Real Analysis
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Lecture - 23
Limit and Continuity of a Function Defined on Metric Space

So, we shall continue with the discussion of limits of functions, let us quickly recall what we did in the last class.

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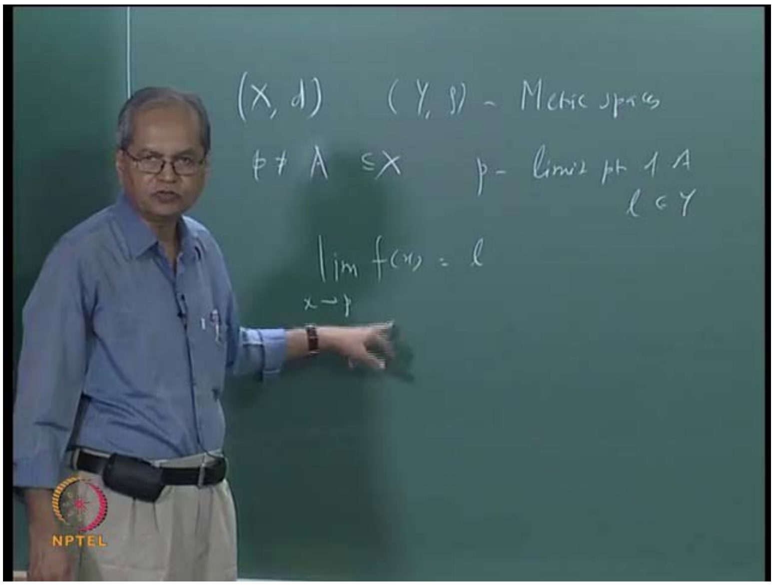

We started with the two metric spaces x d and let us say y rho, those were two metric spaces, and we took a subset E of x, E or I do not know it may be A also subset A of x not empty subset and p as a limit point of A, l some point in y. We define what is meant by saying that the limit of f x as x tends to p is equal to l, we gave the definition in terms of epsilon, delta etcetera I will not repeat that definition. But, we also proved that saying that the limit exist is equivalent to saying that given any sequence x n in A converging to p, the sequence f x in y should converge to l.

This is equivalent to saying that limit of f x is extends to p is equal to l, and one of the major use of this is to show when the limit does not exist. That is suppose you are able to find two sequences x n and y n both converging to p and suppose a limit of f x n and limit of f y n is different. Then obviously this limit cannot exist you have also seen examples of that kind, let us see a couple of more examples, let us for example.

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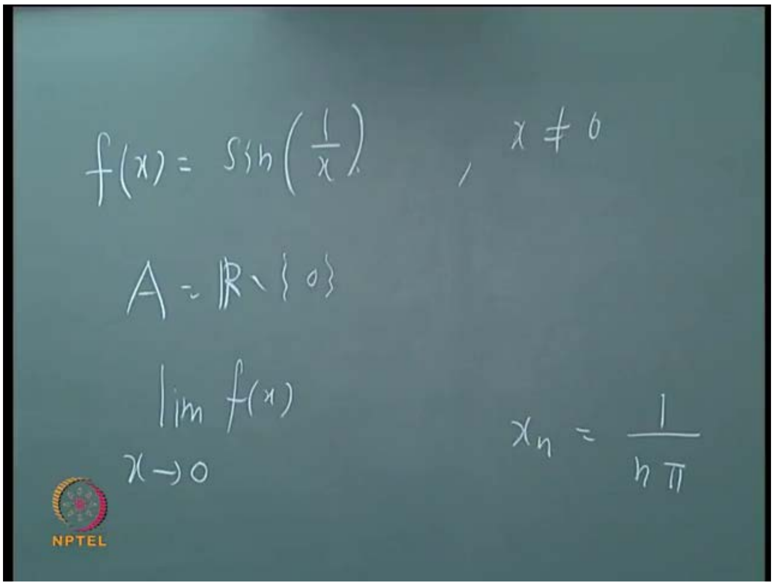

It is one of the famous example, f x is equal to sine 1 by x and suppose we want to discuss whether of course this is not defined at x equals to 0. But, it is defined for every x not equal to 0 correct, it is defined for every real number x which is not 0 and since it is defined for every non 0 x if you take this set a as R minus 0. Suppose, it is the set a as R minus 0, then 0 is a limit point of this set, so we can term of limit of f x extends to 0. So, we can talk of limit of f x as extends to 0, but we can talk does not mean that the limit exist, so what we can do is that for example, to show that the limit does not exist one can look at the sequences.

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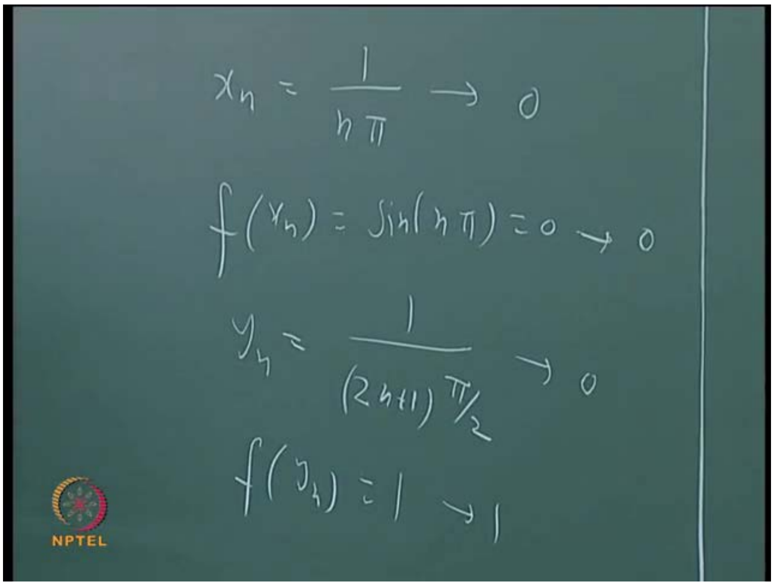

For example let us just take it like this suppose I take x n as let us say 1 by n pi, suppose I take x n as 1 by n pi then what is f of x n, this tends to 0, x n as 1 by n pi that tends to 0. What is f of x n it is sine n pi, f of x that is sine n pi and what is that for every n it is 0, so it is a constant sequence 0, so this tends to 0 that is fine.

Now, let us take something else suppose I take y n, y n is equal to instead of n pie let me take something else suppose 1 by let us say 2 n plus 1 pie by 2. So, that means pi by 2 pi by 2 things like that, this also tends to 0 and what about f of y n f of y n it is sine 2 n plus 1 pi by 2, so that is 1 for all f of y n is 1 for all n, so this tends to 1. So, we have found two sequences x n and y n both converging to 0, if x n converges to 0 if y n converges to 1, so limit cannot exist x goes to 0.

So, this is a simple way of showing that the limit does not exist, when the limit exist then of course, first of all you have to have some idea of what that limit should be. Then one way is that you can use epsilon delta definition and try to show that is that requirement is satisfied let us see one example of that type also, let me just make a small modification here.

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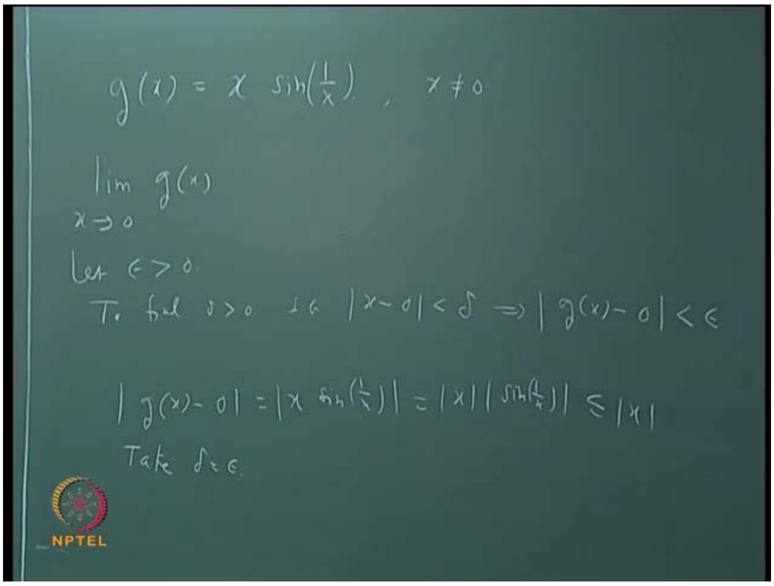

Suppose, I take this g x as this also one of the famous function I will multiply that by x, x sine 1 by x of course, for x not equal to 0 and will take the same set A. Now, let us ask the same question limit of g x as x tends to 0, so as far as this is concerned we have shown that this does not exist and that is the argument. Now, when you want to show the limit exists, first of all you have to have some idea of what this l should be then only you can proceed. In this case what do you expect then we can expect if at all the limit exist it has to be 0, so we should try to show that the limit, is limit is 0.

Now, when the limit is 0 let us say we just want to use the definition, so we shall just take let us say we are given epsilon bigger than 0 and then we want to show that for this epsilon we should want to find delta. To find delta bigger than 0 such that what should happen whenever mod of x minus 0 is less than delta, we should imply that mod of g x minus 0 should be less than epsilon. Let us just take this last thing, mod of g x minus 0 that is same as mod g x, right mode g x minus 0 this is the thing.

But, mode x, mode x sine 1 by x and we already know that modulus of a b is same as modulus of a b. So, that is same as mode x into mode sine 1 by x and now there is one obvious observation here that is this number sine 1 by x, mod sine 1 by x is always going to be less not equal to 1 whatever be x. So, this is always less than or equal to mod x, so what we want is that this mod x should be less than epsilon whenever mode x minus 0 is less than delta.

Now, the choice of delta is obvious we can just take delta is equal to epsilon, so whenever mod x is less than delta mod g x minus, whenever mod x minus 0 is less than delta mod g x minus 0 is less than epsilon. So, that means just take epsilon is equal to delta, just take delta is equal to epsilon and this works. Now, before proceeding further let me also take one more very famous example it is called it is sometimes or some books call it Lebesgue function.

You can define on it on any subset or let us say we define it on, let us say full f R to R will define it f of x, let me use something else let us say suppose I call it h. So, h of x is equal to 1 if x is rational and 0 otherwise, now if you take in this case we will be able to show if you take any x or any p for that matter the limit of h x of x goes to p does not exist and how does that follow. Let us take for the time being let us look at just p is equal to 0, so limit of h x as x tends to 0, now just we want to say the limit does not exist. We use our strategy, use two sequences both converging to 0 both converging to 0, but h of x n and h of y n should go to different limits, that is easy.

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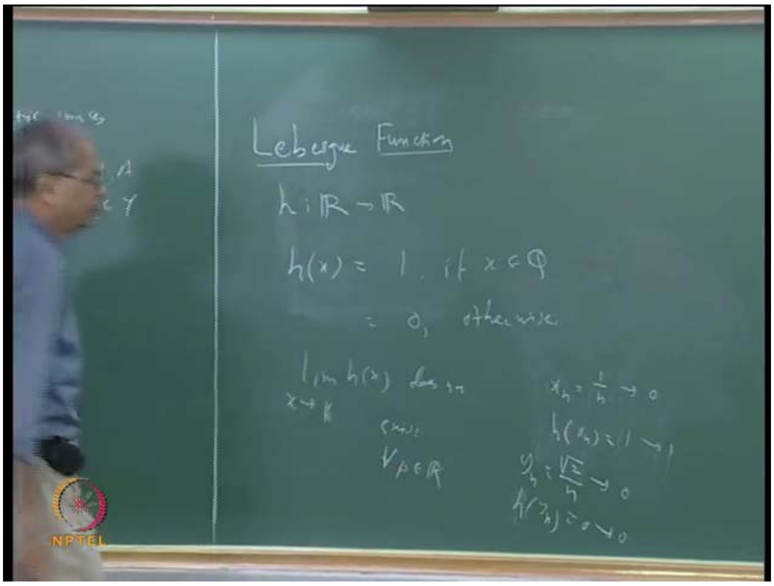

So, suppose let us say x n is equals to 1 by n, x n is equal to 1 by n that tends to 0, what about h of x n, h of x n is 1, for every n h of x n is 1, so h of x n is 1, that tends to 1, alright. Let me take another sequence y n, suppose I take y n as root 2 by n you can take any such sequence, basically it should be an irrational number. So, y n is root 2 by n and this tends to 0 and what about f of y n, f of y n has to be 0, f of y n has to be 0 this tends to 0.

So, again we have two sequences x n, y n both converging to 0, but h x n, sorry this is h y n, h x n and h y n go to different limits. So, the limit does not exist and is it clear to you that it is nothing particular about 0, here if I take any other number, still one can find one sequence of rational number convergence to it and another sequence of irrational numbers converging to it. You can produce the sequence like this, so this Lebesgue function limit of h x does not exist as x goes to p for any p limit of h x goes to p does not exist for any value of p.

So, let me just write it limit x does not exist, does not exist for whatever p for every p in R, alright. Now, let us see a few special types of functions and certain theorems about those functions and we will see again that we shall use the corresponding theorems about the sequences. We shall immediately get conclusions for the theorems about limits of functions, so to do that we shall take this space y as R.

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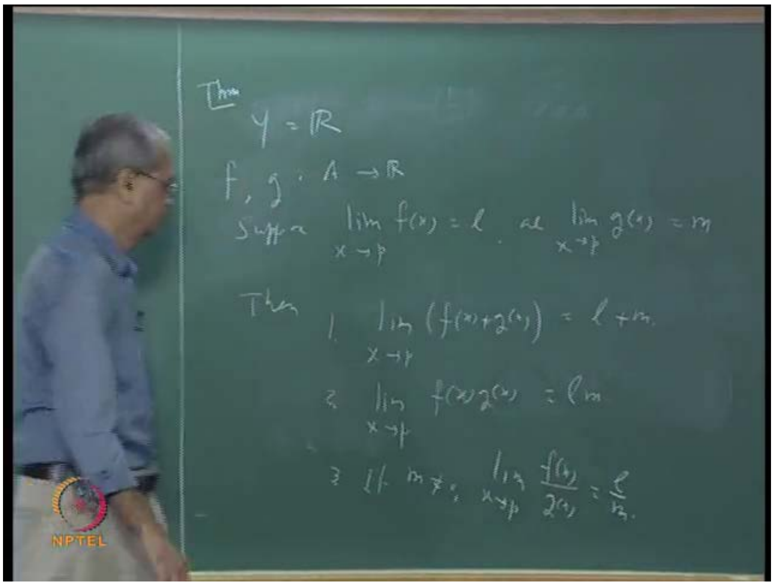

We take this space y as R and let us say all other things are same, I am taking A as a subset of x, p as a limit point and suppose we consider two functions f and g both from A to R, f and g both from A to R. Suppose, limit of f x as x tends to p is l and limit of g x as x tends to p is let us say some value m of course, remember l m are real numbers f x and g x those are real valued functions. There domain may not be R, they can from any metric space, but the values are in R once the values are in R, we can talk of what is meant by f plus g, f into g f by g and things like that.

What we want to say is that, if the limit of f x is l and limit of g x is m, then limit of f plus g as x goes to that is that is l plus m etcetera, so that is a theorem. So, then first thing by the way this whole symbol means when I say limit of f x is x tends to p is equal to l, it means that limit exist and it is equal to l similarly, here. So, otherwise we do not write this equal to anything, so what I want to say is that limit of f x plus g x or f plus g x you can say whatever it is limit of f x plus g x extends to p this is equals to l plus m.

Second thing is limit of f x into g x as extends to p is l into m and finally if this m is not 0, if this m is not 0 then we can talk of limit of f x by g x as extends to p and that should be l divided by m. If m is non 0, limit of f x by g x as extends to p is l by m again I will do say that we shall not spent any time in proving this, just prove the corresponding theorems about the sequences. Let us just see what is the meaning of this, this means that whenever you take any sequence of elements in a let us say x n whenever x n in A converges to the point p, f x n should converge to l and g x l should converge to m right.

Then f of x n plus g x n should converges to l plus m, that is what sequence we know that sequence f x n converges to l and sequence g x n converges to l. About the sequences we already proved that the limit of the sum sequence is same as the sum of the limits, so f of x n plus g of x n will converges to l plus m. Similarly, f x n into g x n should converge to l into m, so no new concept is involved and similarly m is not equal to 0 then f x n divided by g x n should converge to l by m.

Now, you may wonder that not only m not equal to 0, in order to talk about f x by g x this g x also should be different from 0. But, we need not say that specifically because see what we are bothered about is only about the values x, near the point p and if m is not equal to 0. If g x tends to g x tends to m, then for large values n g x n will be different from 0 right, so f x n by g x n or f x by g x will be defined well, defined for x close to p if m is different from 0. So, I just said all these theorem follows by corresponding theorems about sequences and our equivalent theorem about the limit of a function and limit of a sequence. Now, let us go to the next concept incident by depends on the limit and it is that of continuity.

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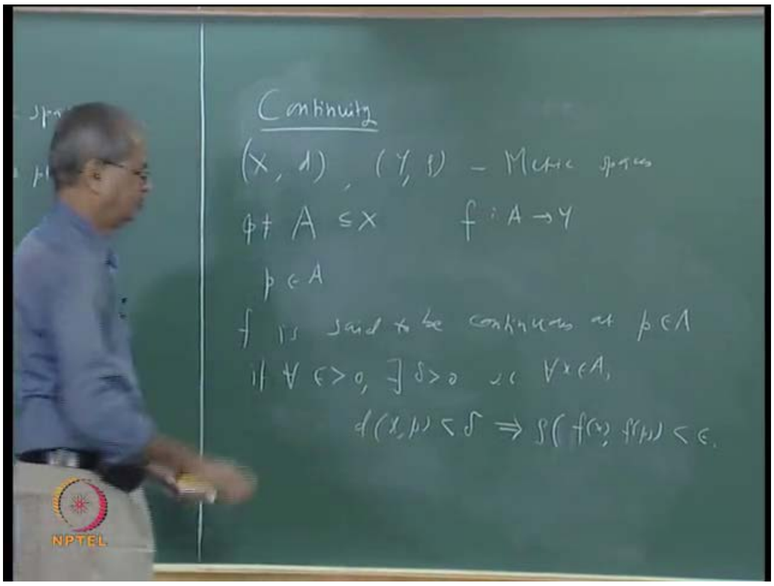

Again, here also we shall take these two metrics as it is x say x d and y rho metric spaces. Again will take a non empty set a in x and f is the function from A to 1 f is a function from A to y and this time will take this point p not as a limit point A. But, p is a point of A, p belongs to A that means the function must be defined at the point A. Then we shall define what is meant by saying that f is continuous at p, so f is said to be continuous at p continuous in p in A if the definition is again very similar to the corresponding definition limits.

If for every epsilon bigger than 0 there exist delta bigger than 0 such that for every x in a if distance between x and p is less than delta, distance between f x and f p should be less than epsilon. So, the distance between x and p less than delta, distance between x and p less than delta this implies distance between this insist the f x and f p those are going to be in y that the distance is rho. So, rho f x, f p is less than epsilon, if f is continuous at every point in A, we select f is continuous on A or in A, so f is said to be continuous on A, f is continuous on A or some books also use in a that is that is minor point if f is continuous at every point in A.

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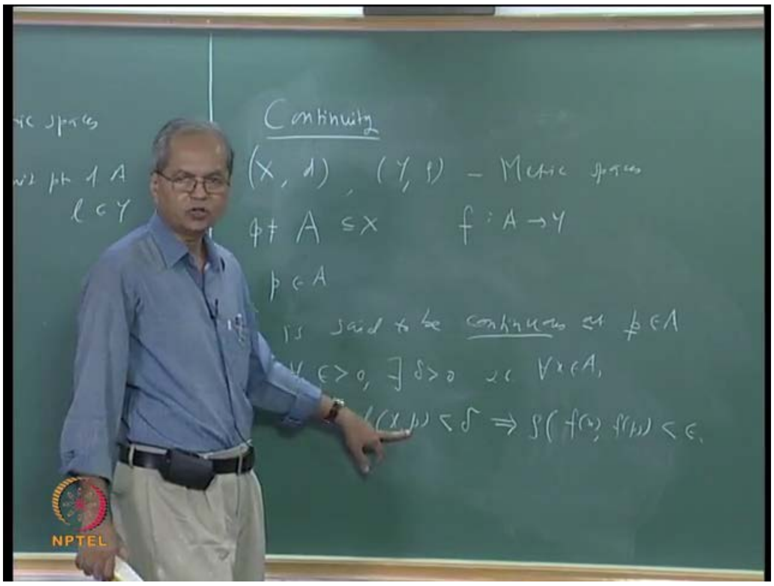

You can realise it this definition of continuity and the definition of limit they are very closely related to each other. But, there are some differences, one difference is that while talking about the limit this point p did not belong to A, the point p did not belong to A it has to be, it is enough it is just a limit point. Whereas, for talking about the continuity the point p must be a point of A, point p must be a point of A and since it is a point of A we can talk of what is f of p.

In case of limit there is no such thing that there is f of p, f of p may not be defined at all, that is the first thing. Secondly once if t belongs to a p may or may not be the limit point p may or may not be the limit point. But, suppose p is a limit point suppose p is a limit point, then you can say this is same as c the limit of f x as x tends to p it is same as f p right, because after all this is in this if I take it is l, it will be in the limit of f x as it tends to p is equal to l.

So, if p is a limit point then saying that these function is continuous at p it is same as saying that limit of f x as extends to p is equal to f p. If p is not limit point then what, suppose p belongs to A, but p is not a limit point remember we had called such a point as isolated point, p is not a limit point means what. There exists some open ball containing p which contains no other point of A, only that point of p.

But, in that case we can say that in that case, we can say that whatever epsilon is given, whatever epsilon is given. Suppose there is, suppose p is isolated points, suppose p is an isolated point. Essentially, what I wanted to say is that if p is an isolated point then the function is always continuous at that point, so let us just make that observation.

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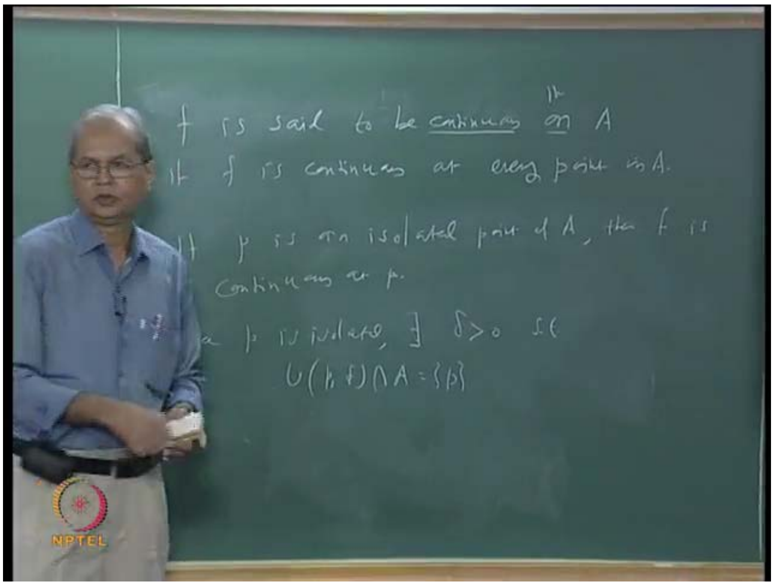

If p is an isolated point of A, if p is an isolated point of A then f is continuous at p, now how does this follow. Let us just see since if p is isolated to point what should happen is that there should exist a ball with centre at p which contains no other point of A. So, we can see, let us since p is isolated there exist suppose I call radius of that point ball as delta, suppose I take radius of that ball as delta. There exist delta bigger than 0 such that open ball with centre at p and radius delta its intersection with A, its intersection with A must be singleton p because it contains no other point of A, it contains no other point of A, intersection A must be singleton p.

Now, it is clear to you from this if I says it that for every epsilon that would exist some delta such that whenever you take any point x in A. If the distance between x and p is less than delta, then distance between f x and f p should be less than epsilon. But, if I take this delta for example, if I take this delta then only x in A which will satisfy this is p then in this case f x and f p distance between f p will be 0.

So, whatever epsilon you take, whatever epsilon you take this delta will work because this inequality distance between x and p less than delta. That is satisfied only by p, only by p and no other because there are no other points near x, so there is no other point near p. So, if p is an isolated point then f is continuous at p right, so in particular for example if you say A as such every point is isolated then the function will be continuous at that point.

For example, suppose A is n set of all natural numbers then every point is isolated point, so any function defined n will be a continuous function right. Let us teach suppose, so that disposes the case of isolated points let us, now look what if it is not an isolated point then it is a limit point. In case of limit point, what should happen is that limit f is continuous at p, then the limit of f x as x tends to p should be same as f p that is what is second required is this definition says.

So, if p is a limit point, if p is a limit point of A then f is continuous at p, f is continuous at p if and all if or this is equivalent say the limit of f x as extends to p is equal to f of m. So, let us again and of course, this does not need any other proof because this set is limit f x is extends of p is equals to f p is basically same as what you have written here. So, basically same as what you have written here alright, so let us again take this as suppose p, see in order to talk about continuity we have to have that p must belong to A. Since p belongs to A, p can be either an isolated point of A or a limit point of A, there is no third possibility.

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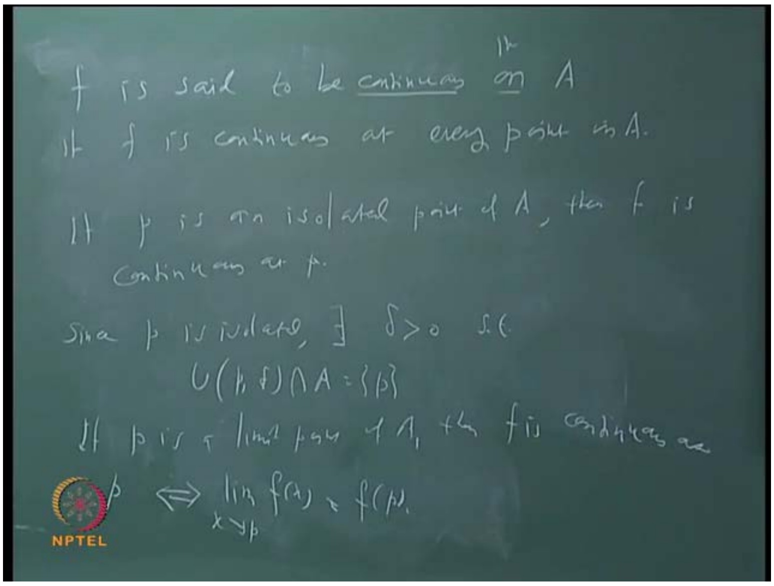

If p is a isolated point of A if it is always continuous, if p is a limit point of A then limit of f x extends to p must be same as f b that is the requirement for continuity. Now, suppose let us ask this question what is the way in which f can fail to continuous of course, let us forget about the case when p does not belong A. If f is not defined at p obviously we cannot talk about continuity at all, but suppose that is the only case suppose p belongs to A if p is isolated point obviously there is no question of failing to be continuous. So, if at all if f fails to be continuous it will happen only at a limit point and in what way it will happen there is two ways, either that this limit does not exist at all or let this limit exist.

But, its value is different from this, so let us again summarise what are the ways of in which a it is different from f of p, but it is different from f of p, now among these two types you can see that the second type is easy to handle. Suppose a limit exist and the value is different from f of p then we can say that we can redefine function can fail to be continuous. First of all, it can fail to be continuous only at limit points it can never be fail to continuous isolated points. There are two ways in which this can happen, when is that limit of f x extends to be exist that is one way. Secondly limit exist, but that function and change the value of f at p and make the function continuous there, that is possible.

So, that is why that kind of discontinuity, if a function is not continuous at a point and we say it is discontinuous, at that point and those points are called points of discontinuity of a function. This second type this I mentioned just now that is called removable discontinuity, the reason is obvious that this discontinuity is you can remove by simply modifying the differential function at that point.

But, if the limit does not exist at all then you can do nothing whatever way you modify definition of f at point p still the function will remain discontinuous there. You can see one more thing, here that though in giving this definition, here we have said that function is defined at A and then p belongs to A etcetera. You can see that in all this definition and whatever discussion we have done, so far compliment of A has no importance at all, right. Whatever happens to the points outside A we are not bothered at all right, so here afterwards we can simply forget about those points.

On top of function going from x to y that is I just regard this A itself as a metric space, I will regard A itself as a metric space and talk about the functions going from A to y which is same as x to y. We shall now give very useful criteria for the continuous functions and let us say that this time I am going to function continuous everywhere, of course we can also give description for the functional base continuous at a point. But, it unnecessarily complicates the thing, so let us let us talk of these and that is in terms of open sets.

As you know open set is a very important concept in metric spaces and if we can talk of something purely in terms of open sets, then that concept can be translated to topological spaces also because you know that we also define. What is mean by topological space, in a topological space there is no concept of distance, but you have a concept of open sets. Now, here we are given a concept of continuity first using distance, but suppose it is possible to give definition using only open sets then we can talk of functions continuous in topological basis also, that is the idea.

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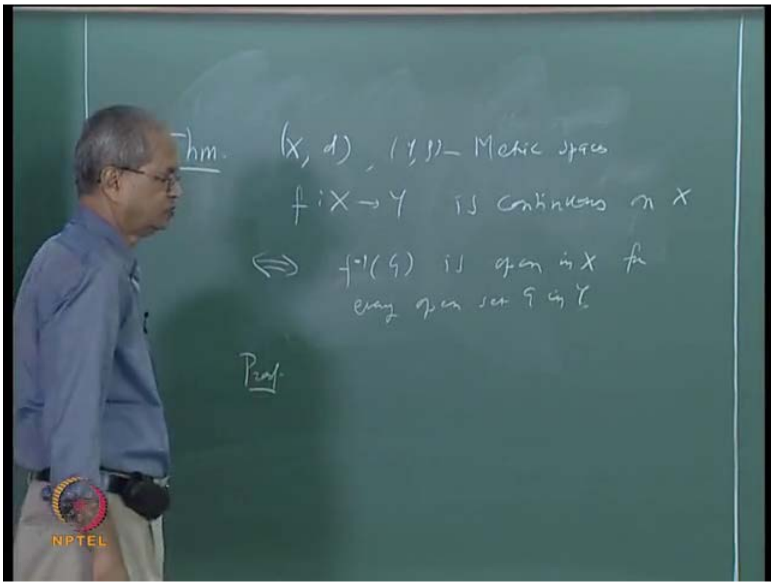

So, let us, let us take that first, so I will write it as a theorem and also it is useful because using this equivalent criteria certain other proofs also becomes simpler. So, let us again say that x d y rho or metric spaces and f from x to y is a function f from x to y is continuous this time I should write continuous on x. But, this is when mention supposes the thing is say if the function is continuous at a point we should say continuous at that point. But, suppose the thing is said simply said f is continuous then it is assured at it is they continuous on x that means continuous at every point of x.

So, f is continuous on x if and all if this is interesting what it says is that if you take an open set in y, and look at its inverse image in x then that should also we open in x that is f is continuous if and only inverse image of every open set is open, right. So, if and all if f inverse g is open in x, for every open set g in y these are about if g is a open set is y f inverse g is open set of x, let us see how we can prove this. Now, we can observe one more thing even before going to the proof of this coming back to this discussion of this continuity here.

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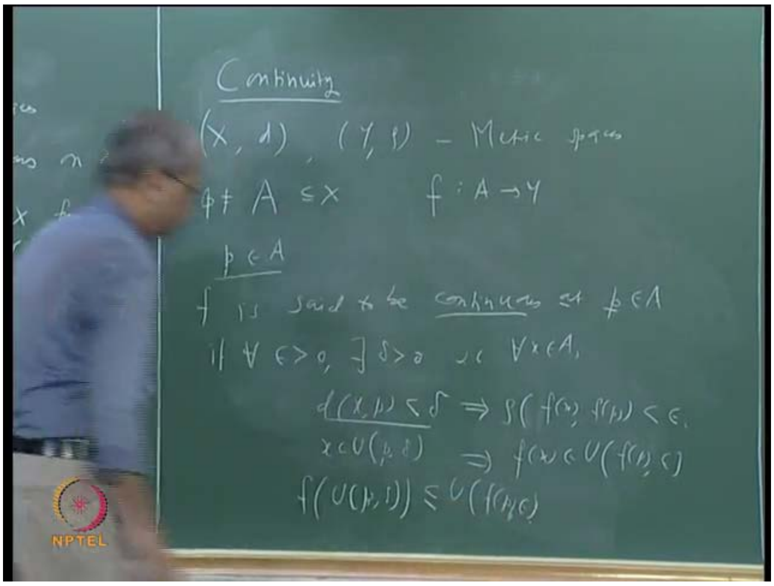

You can say that saying this is d x p less than delta this is same as saying that x belongs to the open ball with centre at p and radius delta, this last thing is. So, this means x belongs to open ball with centre at p and radius delta and what does this mean, this means f p belongs f x belongs to this means f x belongs to the open ball with centre at f p and radius epsilon. So, this means f x belongs to open ball with centre at f p and radius epsilon that is it that mean whenever x is in this ball f x is in that ball is it same as saying that the image of this ball that is f of this whole ball, inside this ball right.

So, this is last the whole sentence whatever we have written, here that can be simple related this f of U p delta is contained in U f p epsilon. So, saying that f is continuous as a limit point means for every epsilon this exist in delta such that f of that open ball with centre delta should be contained in open wall with centre at f p and radius epsilon. We shall make use of this in this proof, in order to prove this let us, let us first use this like suppose f is continuous, suppose f is continuous.

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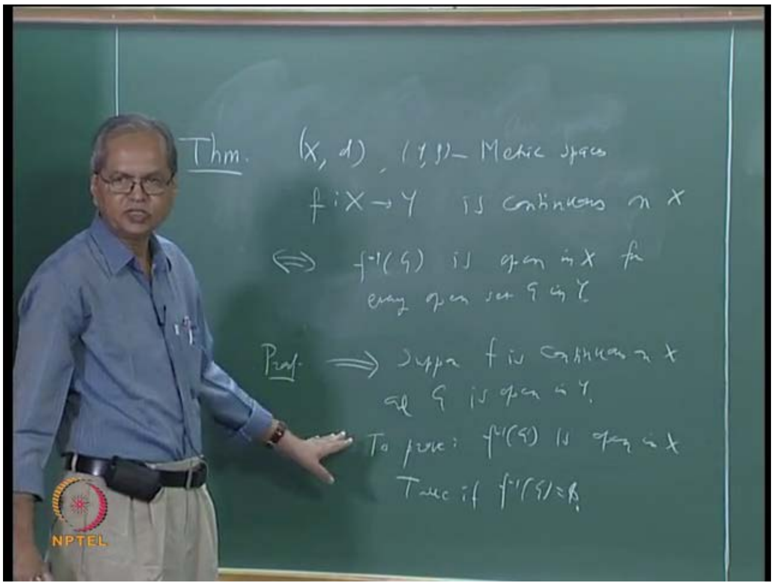

Then we want to say that if g is open in y, if g was open in, so suppose f is continuous, suppose f is continuous on x and g is open in y we want to prove that if f inverse g is open in x that is, this is what we want to prove. To prove f inverse g is open in x, of course if f inverse g is open, there is a thing to be proved if f inverse this is, this is trivial. So, this is true if, so true if f inverse g is empty what is the meaning of f inverse g is empty that means no f x goes to g that is no point in x its images in g. So, that is the meaning f inverse g is empty, in that case nothing to prove.

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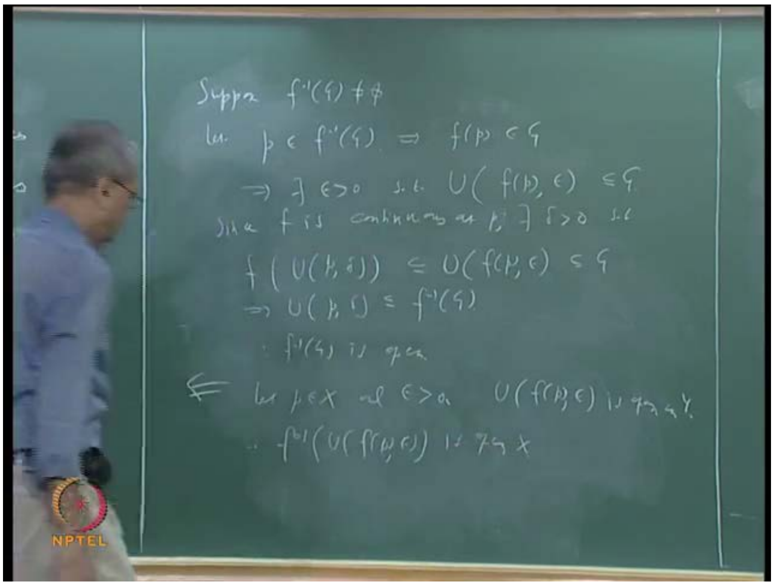

So, next I have shown that suppose f inverse g is non empty, suppose f inverse g is non empty is non empty means what some point belongs to it. So, f of g is non empty and let suppose I take let p belong to f inverse g then we must show that p is a interior point that means we should show that this is the ball with centre at p. Some positive radius such that ball is completely inside f inverse g, but p belong to f inverse g means what, that means this means f of p is in g.

So, this means f of p is in g right, but this is an open set, this is an open set, so there must exist some ball with this is the centre which is completely contained. Suppose I call radius of that ball as epsilon, so this implies there exist epsilon bigger than 0 such that open ball with centre at f d and radius epsilon is contained in g. Now, we assume that if it is continuous, we assume that f is continuous at every point, so at point p also for this epsilon there should exist some delta such that whatever is happen.

So, since f is continuous, since f is continuous at p there exist delta bigger than 0 such that, such that let me such that distance between x and p less than delta implies distance between f x f p less than epsilon. We have seen that means this have lost, that means whenever x belong to U p delta f x belongs to U f p epsilon for which is same as saying that delta bigger than 0. Such that I will write this f of U p delta this is contained U f p epsilon and this is contained in g U f p epsilon is contained in g.

Now, so what we have proved f of this open ball f of this open ball is in g, so does this being open ball is in f inverse g because open ball g if you take any point in this ball, if you take any x in this ball f x is in g, right. So, that means this ball is contained in the inverse image of g, so this implies U p delta is contained in f inverse g. Now, that means that p is an interior point, that means p is an interior point and we have showed that every point p is a interior point that shows f inverse g is open.

Therefore, f inverse g is open is it clear, so we have shown that if g is an open set then f inverse g is also an open set this should happen if f is continuous on x. Now, we want to show that the converse is also true alright, so let us take is this way, now assume that the function f is the property whenever g is open in y f inverse g is open in x then we want to show that f is continuous on x. To show that f is continuous on x means what we must show f is continuous every point in x that is, so let us take any point suppose I call that point p, so p belong to x.

Let us take epsilon bigger than 0, let p belongs to x epsilon bigger than to show that f is continuous at p for these epsilon we have to find some delta. Now, see this U f p epsilon, U f p epsilon is an open set in y open ball with centre at f p radius epsilon this is open in y. So, its inverse image must be open in x that is what we have shown if you take any open set in y its inverse image must be an open set in x. Therefore, we say that f inverse f inverse so that u f b epsilon is open in x, now does p belong to this set it does because f of that is f p, f of that is f p it is in this pole, so it certainly belongs.

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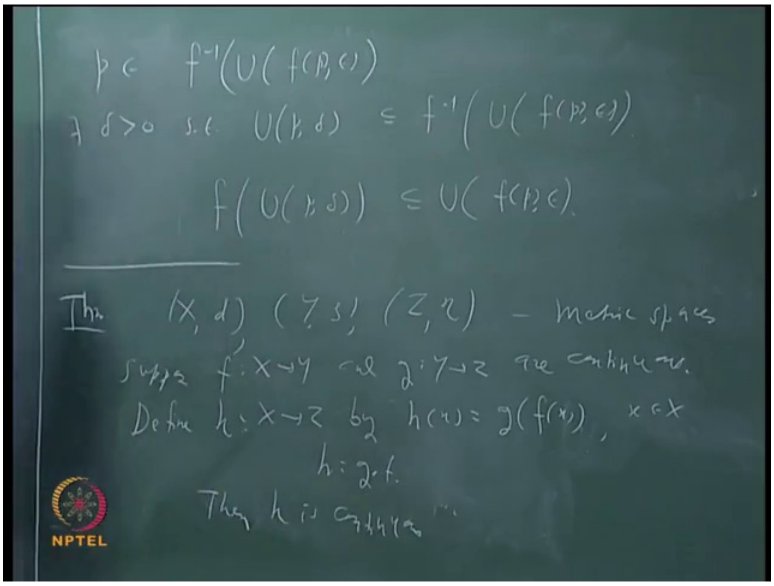

And p belongs to f inverse U f p epsilon, so this simply means f of p belongs to this point that is true whenever we say f inverse of any set simply means that f of p belongs to that set and what is that, that is nothing open ball with centre at f p. So, obviously f of p belongs to that, now p belongs to this set it is open, so what is it mean. It again should mean that there exist some positive numbers such that open ball with that p as a centre in that positive number is completely inside this. So, let us call that positive number as delta, therefore there exist delta bigger than 0 such that open ball with centre at d and radius delta is contained.

In this f inverse of, f inverse of U f p epsilon and this is same as say that f of U p delta is contained, in U f p epsilon and this is whenever distance between x and p is less than delta, distance between f and p is less than delta. Distance between f x and f p is less than epsilon that is, that is same as continuous f p and since p was any arbitrary point in x f is continuous everywhere in x.

So, I just said this describes the continuity completely in terms of open sets right, if you look at this, this has nothing to do with the distress. Of course, in metric spaces open sets are defined in terms of distance that is, but suppose you have some idea of defining open sets without using the distance. Then in that kind of spaces you can talk of what is meant by continuous function by simply taking this as the definition, and that is what is in topological spaces that is what will learn in your course in topology.

Now, let us see how this things make certain proofs also quite simple like for example, now we have taken function f from x to y. Suppose that is a continuous function and let us see let me just write that also as another theorem again a fairly well down theorem. Suppose this time I take three metric spaces, let us say x, d y rho x, d y rho let us say z let us say some metric eta or metric spaces. Actually see, remember let me also say one more thing when it is understood which metric you are talking about or whether the particular reference to metric is not important.

One simply says x is metric space x is metric space y is a metric space strictly speaking one should say x d is a metric space. But, if it is understood which metric you are taking or if the actual reference to the metric is not important for discussion then it is quite customary to say x is a metric space. So, similarly I should have simply said x y z are metric spaces and suppose f from x to y, suppose f from x to y and g from y to z are continuous. Then we can think of a function which goes from x to z which is a composition of these two functions.

So, define h from x to z by h of x is equal to g of f of x g of f of x g of f of x or which is usually denote as this is simply, usually described as h is equal to g converse with f then h is continuous. That is what we want then h is continuous in short what we want to say is that the composition of two continuous function is again a continuous function. Instead of writing the proof detail, I will just give you an idea and then we shall stop with that.

See the idea is simply, this we shall use this criteria, we shall use this criteria to show that the function is continuous we want to set h is continuous. So, h goes from x to z, h goes from x to z, so it is sufficient to show if I take some open set in z then show that its inverse image in x is opening x.

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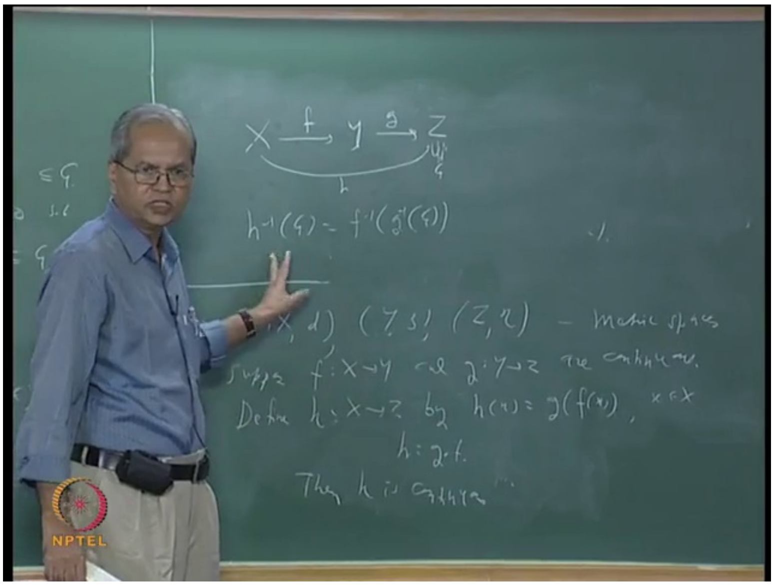

So, I think it is better to explain this is, there see f goes to x from y and g goes from y to z we are taking. Let us say we are taking some open set, we take some open set g in z we want to say that h though this composition is nothing. But, h this g converse with the thing, but h, now what we want to say is that h inverse of g which will be a set in x that is open, that is open. But, what is the argument, this g inverse of g is open in y then f inverse of g inverse of g is open in x because f and g both are continuous.

But, all that we need to observe is that it is nothing but same as h inverse g give the reverse, what you have to observe simply this h inverse of g is f inverse of g inverse of g. If g is open, here since g is continuous g is open in y, and since f is continuous f inverse of this is open in x and that is same as h. Now, that is nothing but elementary set theory because h is if h is defined like this, you can easily show that h inverse is nothing.

But, f inverse converse with if h is g converse with f h inverse is that is fairly elementary side theory and so using that and this theorem we can show that composition of two continuous functions is also continuous. It can also be proved by using the usual epsilon delta definition, I suggest you that you take that as an exercise try to prove it also without using these criteria and you will understand the difference. We will stop with that.