Real Analysis
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Lecture - 22
Baire Category Theorem

We had just begun the discussion of this Baire category theorem, we shall complete the proof of it today. Let us just recall that we had said that set is set B nowhere dense that means a closure, its interior is empty.

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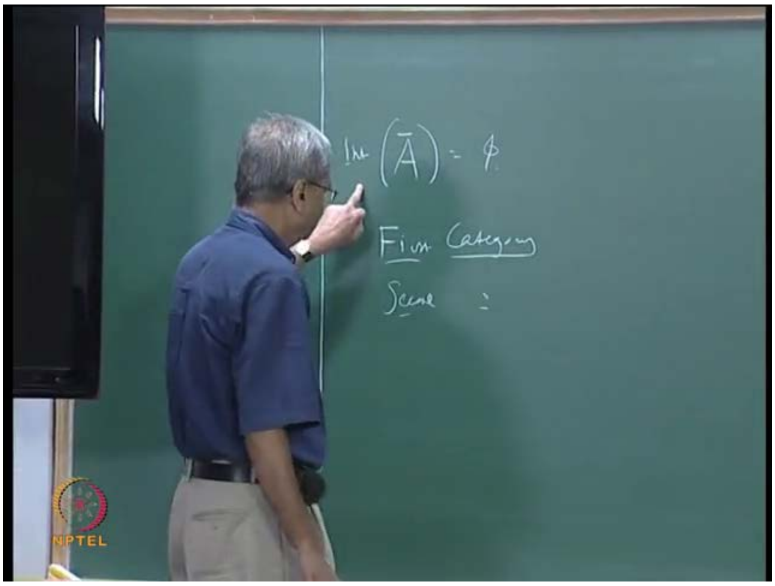

We are also seen the examples of nowhere dense set in last class. We also seen that if the set is said to of the first category. If it can be expressed as a countable union of nowhere dense sets and otherwise it is of second category. Second category means of not of first category. So, that is the second category we shall also recall a few things what exactly is bidding of this and of course, Baire category theorem. That let me take the statement once again.

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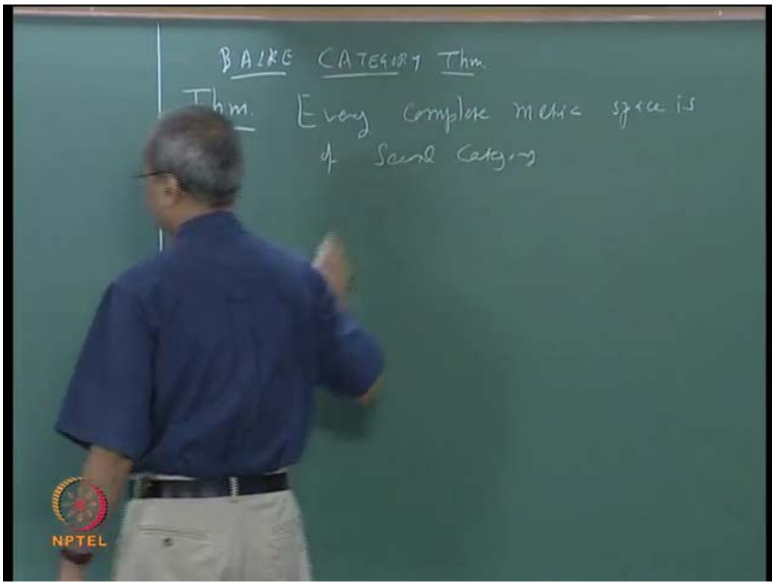

The statement simply says that every complete metric space every complete metric space is of second category. This is called Baire category theorem or simply sometimes Baire theorem. Now, before going to the proof of this theorem let us see couple of this that follow from saying the set is nowhere dense. For example, we mean by definition it means set is nowhere dense. If interior of a closure is empty that means a closure does not contain any interior point.

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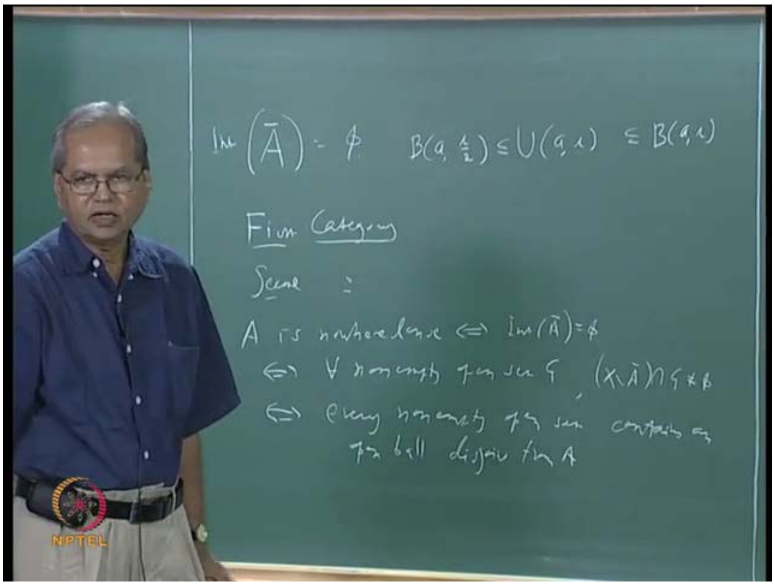

Let me just write here a is nowhere dense this means by definition this means interior of a closure is empty, this means a closure does not contain any interior point. So, which also means that if you take any non empty open set. Then if you take any non empty open set, then that non empty open set cannot be subset of a closure. Because, if it is a subset of a closure then that will be also in the interior.

So, I will also contain interior point, so if it means if you take any non empty open set. Then there will be some point in the non empty open set, which is outside a closure. So, let us just say that is this if you can say, if A n only G for every non empty open set G every non empty open set G. What we can say that X minus a closure intersection G will be non empty. So, otherwise what will it if this is empty it will G in the G inside a closure. That is not possible, because that will be in the a closure contains some interior point, now x. Now, if this set is non empty, is it clear? This is a open set, because a closure is open G is open.

So, if this is non empty it open set it means it will contain a ball. It will contain a ball that ball is in X minus a closure. As well as in G that means this what it mean if you take any non empty open set. It will contain A n open ball, which is disjoint from A, which is disjoint from A. So, A is nowhere dense if and only if what I want to say this if and only if every non empty open set. Every non empty open set contains a open bond contains A n open bond disjoint from A. This is something we shall be using fairly often in this proof then second thing is again A n element the observation. That is suppose you take any open ball with centriod A and radius R.

Then of course this open ball is contained in the closed ball with the centriod R. That is triver this contained in the closed ball with centriod A and radius R what I also want to use is the following. That if I take some radius smaller than this. Let us say R by 2, then the closed ball with centriod A and radius R by 2 is contained in the open ball with centriod A and radius R. What I want to always wanted use this fact. Suppose, you take the closed wall with centriod A and radius R by 2. I mean not less than R by 2, any numbers strictly smaller than R. Then that close wall with centriod A and radius R by 2 is contained in the open ball with centriod A and radius R. Now, with this preparation let us come to the proof. So, let us start with a complete matrix space.

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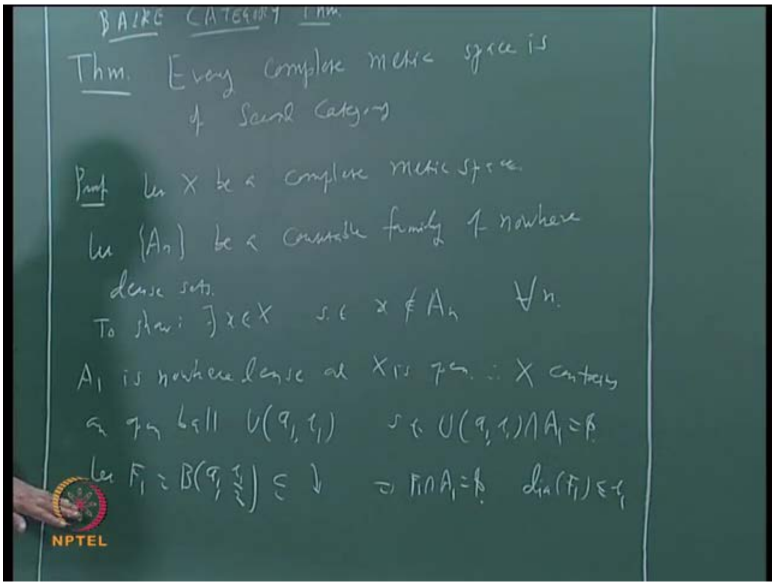

So, let X be a complete matrix space. Then we want to show that it is of a second category, which means that it is not our first category, which means what, which we want to show that. It cannot be expressed as a countable union of nowhere dense sets X cannot be expressed as a countable union of nowhere dense sets. What is the meaning of that? It means it should take any countable family of nowhere dense sets. Its union is not X again it means, what it means That if you take any countable family of nowhere dense set. That will exist some point in X, which is not in any of those sets. This is what we need to prove, proving in this Baire category theorem, is that clear?

So, let us say that let A n be a countable of nowhere dense set countable family of nowhere dense sets. We want to show the following. We want to show that exist in X. Such that X does not belong to any of the A n does not belong to the A n for all X. This is what we want to show if we show that proof will be complete. How are we going to construct? Show proceed it we shall be using this fact for we were seen. If any set is nowhere dense and if you take A n open set then that open set should contain a open bound, which is disjoint from nowhere dense. This is what we use repetively will use for each of this sets A n. So, start with A 1 start with A 1, so A 1 is nowhere dense.

A 1 is nowhere dense and X is open take the first open set as X and X is open and X is open of course, X is non empty and open X is non empty. We wanted to start with the non empty open set. So, from this what should happen. So, X contains a open bond, which is disjoint form. So, therefore, X contains A n open ball suppose I call that I call open suppose the centre is A 1 and radius is R 1. So, let us say open ball U a 1 r 1 such that U a 1 r 1 intersection A 1 is empty. Then I shall use this fact I will take a close ball with this same centre, but radius smaller than R. Let us say R 1 by 2, so and let me call that let me call that set F 1.

So, let F 1 be the close ball with radius A 1 centre A 1 and radius R 1 by 2. Then F 1 is contained in that U a 1 r 1 F 1 is contained in that F 1 is contained in that U a 1 r 1. Hence, F 1 intersection A 1 is empty F 1 also does not therefore, F 1 intersection A 1 is empty. Now, F 1 is a closed ball and its radius is R 1 by 2. So, we can diameter is less than or equal to R 1 is diameter is less than or equal to R 1. So, in fact sequel to diameter of F 1 is less than or equal to R 1. Also, what can we say about F 1 interior? F 1 interior will suddenly contain pen ball with the centriod A 1 radius R 1 by 2. So, in particular it will contain A 1, so F 1 interior is non empty interior of F 1 is non empty. So, I can say interior of F 1 is non empty.

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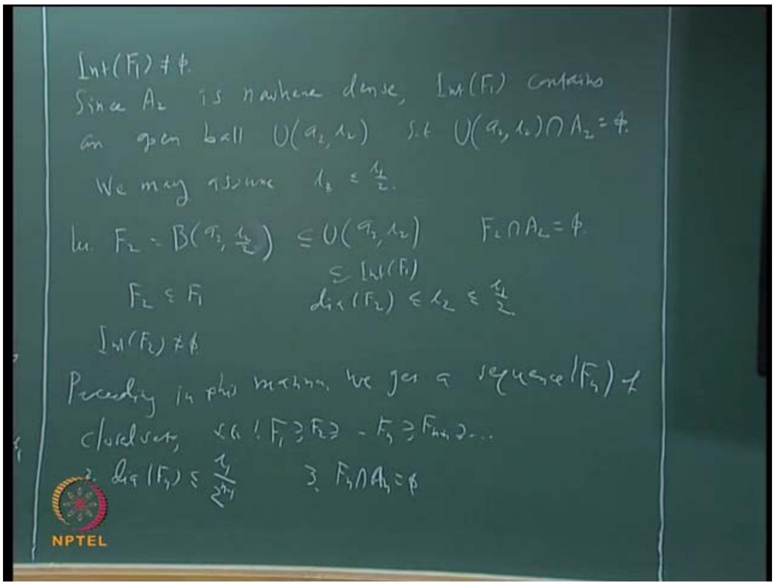

Now, interior of F 1 is non empty it is a non-empty open set interior is always a open set. Now, consider next set A 2 a two is nowhere dense and this is a non empty open set. This is a non empty open set. So, this will contain this will contain A n open ball, which is disjoint from A 2, which is disjoint from A 2. So, since A 2 is a nowhere dense since A 2 is a nowhere dense interior of F 1 contains a open ball. Suppose, I call that centre as A 2 and radius R 2 that is disjoint from A 2 that is disjoined from A 2. Such that U A 2 R 2 intersection A 2 is empty we can also make one more thing. Suppose, this is true U A 2 R 2 that what is the thing that U A 2 R 2 is contained in this interior of F 1. It is its intersection with this A 2 is empty.

If I take any radius smaller than R 2, but the same centre will that still be true. I can assume I can make radius smaller than that is radius R 2 and still it be S 2. So, what I will do is I will take R 2 to be less than this R 1 by 2. I will take R 2 to be less than R 1 by 2 in fact that. So, we may assume R 2 to be less not equal to R 1 by 2. Then we do whatever we did here I will choose a next set F 2 as the closed ball with centre at a two and radius R 2 not R 2 by 2. Because, we want R 2 by 2, because we want that to be contained in this. So, that is contained in U A 2 R 2.

So, what is the property of this F 2? Now, F 2 intersection A 2 is empty F 2 intersection A 2 is empty. So, F 2 intersection A 2 is empty. Second thing is F 2 is contained in U A 2 R 2 F 2 is contained and U A 2 R 2 is contained in interior of F 1. So, this is contained in interior of F 1. So, in particular we can say that F 2 is contained in F 2 is contained in F 1 Further, diameter of F 2 diameter F 2 is less not equal to R 2 and R 2 is less nor equal to R 1 by 2. Now, what is to be done after that is clear you proceed in the same way. Next, you take consider interior of interior F 2 interior of F 2 is non empty. Since, interior F 2 is non empty open set you look at the next set A 3, A 3 is no where dense.

So, interior F 2 will contain A n open ball, which is disjoined from A 3. Similarly, you adjust the radius say take the radius. Let us say that open ball is let us say A 3 R 3 take R 3 to be less not equal to R 2 by 2. Consider next set F 3, which will be b A 3 R 3 by 2. Suppose, you do that what are the properties that that F 3 will get F 3 will be contained in F 2 F 3 will be contained in F 2 F 3 intersection A 3 will be empty. Diameter of F 3 will be less not equal to R 2 by 2. Hence R 1 by 4, so you can say R 1 by 2 to the power 2.

So, continuing in this way will get a sequence of closed sets will get a sequence of closed in fact will generate decreasing family of closed sets. What is the property that each F n will be contained in F n minus 1 and F n intersection A n is not empty. Diameter of F n is less nor equal to R 1 by R 1 by 2 to the power n minus 1.

So, let us just recall these things, so proceeding in this manner proceeding. In this manner we get a sequence F n of closed sets, such that F 1 contains F 2 contains F 3 etcetera. Such that you can say F 1 contains F 2 etcetera etcetera F n contains F n plus 1. That is first property let me write this is the first property. Second property is that diameter of F n is less nor equal to R 1 by 2 to the power n minus 1. Third property is F n intersection A n is empty. Now, it is clear how to proceed after this in for in fact why we have constructed this. Now, we would not use Canter's intersection theory. So, what we have seen is that in a complete metric space, if you have decreasing family of closed sets like this and if diameter of F n tends to 0.

That is what is happening here diameter of F n is less R 1 by that is the reason for adjusting the radius at each stage. So, this diameter of F n tends to 0 as n tends to infinity. So, by Canter's intersection theorem intersection of F n will contain exactly one point. So, by Canter's intersection theorem intersection of F n belonging to n this is contains intersecting A 1 point.

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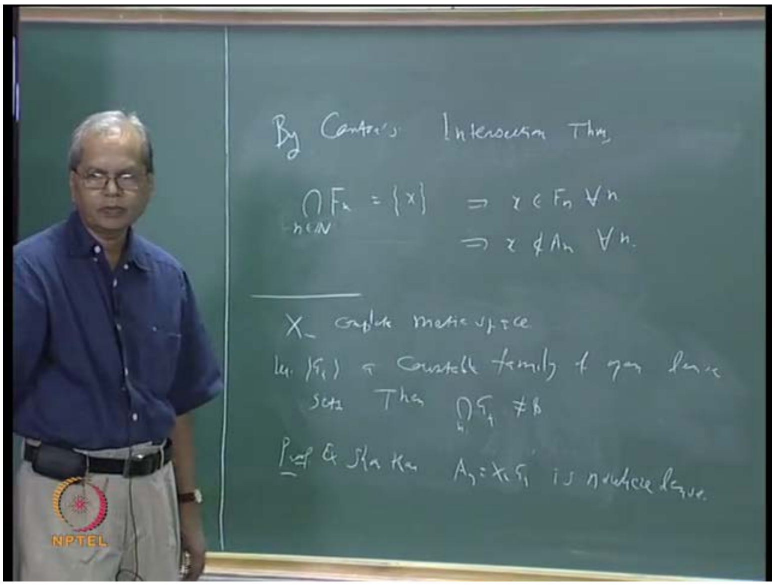

So, this the thing, but single term X and X is in the intersection of F n means what it means X belongs to F n for all X. So, this implies x belongs to F n for all n, but each F n is disjoined from A n. So, X cannot belong to any of the n's, so this implies X does not belong to A n for any n. That is what we wanted to show, we wanted to show that there exist some X, which is not is any of the n's. That is what we have shown. So, that completes the proof that every complete metric space is of second category. You can notice that, we have made a crucial use of this Canter's intersection theorem using the property of lower dense sets. We have constructed a decreasing family of closed sets such that each F n is disjoined from A n. Also, the diameter of F n goes to zero and that gives this point X, which is in each of the f n. Hence, outside every F n all right?

Now, there is another formulation of this Canter's theorem not Canter's theorem this Baire category theorem. That is in terms of open sets many books give that version as a cat as the bearing category theorem or simply bears theorem that is as follows. So, another version of bears theorem suppose X is a complete metric space complete metric space. Suppose, you take countable family of open dense sets, suppose you take a countable family of open dense sets.

So, let G n be a countable family of open dense sets that is each G n is open and each G n is dense in X then what theorem says is this. Then intersection G n is non empty in other words in a complete metric space countable intersection of open dense sets is non empty. That is A n equivalent formulation of bear category theorem. Why it is equivalent? How does it follow from this? There is only one very obvious thing to be looked. You just notice that if G n is countable and dense. I will just give the proof to you as A n exercise I will just give you this as A n hint show that.

Suppose, take A n as X minus G n you take A n as X minus G n then show that A n is of first category show that A n is nowhere dense, show that X minus G n is nowhere dense that is a exercise. Let us say you have done this exercise assume that you have done this shown that a compliment of G n is nowhere dense then is it clear? That it will immediately, now follow from this. Because, according to this there should exist some point X, which is not in any of this n's. So, that must be in the intersection of G n that point must be in the intersection of G n. It is not in any of the G n's any of the A n's. It means it is in all of this in each of the G n's X does not belong to n means X belongs to G n.

If this happens for each n X is in intersection of G n. So, let us, now proceed to next topic namely see till now. We have taught of various properties of metric spaces by remaining inside only one metric space. That is we are taken sequences in metric spaces sets in metric spaces open sets, closed sets, closure interiors. You say at each time every everything was happening inside one metric space. Now, we will take two metric spaces at the same time and consider the functions going from one metric space to the other Consider limits of such functions see till now we have considered limits of sequences off course. In your calculus course you talk of limits of functions.

So, we also want to talk of limits of functions in that sense, but this our functions did not just go from R to R are some subset of r to r itself, but from any metric space to some metric space of course. In particular both those metric spaces may be real line. So, usual concepts of limit will turn out to be special cases this concept of a limit, which we shall be discussing.

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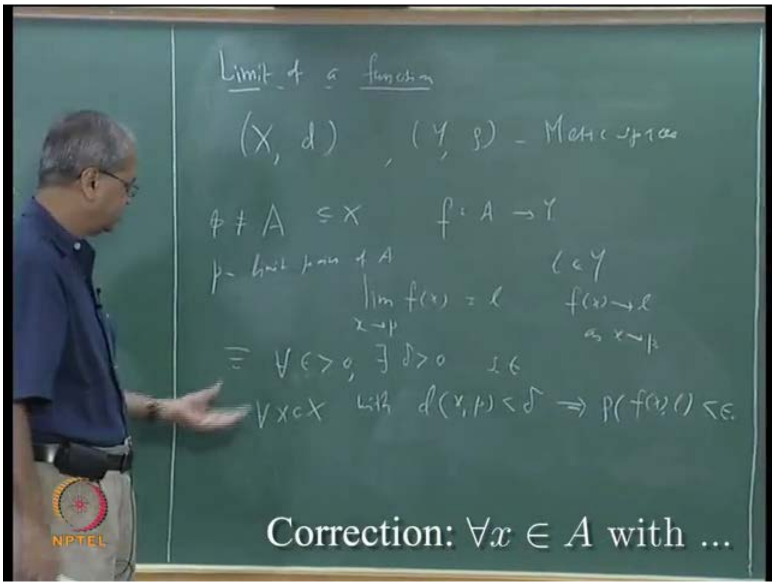

So, let us now take two metric spaces X, let us say X d and y let us say row are metric spaces. So, we are going to discuss, now what is meant by limit of a function. So, we should take a function F going from X to y if you take a function f going from X to Y but as know for the consideration of limit function did not be defined for the whole of X. We can also talk about function going from some subset of X to Y.

So, let us take some subset of X say let us say a is subset of X. Of course, to avoid to reduce let us take A to be non empty and F is a function, whose domain is a and co domain is Y F is a function, whose domain is a and co domain is y. We want to say what is meant by saying that limit of f x as X goes to some point is equal to some point in Y. Now, that point that is suppose I call that point let us say the point p.

Suppose, I want to say what is meant by saying this limit of f x as X stands to P equal to l, what is the meaning of this symbol? That is what we want to say. So, first of all in order to be able to talk about this there should be something true about this point P. P did not belong to A that means F did not be defined at that point P, but we require is that F must be defined on every labor hood of p.

That is if you take any open set containing P then that should contains some point from A If P is itself not in a then it means it P must be a limit point of A P must be a limit point of A. So, that is that is the requirement, so P is a limit point of A. Of course, this l must be a point in y this l must be a point in Y. So, we talk about what is meant by saying that limit of f x as X stands to this limit point P is this point l from Y. Again, the idea is basically the same that is it means that, whenever X goes arbitrary closed to b f x should go closed to l.

That what we will make precise by using this concepts of these to distances that means f x goes a closed to l means. What yet this distance between f x and l must be less than epsilon and X goes closed to P means, what it means distance between X and P must be less than some again. Some arbitrarily small number, which we will call delta. So, this symbol means that limit of f x is X stands B is equal to l this means for every epsilon bigger than 0.

There exist delta bigger than 0, such that for all X for all X in X with a distance between X. P less than delta distance between X and P less than delta this implies the distance between f x. l is less than epsilon, but in Y the distance is row. So, row f x and l less than epsilon. We also say that either this symbol of f x as X stands to P or this. Also, X spaces saying that f x stands to l as X stands to P f x stands to l as X stands to P. Of course, such a limit may or may not exist. So, our next objective will be to decide given a function how does one know whether limit exist or not. Of course, if the limit exist at this point will have just to prove that given any epsilon, you have to find out delta depending on that epsilon. Such that whenever distance between X and P is less than delta distance between f x and l is less than epsilon.

Since, you would have done lots of such things in your calculus under graduate calculus course with real valid functions. We shall not going to that kind of examples here we should not spend to must time on that. That is also discuss the other question how does one show that there limit does not exist again? If you go by this definition you will have say that since limit exist means something happens for every epsilon. So, limit does not exist means for some epsilon, these things do not happen that means. We can say there exist some epsilon let us say let us call epsilon 0. Such that what should not happen that for that epsilon 0 there is no delta satisfying this. That means whatever delta you take whatever delta you take, because always find some X. Such that distance between X and P is less than delta, but that is bigger nor equal to epsilon 0.

That can be done that can be done, but usually that is quite difficult. It Baire can turn out to be quite difficult in several situations. So, what we will try to do is a we shall try to find a better way of showing that the limit does not exist? Whenever, it does not exist. To do that we what we shall do is that we shall connect, this concept of limit with the one concept of limit, which we have already discussed. Namely, the limit of a sequence we want to establish a connection between this idea of a limit of a function. The idea of a limit of a sequence that we have already learnt once we do that. Then showing that the limit exist or limit does not exist. We can translate all those things in the language of a sequence sequences. Also, several theorems about the limits of functions can be translated by in the language of sequences.

We can use the theorems, which we have already proved about a sequences and show that many of these things. We can quickly dispose of that is the idea. So, what is the connection is the following this is true that is limit of f x is extends to P is equal to l. If you take any sequence in X any sequence in A, which converges to P. Suppose, that sequence is X set then f x n should converge to l in Y. That is the connection in fact that is what expresses the entity idea also clearly. That if X goes close to P f x should go close to l. So, let us just write that is a theorem once that is proved many things we can be said immediate, so theorem.

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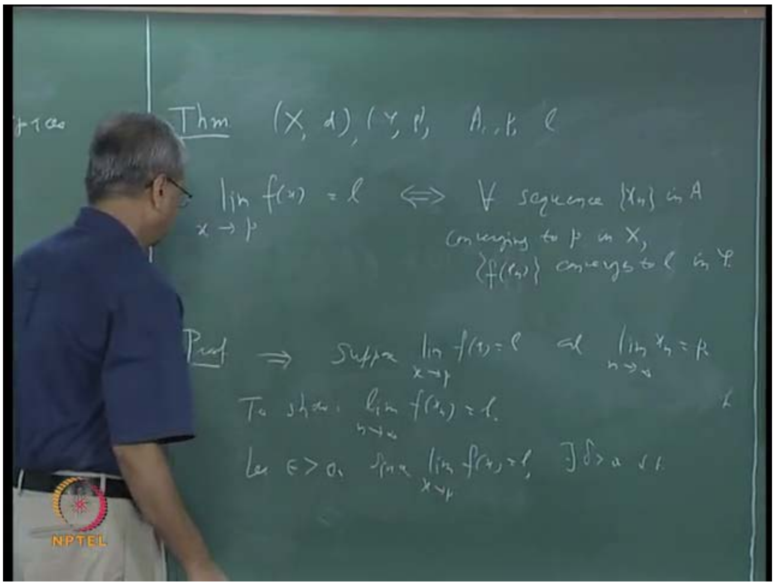

So, let us say that X Y A f P etcetera are all as in here that is I will not write that again that is X d y row, A P l. All those things are as here. Then what is the theorem says that limit of f x as X tends to P is equal to l. If only for every sequence X n in a converging to P. Whenever, you take any sequence X n converging to P f X n. Let us say converging to P in X. That means d X n P goes to 0, that means d X n P goes to 0 F x n converges to l in Y.

Let us see how this can be proved. Let us look at this way of the suppose we know that there is a limit of f x as X tends to P is equal to l. Then we should show that for every sequence X n in a converging to P in X F x in converges to Y. So, let us assume this. So, suppose limit of f x as X tends P is equal to l and 0, will simply write this limit of X n as n tends to infinity is P X n is a sequence in A. Limit of X n is P, then we must show that limit of F x n is same as l limit of F x n is same as l. What does it mean? Let me just write of the it to show limit of F x n as n tends to infinity is l by definition. What does it mean? That given any epsilon bigger than 0. We should find out some n 0. Since, that whenever n is bigger nor equal to n 0 distance between f x in and l is less than epsilon.

So, we start from epsilon, so let epsilon be bigger than 0, but since we know that limit of f x as X tends to P is equal to l for this epsilon. There exist some delta for this epsilon there exist some delta. So, then since limit of f x as X tends to P is equal to l there exist delta bigger than 0.

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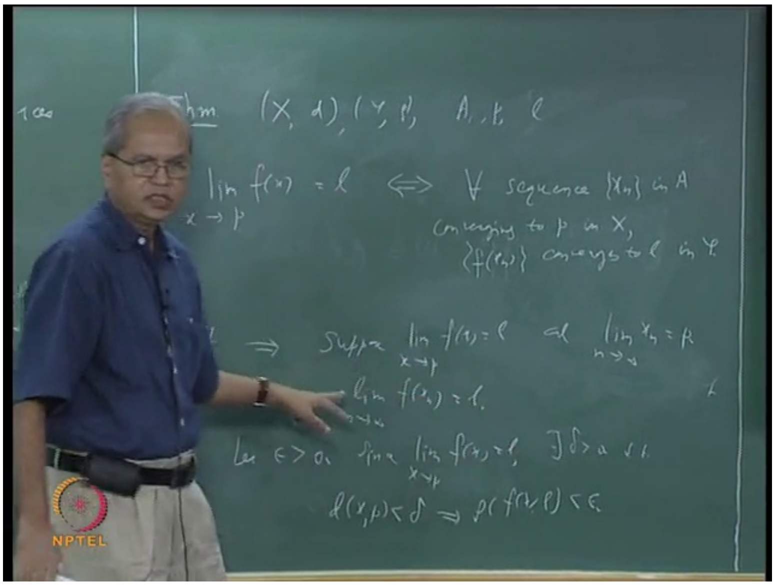

Such that I will simply write this that is such that whenever distance between X and P is less than delta. Whenever, distance between X and P is less than delta then for that X or distance between f x and l that is row f x and l is less than epsilon, but we now consider the sequence. We also at limit of x n as if it is P. So, since X n tends to P as n goes to infinity for this delta bigger than 0. There will exist some n 0 such that whenever n is bigger nor equal to n 0 distance between x n and P is less than delta.

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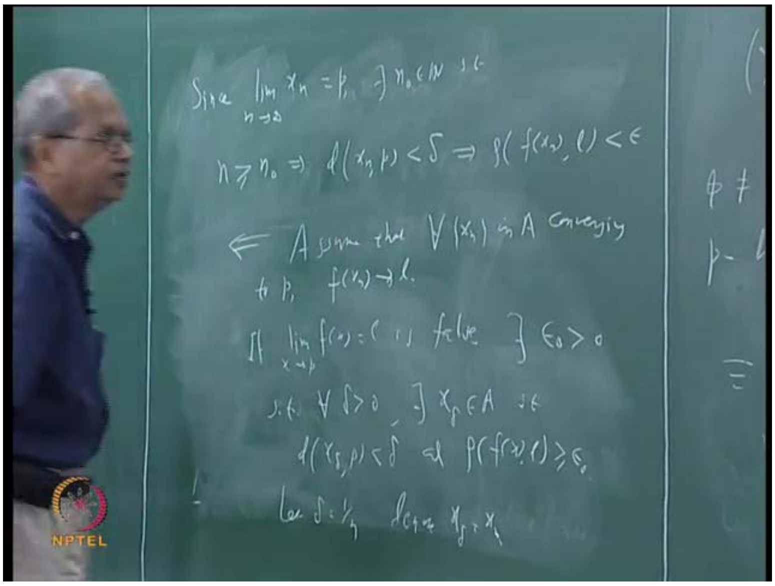

We can say that since limit of x n as n tends to infinity is equal to P. We know that whenever this happens for every epsilon bigger than 0. There exists some n 0 etcetera I am using that epsilon instead of that epsilon I am taking that big delta. So, there exists n 0 in n such that n bigger nor equal to n 0 implies distance between x n. P is less than delta, but if distance between what we know from here is that for any X. If distance between X and P is less than delta then the distance between f x and l is less than epsilon. You apply it to that x n. Now, we know that distance between x n and P is less than delta as distance between f x and l must be less than epsilon. So, distance between f x and n must be less than l. So, that proves family that is whenever the limit of f x is X tends to is equal to n.

Then this implies if you take any sequence x n in a converging to P if the sequence. If x n must converge to l. Now, let us take the other way. So, what does this mean it means that we know that whenever you are given a sequence x n in a converging to P. If x n converges to l and using that we must show that limit of f x is to P is equal to l. Now, suppose that is false. Then let us say that assume as shown that for every x n in A for every sequence x n in a converging to P. If x n converges to l.

Now, if this is false if limit of f x as X tends to P equal to l is false. What should happen as just, now I have said there should exist some epsilon 0 or epsilon 0 whatever you call that. There should exist some epsilon 0 there exist epsilon 0 bigger that, such that for every delta bigger than 0. What should happen is that there exist some X such that distance between that X and this P is less than delta, but distance between f x and l is bigger than this epsilon 0. Such there exist delta bigger than for every bigger than 0 there exists some X. Of course, this X may depend on that delta. So, I shall call it X suffix delta X delta in a such that such that distance between X delta and P is less than delta and distance. This distance is 0 row f x l is bigger nor equal to epsilon 0.

Now, remember this is true for every delta I can find some X delta. Such that distance between X delta and P is less than delta and distance between f x and l is bigger nor equal to epsilon set. Now, what I will do is I will take this delta as 1 by n take various variety of delta. So, let delta be equal to 1 by n. So, for this delta, so for this delta there should exist some X delta. Denote that X delta is one x n let delta be equal to 1 by n. Denote this X strictly speaking I should say X suffix 1 by n instead of that, I will take x n Denote X delta should be equal to x n.

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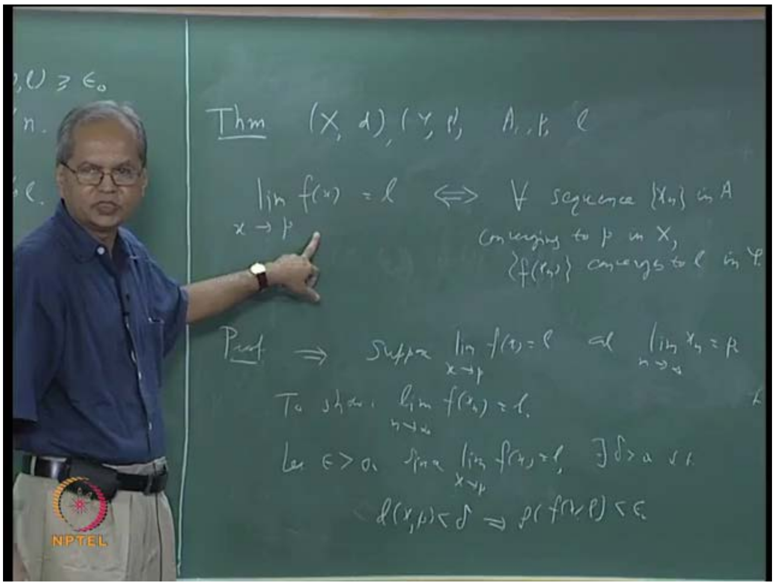

Then what should happen? then distance between xn and P is less than 1 by n distance between and P is less than 1 by n. Distance between f x n and l is bigger nor equal to epsilon set distance between f x and l is bigger nor equal term zero for every n. What does this mean? Distance between x n and P is less than one by n for every n means what this means x n power is to P. So, this implies that x n converges to P this implies that x n converges to P. What does this mean that f x and distance between f x and n is always bigger nor equal this number that. So, that can never be, so arbitrarily small. So, f x n does not converge to n. So, this tends f x n does not converge to n, but our assumption was that. Whenever x n converges to p f x n should converge to n.

So, we have got a contradiction to that, so whenever this happens whenever x n in A converges to p f x n converges to n. That must imply that limit of f x as X tends to P is equal to n. So, both these things are equivalent and other side what is the advantage of this. Now, that we have established the equivalence of these two concepts of limits limit of a function and the limit of a sequence of course. They are not exactly the same, but you can express one in terms of the other. So, the advantage is that in order to prove anything about the limit of a function, I can use the corresponding theorems about the limit of a sequence. Take immediately get direct theorem. For example, let me just take one example here, which will immediately follow from, whatever we have learnt from the earlier sequences.

That theorem is that limit of a function if it exists is unique limit may or may not exist, but the functions cannot have two different limits. So, we can say limit of f x as X tends to P if this limit exists. It is unique what will be the proof suppose limit is not unique. Then let us say that this limit is l also and m also and l is different from m. Then what will be the limit for every sequence? x n convergent to P the sequence f x will converge to l also and m also, but we have already shown that, if a limit of a convergent sequence is unique. So, limit look at h t e sequence f x n if it converges to different point l and m those two points must be the same. So, using the corresponding theorem of what the limit of sequences we can prove this theorem immediately.

So, do not just, no need to do anything any separate work here. There is another use of this theorem and that is the following suppose it. So, happens that you can find two sequences say let us say sequence x n and y n suppose both of them converge to P. If x n and if y n converge to two different limits. Then what will that mean, so that obviously that cannot happen. If limit of f x is X tends to P is if this limit exists. So, one of the ways of showing that the limit does not exist. In fact, I would say that easy way of showing that a limit does not exist is that you find two sequences x n and y n. Both converging to P, but limit of f x n and limit of f y n are different. That is the easiest way of showing that a limit does not exist. Let us just see one or two examples of this and that then we will stop with that. Let us for example let us think well known example.

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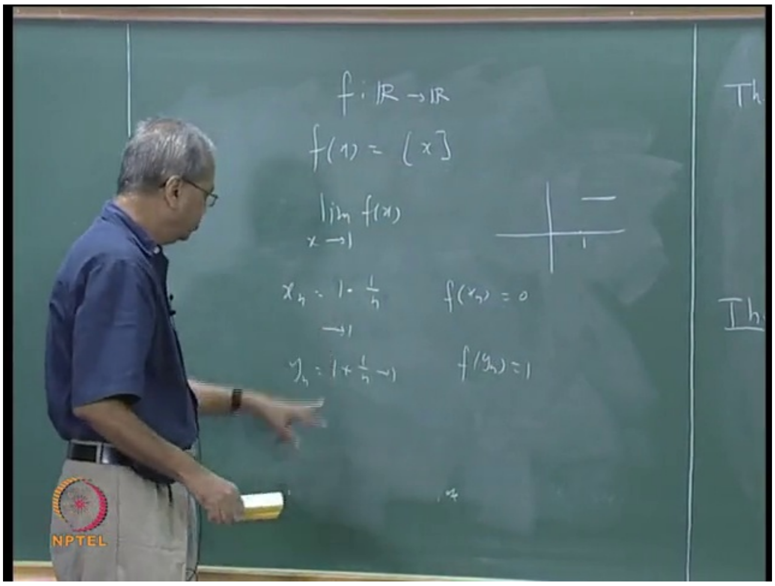

Suppose, is the, so called step function let us say f x is let us say. It goes from r to r and let us say f x means integral part of X. It is the greatest integer not greater that X it is a part of X n. I think let us suppose I want limit of f x as X tends to one limit of X is X tends to one basically do, we this particular cases. Look at the graph, suppose we look at the graph, when X goes from 0 to 1 f x is 0.

When f between 0 and 1 f x is 0 and say after between 1 and 2, it is 1. Now, what I say is that you look at a sequence. Suppose, I take a sequence x n as 1 minus 1 by n. So, we take a sequence x n as 1 minus 1 by n, what is f of x n 1 minus 1 by n? Will lie somewhere here for every n 1 minus 1 by n, will lie somewhere here. So, f of x n is zero f of x n is 0, this tends to 1 minus 1 by n tends to 1 and f of x n is 0 for all n.

So, f of x n tends to 0, now take y n as 1 plus 1 by n, 1 plus 1 by n also tends to 1 plus 1 by n also tends to 1. What about f of 1 f of y will be, because 1 plus 1 by n is going to lie this side. So, f of y n is 1. So, x n and y n both converge to 1, but f x n converges to 0. If y n converges to 1. So, the limit does not exist just we had simply let me give one exercise.

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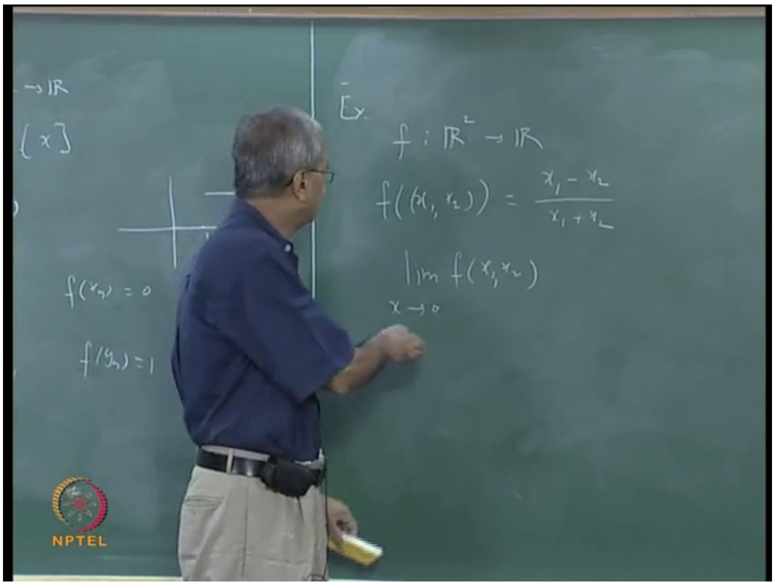

Let us say this time am taking f going from R 2 to r take any metric on R 2 usual metric or l one metric. Suppose, I take f of X x let us say X is rough form X 1 X 2. So, f of x 1 x 2. So, suppose f of x 1 x 2 is let us say something in this x 1 minus x 2 divided by x 1 is x 2. Now, this is not defined at X equal to 0, but does not matter it still i can talk of limit of f x 1, x 2. Let us I will simply say as X goes to 0, X goes to 0 means X zero of R 2. That means x 1 x 2 goes to 0 1 can talk of this. Then you would have seen this kind of problems in your advance calculus course.

When I suddenly and we know that suddenly we does not exist and you would have seen some argument for this, but now look at use this theorem. Use this theorem to show that this limit does not exist, what will be what will be the way of showing that. We will have to construct two sequences in R 2 both going to 0, but f of 1 sequence goes to different limit. f of other sequence goes to different limit just to this as A n exercise. I will stop with this.