say is continuous and that will hold for each and what will be the reverse argument.

Suppose we assume that each f continuous then again similar thing which we have done, that is if each of this let us say if each of this can be made less than let us say epsilon by k than the sum will be less than epsilon. So, we can say that if each f say is continuous we can always find a delta such that whenever distance between x and x naught is less than delta mode f say x minus f x naught is less than epsilon by k. This we can do for each k this we can do, sorry this we can for each and then if each of this is less than epsilon by k then sum is less than epsilon.

So, that will mean that norm of f x minus f x naught is less than epsilon, so basically the whole thing follows from this. Once you realise this you will see that if I take some other norm, here instead of this let us say norm suffix 1, suppose I take norm suffix 2 or norm suffix infinity whatever you do the similar argument will work. Now, let us also recall one more thing we have seen that a function is continuous this is, this implies that if you take let us, let us.

Let me start from this, suppose let me take any other way suppose f from x to y is a continuous function. Then we know that, from this it follows that if you take any convergence sequence in x, its image is y must be convergence sequence because we have seen that equivalent of two definitions of limit correct. So, f is continuous means for any let us say for any x naught limit of f x has extends to f x naught. So, if you take any sequence x n converging to a x naught f of x, n must converge to f of x naught, so we can see that f from x to y is continuous this is actually if and only if for if x n for every convergence sequence x n.

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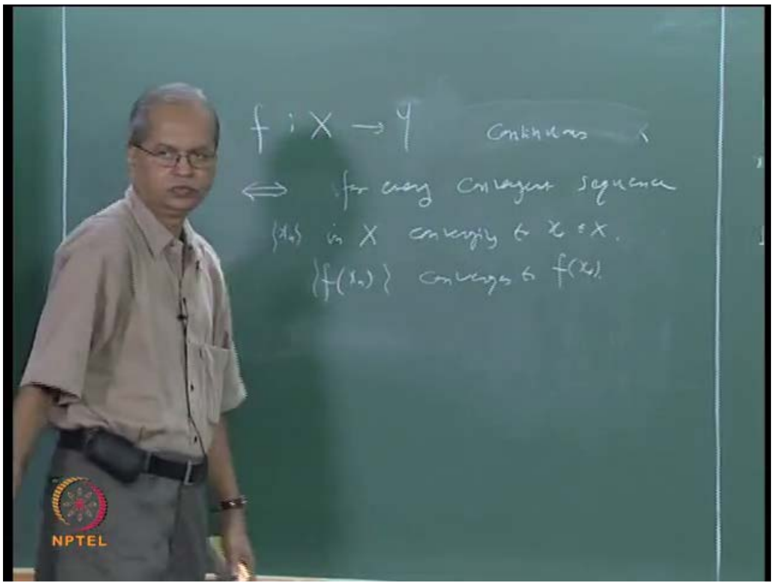

Let us say sequence x n in x converging to, converging to let us say x naught in x f x n converges to f x n, so continuous maps of the property that image of every convergence sequence in x is a convergence sequence in y. Let us ask next question what can we say about Cauchy sequences, can we say that image of every Cauchy sequence is also a Cauchy sequence, to answer this question best thing is will look at an example.

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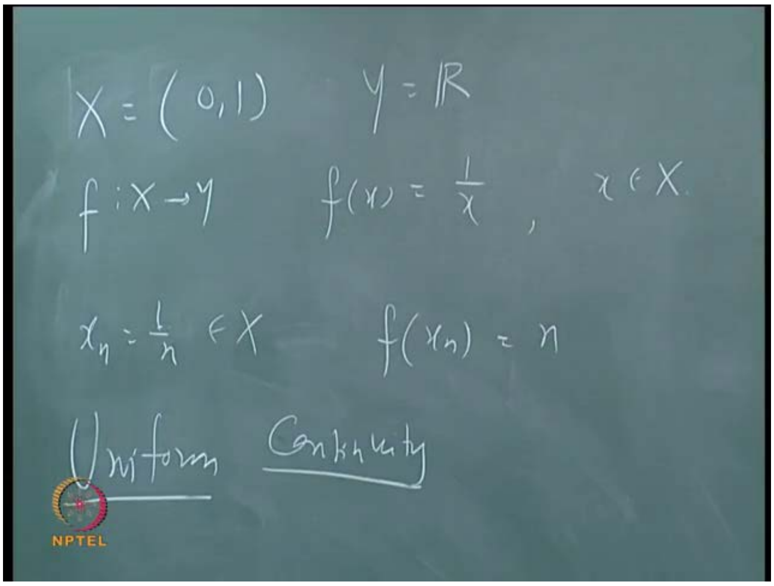

I will take x as the open interval 0 to 1 with the usual metrics and say and I will take y as R y as R then define f from x to y by f x is equal to 1 by x is that a continuous function, that is a continuous function. Suppose at x n is equal to 1 by n that is the sequence in x, of course it is not a convergence sequence, it is not because 0 is not in x it a it does not converge to any point in x. But, that is not important is it a Cauchy sequence, it is it is a Cauchy sequence what is f x, n f x n is n, f x n is n is that a Cauchy sequence in R it is not, it is not.

So, what does it mean in general image of Cauchy sequence under a continuous function need not be a Cauchy sequence? Though we know the image of a convergence sequence or continuous function will always be a continuous function, image of a Cauchy sequence did not be a Cauchy sequence. Basically, to take care of this kind of things and of course for few other things, we now discuss a little stronger type of continuity that is called uniform continuity. Now, even before coming to this definition uniform continuity let me also introduce one or two more terming more definitions, let me remove this for the time being.

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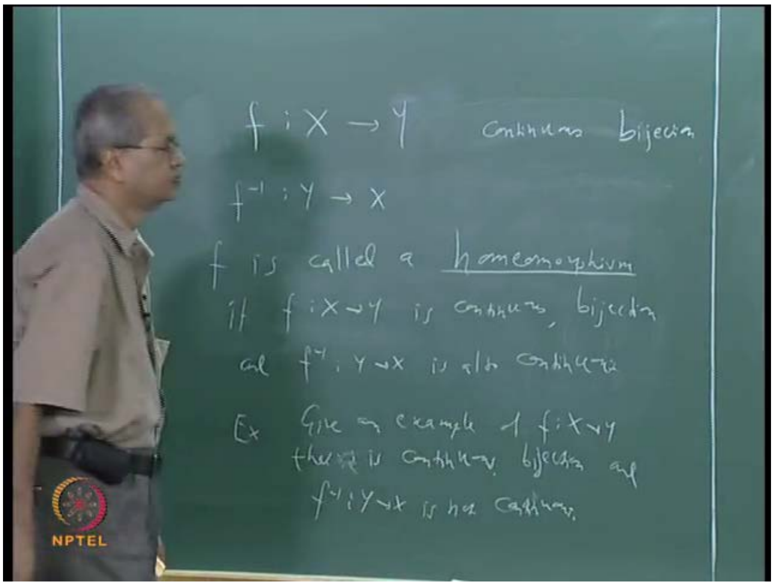

Suppose, f x n y are two metric spaces and f from x to y is continuous, let us say, suppose in addition f is also a bisection suppose f is a continuous and bisection, bisection means it is, it is 1 1 and 1 2. We have, we have known that if of a function is bisection we can define what is mean by inverse function, so then we can define then f inverse is a function from y to x. But, f inverse may or may not be continuous, if f is continuous in bisection inverse function exist.

But, in inverse function may or may not be continuous in general unless we put some additional on x y f etcetera. But, when it is also continuous that map f is called homeomorphism, so we will say that f is called a homeomorphism if f from x to y is continuous it is bisection and f inverse from y to x is also continuous.

We can use obvious examples of homeomorphisms, for example map going from x to x that is continuous, its inverse is also continuous also we can also construct a non trivial example is not very difficult. Let us use as an exercise, construct an example of a function which is continuous and bisection, but whose inverse is not continuous. So, let me just do it as an exercise, so give an example of f from x to y that is continuous bisection.

But, f inverse is not continuous that means it is continuous bisection, but not homeomorphism f inverse from y to x is not continuous is this clear what is mean by homeomorphism. What should happen is that the function f should be continuous inverse should exist, so it is a bisection and inverse also must be continuous. Now, if you are given two metric spaces we say that those two metric spaces are homeomorphic to each other we say y is homeomorphic to x if there exist homeomorphism from x to y.

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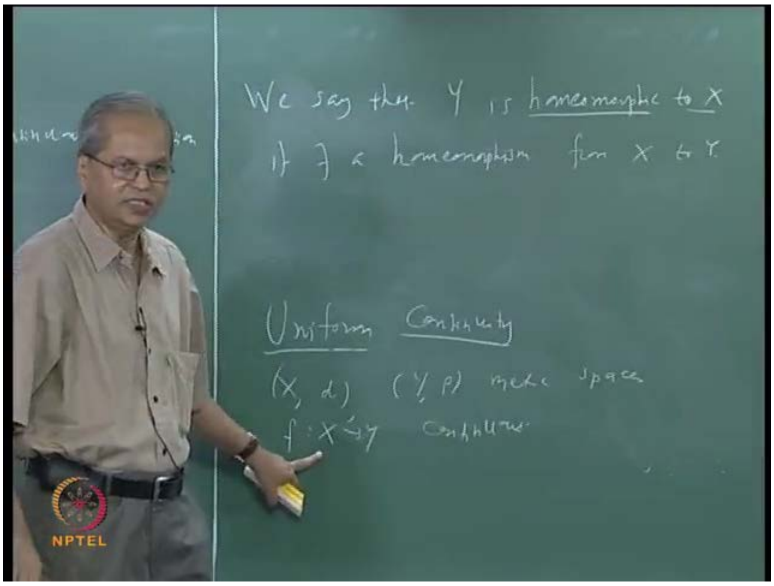

So, given we say that we say that y is homeomorphic to x y is homeomorphic to x if there exist a homeomorphism from x to y. Of course there is nothing unique about homeomorphism, given two metric spaces there may, there may, there may not exist any homeomorphism there may exist several homeomorphisms also. Now, this establishes a relation between the set of all metric spaces, if you take class of all metric spaces this says how two metric spaces are related to each other this is it clear to you this is an equivalence relation x is homeomorphic to itself.

Thus, identity map is a homeomorphism, if x is homeomorphic to y does it y is homeomorphic to x, x is homeomorphism going from x to y f inverse y is also homeomorphism going from y to x what about transitivity. Suppose we know that x is homeomorphic to y and let us say y is homeomorphic to z, will it follow that x is homeomorphic to z. Obviously what you do is that just take the conversion of two homeomorphism and show that composition of is also homeomorphism and to do that we shall use the theorem that we reproduced that composition of two continuous maps is continuous.

So, this is an equivalent solution and as we have seen earlier any solution will partition the given class into whatever called equivalence classes. So, every metric space in that equivalence class will be homeomorphic to all other metric spaces in that equivalence class. Now, let us come to this uniform continuity, now here we talk of uniform unlike we can talk of continuity at a point. But, we talk of uniform continuity on a set uniform continuity on a set of course in particular it can be a set, but it will be trivial. So, let us just take this case suppose x and y are two metric spaces and as I said yesterday when the metric underlying metrics are not very important for discussion, I will not say explicitly that x d is a metric space or y rho is a metric space etcetera.

Whenever that is required we will, we will see that for the time being let us let me take this, let us say x d and y rho are metric spaces. Suppose f is a map which goes from x to y f is a map which goes from x to y and suppose let us say it is continuous, suppose x to y is continuous. Now, let us see recall the definition of continuity, here once again what we have said in the definition is that is that given any epsilon bigger than 0 that is f is continuous means f from x to y continuous means, f is continuous at each x f.

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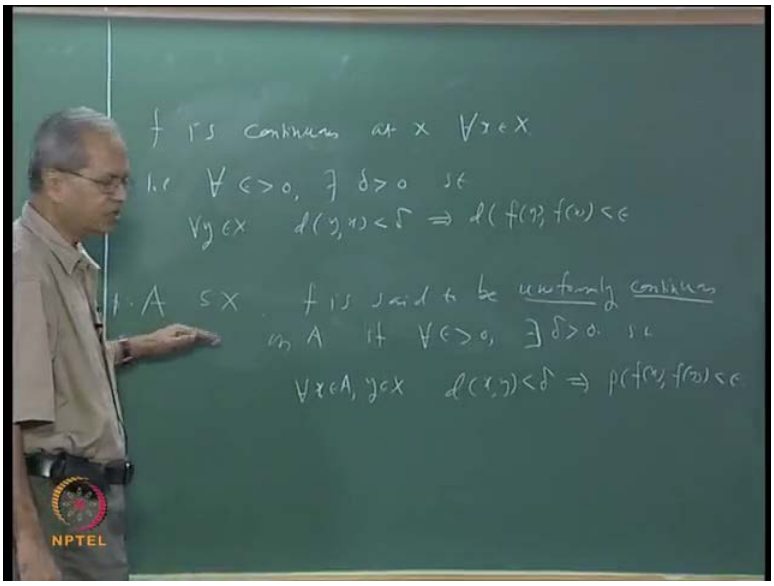

If this means f is continuous, f is continuous at x for every x in x for every x n, x f is continuous at x. Now, what is the meaning, it means that that is for every epsilon bigger than 0, for every epsilon bigger than 0, there exist delta bigger than 0 such that for all y in x distance between y and x less than delta implies distance between f y and f x is less than epsilon. Now, the whole idea of uniform continuity deals with how does this delta depend on epsilon, we know that delta may depend on epsilon if epsilon is changed delta is to be changed.

But, it is not my definition is clear, the definition of continuity that delta will depend on this x also in general, for example suppose at some other point is x 1 or x 2. Then it is not, it is not necessary the same delta will work for x also right, so if it is, so happens you can find in some delta which works for all elements. In some sets which works for all elements in some set or in particular for the whole of x then we say that f is uniformly continuous on that set A and if A is equal to x, we say f is uniformly continuous on x.

So, let us say that let me just take this definition, so suppose now A is a subset of x f is said to be uniformly continuous on A, uniformly continuous on A. If for every epsilon bigger than 0 there exist delta bigger than 0, such that this delta will work for all elements in A, this delta will work for all elements in A such whenever take any x in A and any y in x. So, what we say this is important such that, such that for every x in A, for every x in A and let us say y in x distance between x and y less than delta, this implies distance between f x and f y less than epsilon.

Remember once again uniform continuity is always discussed with respect to a set, we always say a function is uniformly continuous on a set, whereas continuity we can say function is continuous at point or on a set, both are possible. Of course the special case is when a is equal to x when A is equal to x, then f we say f is uniform continuous on x A, of course A can be a singleton set. But, it only be that f is continuous, at that point if A has only one point obviously whatever delta works for that point it works for all point in that set A.

So, if A is only one point then any function continuous at that point is also uniformly continuous on A is it also clears a small modification of this. We can also make this work if A is a finite set, suppose A is a finite set if there are finite number of points given any epsilon we can find let us say there are, let us say there are n points A , A 2, A n. So, for each A 1 each of these A 1, A 2 find delta 1, delta 2 delta n, suppose delta 1 works for A 1, delta 2 works for A 2, etcetera delta n works for A n. Then what you do obviously you take the minimum of all those deltas, you take minimum of all those deltas that will work for, that will work for every point in that set.

So, what we have essentially said just, now is that every function which is continuous it will always be uniformly continuous, on a finite set every continuous function will always be uniformly continuous on a on a finite set. So, that is not very important, so really uniform continuity really matters when we take A as a infinite set and that may or may not happen a function may be continuous. But, may not be uniformly continuous let us see some examples, so that these ideas will be clear. So, let me start with familiar spaces, so let us say f R to R by the way let me again go back to our favourite space discrete metric space.

We already seen that every function define or discrete metric space is continuous is it also clear that it is continuous, uniformly continuous also. Suppose x, suppose x is a discrete metric space then what you can do is that whatever be the epsilon given I can take this given delta to be half. Suppose it take delta to be half then d x y less than delta can happen only if when x is equals to y and in that case distance between f x and f y will be just 0. So, on a discrete metric space not only that every function is continuous, but it is also uniformly continuous.

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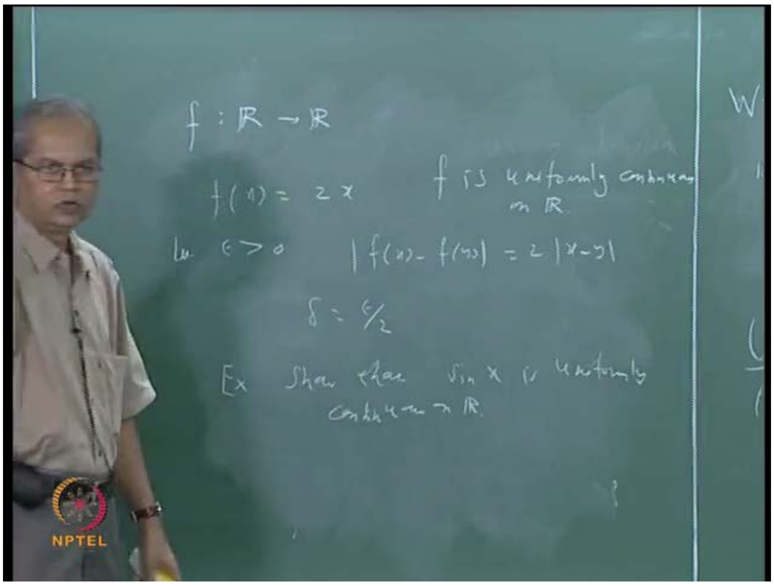

Now, let us take this space and let us look us this function f x is equal to 2 x, we know that this is a continuous function, the question is whether it is uniformly continuous or not. So, to do that let us look at say take some epsilon let epsilon be bigger than 0 then we have to find delta such that whenever mod x minus y is less than delta mod f x minus f y should be less than epsilon.

So, mod f x minus f y this is a thing, but more 2 x minus 2 y, so there is nothing but two times more x minus y and we want to choose delta in such a way that whenever mod x minus y is less than delta this should be less than epsilon. Now, the choice here is obvious you take just delta is equal to epsilon by 2 delta is equal to epsilon by 2 and this delta will work regardless of whatever x and y.

As soon as mod x minus y is less than, less than epsilon by 2, mod f is minus f y is going to be less than epsilon, so this is an example of a uniformly continuous function. So, f is uniformly continuous on R is uniformly continuous on R, now let us see some other function which is not uniformly continuous. Now, before proceeding further let me give you an exercise to try on your own show that sine x is uniformly continuous on R.

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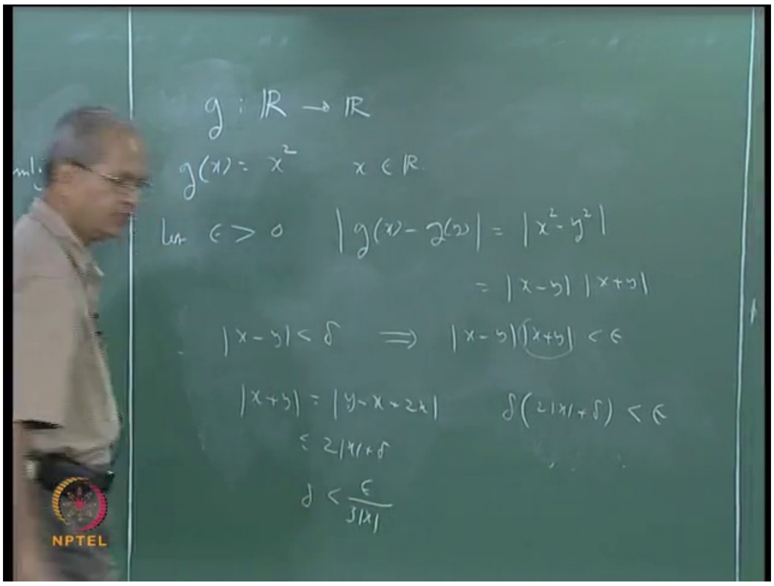

Let us take another function g from R to R suppose I take g of x is equal to x square, now let us try to do the same thing that is given epsilon bigger than 0, try to find delta, find the requirement. So, let epsilon be bigger than 0, let epsilon be bigger than 0 then what is g of x minus g of y, so if we look at g of x minus g of y absolute value of this that is the thing, but mod x square minus y square that is the thing. But, mod x square minus y square and this is the thing, but we can write this as mod x minus y into mod x plus y. Now, what we want then mod x minus y is less than delta mod x minus y is less than delta, this whole thing should be less than epsilon.

So, how do we choose delta like that, but y is something that will depend let us say what we want is this mod x minus y less than delta this should imply mod x minus y into mod x plus y this is less than epsilon. So, what we have to do is that let us see that whenever mod x minus y is less than delta what is the maximum possible value of this x factor mod x plus y. So, we can say that mod x plus y, we can say that this is the I can write this as mod x minus y plus 2 y or better still let us say we are talking about continuity at x we can write it as mod y minus x plus 2 x.

So, this will be less than or equal to two times mod x plus delta, two times mod x plus delta, so if I take, if I choose this will be what I want is delta into two times mod x plus delta, this should be less than epsilon, this should be less than epsilon. Now, we can see the idea the one the way the way in which one does is that see, we know that if any particular delta works, any delta smaller than that that will also work, any delta smaller than that will also work. So, I can also make another decision that I will choose delta smaller than mod x, of course to choose delta smaller than mod x we must require that mod x is not 0, we can make that if mod x is equal to 0 that means x is 0.

For that, we shall think of something else, for that we shall think of something else that case we can discuss separately. Suppose we take the case mod x is bigger than 0 then if I choose delta in such a way delta is less than mod x then this whole thing will be less than 3 times mod x. So, that means choose delta in such a way that delta into 3 times mod x is less than epsilon, so that means delta should be less than epsilon by 3 times mod x that is a requirement and we also want delta is less than mod x.

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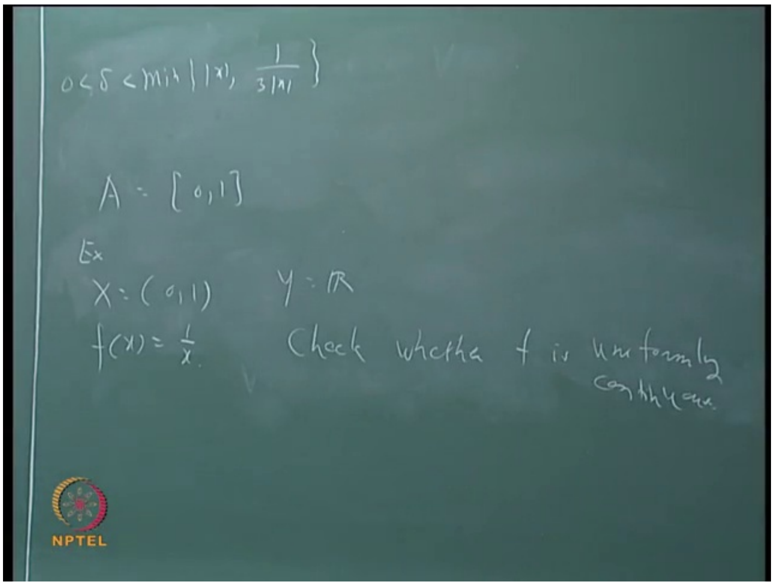

So, we can say that look at minimum of this two numbers, minimum of mod x and 1 by 3 times mod x and take delta to see if mod x is bigger than 0, if x is not zero this number is strictly going to be bigger than 0. Choose any delta which is smaller than this, that is 0 less than delta less than this that delta will work, that delta will work. But, it is clear from this discussion that this choice of delta depends on x choice of this delta depends on x if you change the point x, the value of delta will change value of delta will change. So, we cannot the same delta will not work for any arbitrary x, so this function is not uniformly continuous, this function is not uniformly continuous on R.

But, on the other hand suppose I take some close interval in R, instead of taking the whole of R suppose I take a as simply 0 to 1, suppose I take a as simply 0 to 1 then what you can do is that take the maximum possible, take this minimum over all x which belong to that. Then take minimum value of that and then we can find delta which works for all elements, so it is not uniformly continuous on R.

But, it is uniformly continuous on say its like this it is uniformly continuous on sets like this, so uniform continuity as I said earlier uniform continuity is a matter depends on the set on which you are taking from uniform continuity. If you change the set, if function may be uniformly continuous on some set A it may not be uniformly continuous on some other set B. Now, let us go back to the whole point we started with we said that given an arbitrary continuous function image of convergence sequence is always convergent.

But, image of a Cauchy sequence need not be Cauchy, now does uniform continuity corrects that is can we say that if a function is uniformly continuous then image of a Cauchy sequence is again a Cauchy sequence. Now, before going to that question let me again give you an exercise the same function which we discussed take x is equal to 0 to 1 and y is equal to R and f of x is equal to 1 by x. Check whether f is uniformly continuous this is an exercise check whether f is uniformly continuous, now coming back to that question.

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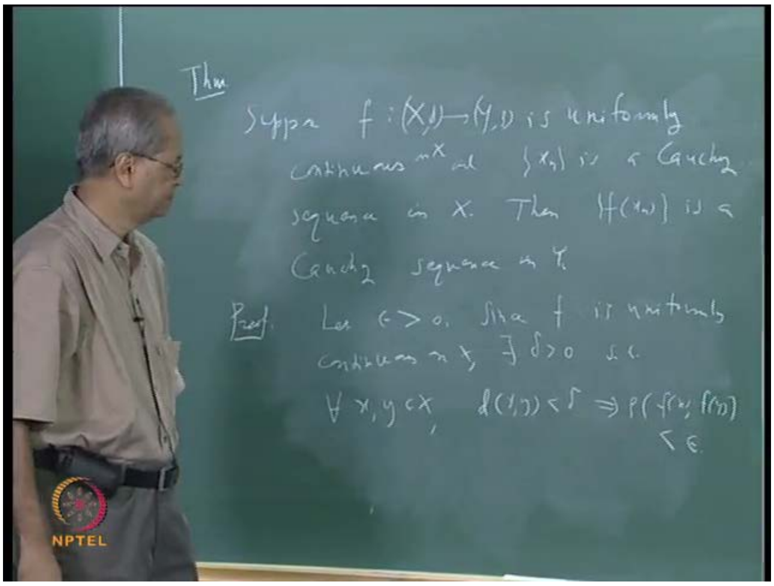

So, suppose f from x to y is uniformly continuous, uniformly continuous and x n is a Cauchy sequence in x sequence in x, x n is a Cauchy sequence in x. Then f x n is a Cauchy sequence in y, so this shows that under uniformly continuous function image of a Cauchy sequence is a Cauchy sequence. So, anyway how does one show that something is a Cauchy sequence, just go by the definition take epsilon bigger than 0. Then we have to find n, 0 says it whenever n, m and n both are bigger than equal to m 0, distance between f of x n and f of x m should be less than epsilon.

So, let epsilon be bigger than 0 then for this epsilon we will have to find n 0 that is find that requirements. But, we know that f is uniformly continuous and x n is a Cauchy sequence, so these things we should use these things we should use. First of all, f is uniformly continuous that means for this epsilon there will exist some delta such that whenever distance between the two points, here is less than delta distance between their images, there is less than epsilon.

So, since f let me, now use some metrics also here x d and y rho, since f is uniformly continuous, of course uniformly continuous on x uniformly continuous. There exist delta bigger than 0 such that for any two points x and y, for any two points x and y in x distance between x and y less than delta.

This implies distance between f x and f y is less than epsilon and remember, here that it is important that to know that this delta does not depend on what is x and y this is important this part is important for every x and y in x this same delta works. Unlike that function there where delta will depend it on the corresponding choice of x, now we know that x n is Cauchy sequence. So, what I can say is that for this positive number delta, there will exist some n 0 such that whenever n and m is bigger than or equal to 0, distance between x n and x m is than delta.

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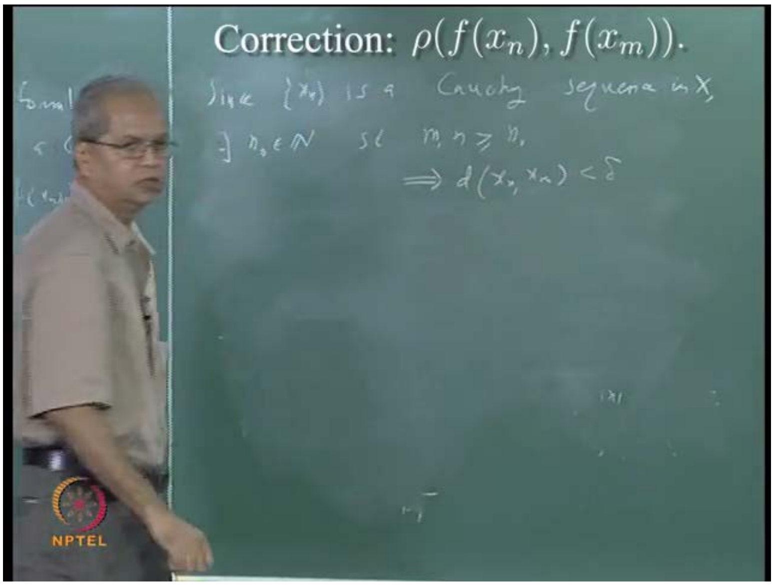

So, since x n is a Cauchy sequence in x, since x n is a Cauchy sequence in x there exist n 0 in n such that m and n bigger not equal to n 0 implies distance between x n and x m is less than delta. Now, if distance between x n and x m is less than delta what about distance f of x n and f of x m that should be less than epsilon because you take x as x n and y as x m here. So, this implies distance between f of x n and f of x m is less than epsilon.

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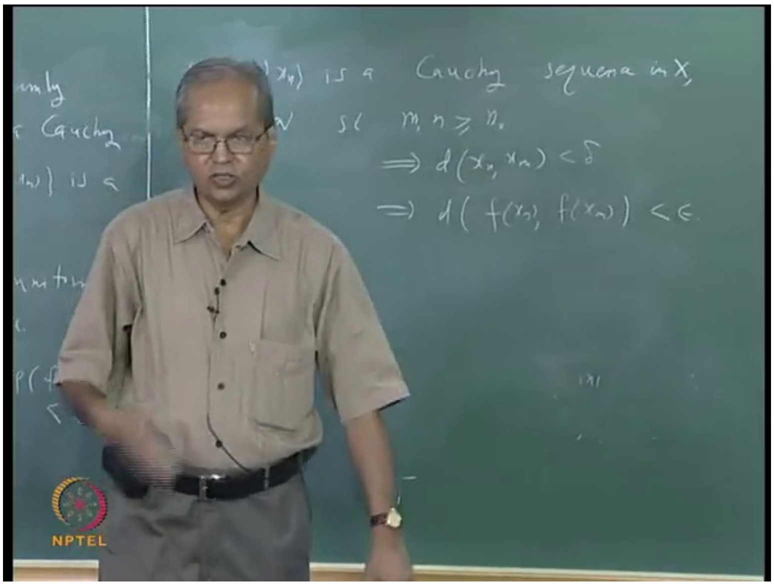

So, we have proved that image of a Cauchy sequence, under uniformly continuous function is again a Cauchy sequence which is this is a property, which is not true for an arbitrary continuous function. So, uniform continuous function it is a stronger type of continuity, obviously every uniformly continuous function is continuous. But, the converse is false we have seen in, we have seen it in the examples. Now, let us see some common examples or popular examples of uniformly continuous functions, in fact some of these types are also given certain particular names.

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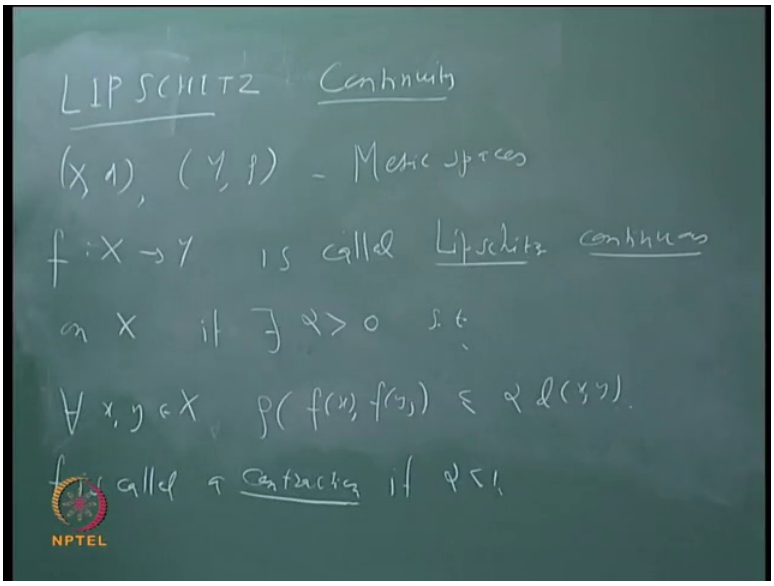

One of those is what is called Lipchitz continuous function, Lipchitz continuous Lipchitz is again a name of a mathematician perhaps you may have heard of Lipchitz continuity while discussing differential equations. So, again let us say that take two metric spaces x d and y rho metric spaces and suppose we take function f from x to y function f from x to y this is called Lipchitz continuous, is called Lipchitz continuous on x this is called Lipchitz continuous on x. If there exist some alpha bigger than 0 if there exist some alpha bigger than 0 such that, such that distance between if suppose take any two points in x and y.

Here, the distance between f x and f y is less not equal to this number alpha times distance between x and y. So, what should happen is that for every x and y in x, for every x and y in x distance between, so rho f x, f y is less not equal to alpha times distance between x and y. Now, it is obvious that every Lipchitz continuous function is uniformly continuous because you can just take delta as epsilon by alpha.

Suppose, you take delta as epsilon by alpha, whenever this is less than delta distance between f x and f y will be less than epsilon. So, every Lipchitz continuous function is uniformly continuous, of course it is natural to ask, here what about the converse. But, we shall not go into that question, now another special case of this if this constant alpha is strictly less than 1, and then this map is called contraction.

Lipchitz continuous function is called contraction if alpha is less than 1, so f is called contraction, is called contraction if alpha is less than contraction or contraction map it is called contraction or contraction map. See suppose I had not defined Lipchitz continuity at all, and suppose I want to define contraction directly. Then what I should have said is that there exists alpha such that 0 less than alpha, less than 1 and rho f x, f y less not equal to alpha times d x y for every x y in x that will be definition of contraction.

Here, since we already find is Lipchitz continuous function contraction is a special case of Lipchitz continuous function and you can also see why it is called contraction is called contraction because the distance between the images will be strictly less than distance between x and y. So, the distance is shorten that is why it is called it is called contraction.

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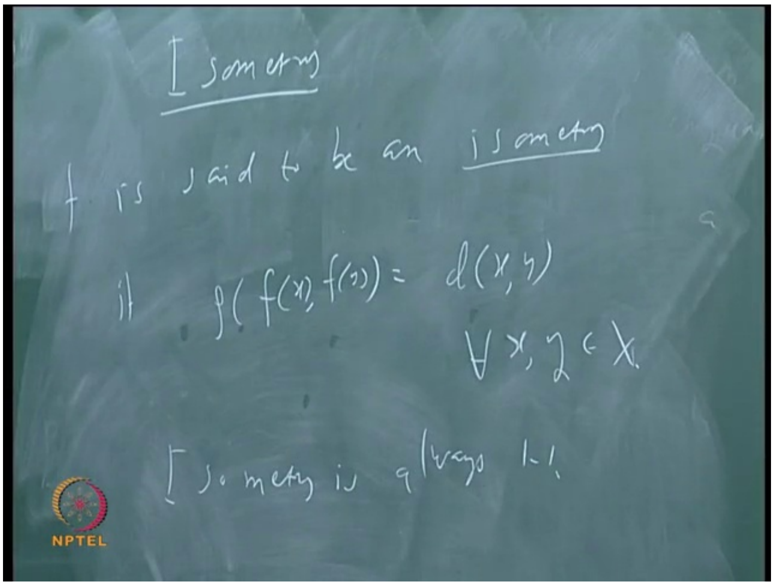

I will just make one more definition and then we will stop with that, this is something which you come across very often and it is what is known as isometry, isometry is a map that resource the distance f is called an isometry if distance between f x and f y its same as distance between x and y. So, f is said to be an isometry if distance between f x and f y is same as distance between x and y for every x, y, x for all these things Lipchitz functions contractions isometrics all of this are examples of uniformly continuous functions.

We shall come across examples like this it is also, it is clear from the definition of a isometry that isometry will be always 1 1. That is clear because suppose f x is equal to f y it will be in the true 0 that will be in the distance between x and y in 0 that will be x is equal to y.

So, Isometry is always 1 1 it may or may not be onto, isometry is may or may not be onto, but if it is also onto we say x is isometric to y, we say that x is isometric to y. If there is onto isometric, isometry will be always 1 1 by definition is an onto isometry means automatically becomes by section. Then we say that x is isometric to y and this also establishes a relation between two metric spaces just like homeomorphism when we say x is homeomorphic to y. So, similarly we say x is isometric to y if there is an isometry from x onto y this is also an equivalence relation, we will stop with that.