Real Analysis

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Lecture - 21
Completeness

So, we were discussing the property of Completeness that metric space may or may not have. Let me recall that metrics paces said to be complete, if every Cauchy sequence in it converges to appoint in that metric space. We also seen some examples of complete metric spaces. Also, some examples of incomplete metric spaces. Now, coming to the examples of complete metric spaces. We have seen that discreet metric spaces is always complete whatever be the under lying set x. In the next set of examples that we have seen was that R and C. These are the complete metric spaces with the usual metric and next we are seen that these two.

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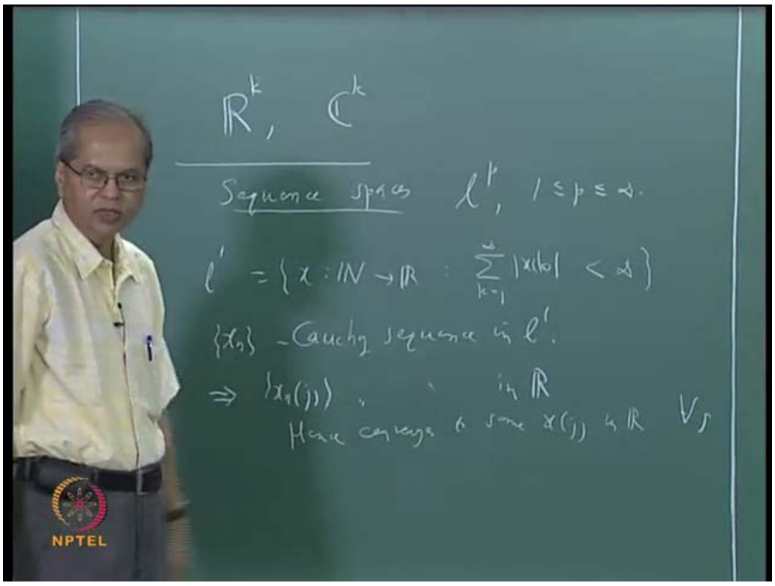

These spaces R k, C k in fact a familiar spaces. These are all complete metric spaces in the next group of metric spaces, which we are discussed. Let us look at this sequence spaces this spaces l p were one less nor equal to p less nor equal to infinity its, so happens that all this spaces are also complete. Let us just look at this space l 1 and proof will be more always similar to the proof. In the case of l 1 in other spaces also, l 1 let us I call it is a spaces of all sequences x of l to R such sigma mod. Let us say x k going from one to infinity is finite that this is a convergence series. So, to show that this is complete will have to consider a Cauchy sequence x n in l 1 Cauchy sequence in l 1.

So, if this is a Cauchy sequence what we can observe from here is that or something that we have already observe, that these implies that for each x n j this is a Cauchy sequence in R. If x n is a Cauchy sequence in l 1, if you take the z component that will form a sequence x n j, that is a Cauchy sequence in R. Since, we already seen that R is complete this converges. So, s converges to some x let us its some x j in R and this happens for each j this happens for each j. So, you get a sequence x going from n to R for each j you have some x j, what remains to prove this new sequence x is in l 1. This sequence x n converges to x in that l 1 metric induce by this l 1 R to do that.

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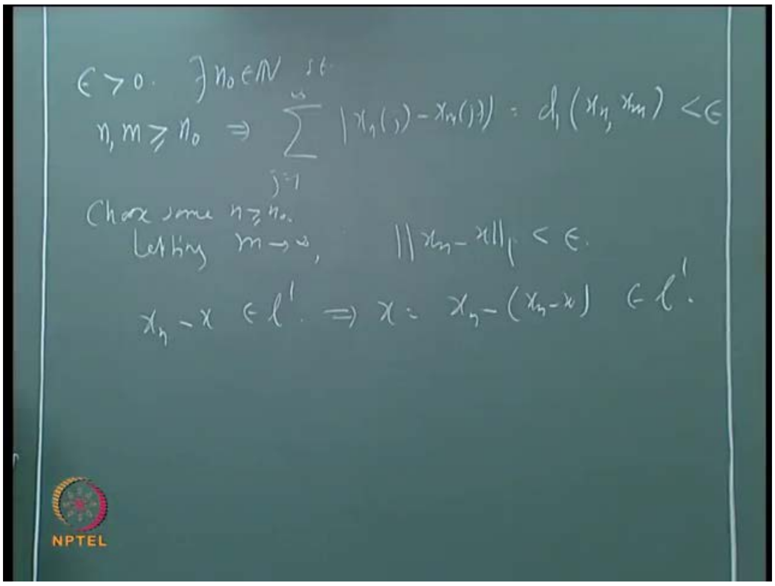

Again, will we see that x in Cauchy this means that, suppose you are given some epsilon bigger than 0 then there exist n 0 in n. Such that if you take to induces n and m bigger nor equal to n 0. Then this means distance between x n and x m that is less then epsilon distance between x n and x m that is less then epsilon, but what is the distance between x n and x m? It is nothing but norm of x n minus x m suffix 1 norm of x n minus x m suffix one. What is norm of x n minus x m? That is sigma mod x n j minus x m j going from one to incite, this is this is nothing but what we had call set d 1 between x n and x m. This must be less than epsilon this must be less than epsilon.

Now, the argument usually given after this is that keep some 1 n fixed choose someone n bigger nor equal to n 0 keep it fixed. Let m go to infinity in this in equality. So, doing that what you will get is that this x m j will go to x j, see this these in equals to true for every m. So, suppose you keep n fixed this in equalities to true for every m. Hence, it is true in the limit also, that will imply that x m j goes to x j.

So, that will imply the distance between x n and x is less than epsilon. In other words it will imply that distance sigma mod x n j minus x j that is less than epsilon. So, what it will implies that, so we can say that letting m tend to infinity will get norm of this x n minus x suffix 1. That is less than epsilon that is keeping n fixed that is choose some n bigger nor equal to n 0. That is choose let us say choose some n bigger nor equal to n 0 and then let m go to infinity.

Of course, this argument as be to made precise, but it is I think we are used to that kind of agreements by now. So, using that will get that norm of x n minus x is suffix 1 is less than epsilon, which means. Suppose, we write in the full format will mean that sigma mod x n j minus x j that is less than epsilon? So, in particular it will mean that this sequence x n minus x that sequence is in l one.

That is if it is a sequence x n j minus x j that is in l 1, but x n is already in l 1. So, that will give that x is in l 1 that will give that x is in l 1. So, this will say for example, the x is nothing but we can say x n minus x n minus x and x n is in l 1 this is also in l 1. So, this let us some is also in l 1, so this is also in l 1. Again, this same inequality will show that x n convergys to x in inferior. Here, we have shown that x n there exist n 0. If you take any n bigger nor equal to n 0 norm of x n minus x suffix 1 is less than epsilon. So, that is the that shows that l 1 is complete l and with slight modification you can show that all the other l p's are also complete. Slight modification of this proof will work for a any other value of l p’s.

Now, among the sequence spaces the space, which is not complete is this space C naught naught. Let us recall what is C naught naught, C naught naught was this space of sequences were all, but finitely many of x n's and zero. That is for each sequence x every sequence x in becomes constant sequence of zero after some time those sequence, which are eventually 0.

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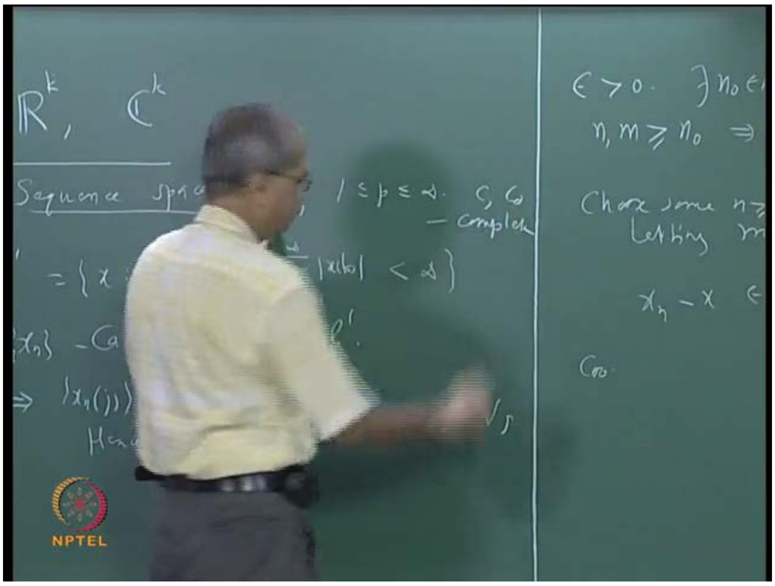

By the way l p as well as this c, c naught these are complete, all of these l p's as well as the sequence space. That is the set of all convergent sequences similarly, c naught the space of all sequences which come as to 0. These are complete metric spaces and proves will be both are less similar to what we have done now so among the sequence spaces this C naught naught is the example of incomplete metric space.

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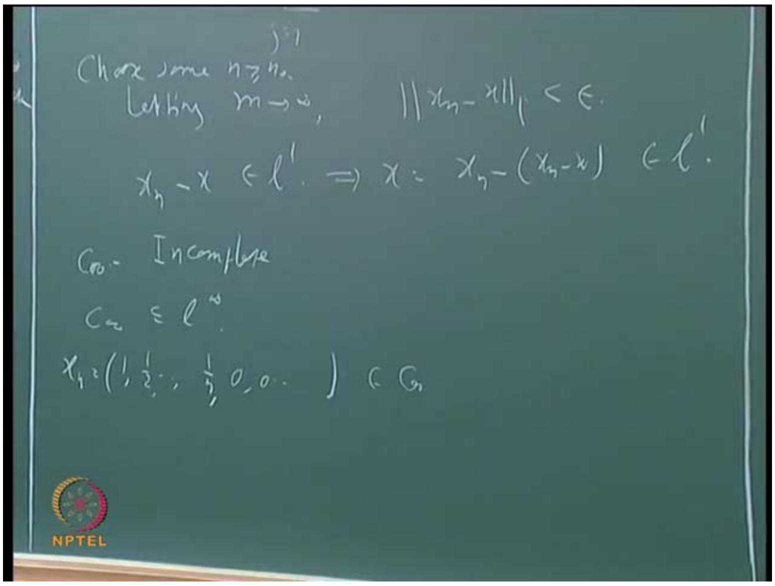

Of course, since C naught naught can be viewed as a sub space of any these, we should decide what metric we are giving on C naught naught. Suppose, we are regards C naught

naught as a sub space of l infinity, suppose we regards C naught naught that means we are giving this supreme. Of course, we can as also regard a sub space of l 1 also right now to show that a space is not complete. There is only one way you have to exhibit Cauchy sequence given example of Cauchy sequence, which does not converse. Because, till now we have only done that subsequently. If we do when we do some properties of complete metric spaces, then we can another way of showing that is spaces not complete. Will be to show that metric space does not have that property, but now we can use only a definition.

Let us take some example suppose we take x n as this sequence, say 1, 1 by 2 etcetera 1 by n and then 0, 0, 0 that is first n and three z are 1, 1 by 3 z to 1 by n at subsequently the sequence become 0. So, this becomes C naught naught were this becomes c naught naught. Now, first of all we have to see that this is a Cauchy sequence. Now, you see that this is a Cauchy sequence again as I say since we are regarding it is infinity.

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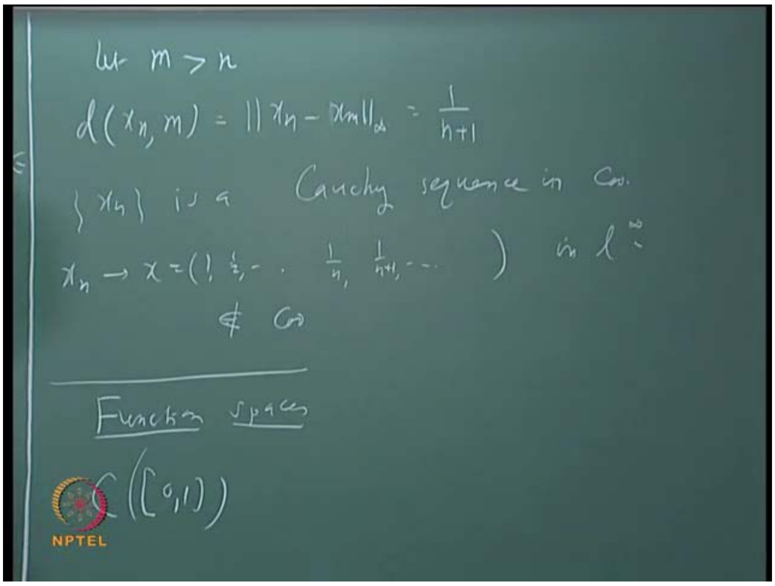

We have to look at distance between x n and x m and show that for large values of n and m. That distance is small distance between x n, x n is nothing but the this we are taken this x suffix infinity in norm. Let us take some let us say m bigger than n. So, x n is 1, 1 by 2, 1 by n up to 0 and x n will be 1, 1 by 2. Let us say 1 by m up to 0, 0, 0. So, what will be x n minus x m? That is tequilas start from the first n and it is will be zero. When 1 by m plus 1 sorry 1 by n plus 1, 1 by n plus 2. It will go up to 1 by m and then again 0, 0, 0.

So, what will be the norm of that? Remember, norm is norm of any x is maximum of mod x. Here 1 upon n plus this will be 1 by n plus 1. Now, can let be made arbitrary small, suppose given any epsilon can be always use some n 0, such whenever m and n is bigger than that n 0. This number becomes less than epsilon, that is true. Because, 1 by n plus 1 tends to 0. So, this something we can always do, so that is clear. Then this x n is a Cauchy sequence is a Cauchy sequence in c naught naught, but the question is does it converge?

Of course, it converges in an infinity is complete, that is what we have seen. It converges an infinity it convergys in an infinity to which sequence? It will be the sequence 1, 1 by 2 1 by n for all. That sequence obviously does not belongs to c naught naught, that is clear. So, in let us let us just xn converges to x, what is x? x is the sequence one 1 by 2 etcetera 1 by n plus 1 in an infinity.

So, this element x that is not in c naught naught. Because, in C naught naught what we require that after some space all entries should be 0. That is not happening here, now can it converges to some element in C naught naught also obviously not. Because, that will be in that it converges to two different elements in an infinity. So, we have a shown given example of Cauchy sequence, which does not converges. So, that proves that this space C naught naught is incomplete. Now, let us just look the last clue namely functions spaces. Here I will just make a small comment here that is suppose, we are look that the space let us say C is 0 1.

Of course, we can take any interval a to b, remember this is a space of continuous real valued functions on the interval 0 to 1. On this spaces we had consider two different metric, one metric was this given by norm suffix infinity. So, distance between two functions f and g was supermom of mod f x minus g x for x in 0, 2, 1. Now, with respect to that this is complete. So, C 0 1 with respect to this norm is complete, but this proof we shall do little later. Because, the convergence in this particular norm or this particular metric is what is usually called uniform convergence. We shall discuss the properties of uniform convergence of sequences etcetera little later.

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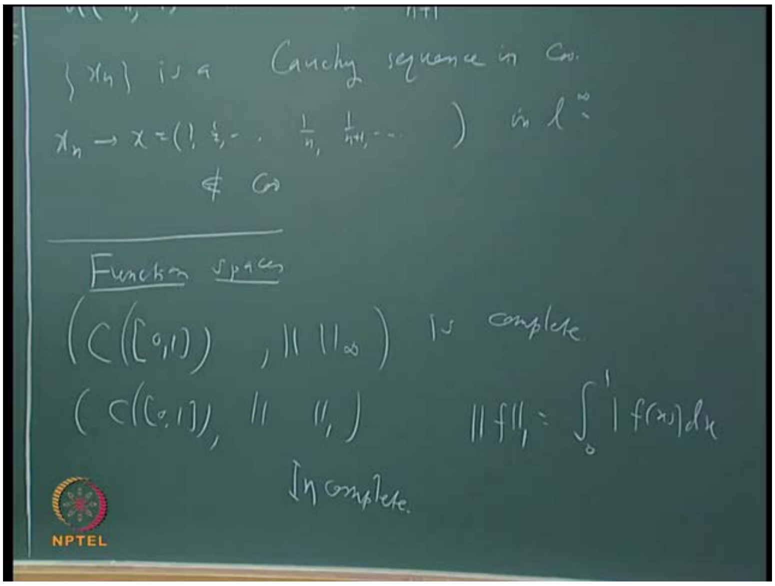

So, that time we shall prove that this spaces complete, we also discuss one more norm on this. That is C 0 to 1 norm suffix 1, what was this norm? Let us I call that ah norm suffix 1 of f this was defined as interregnal 0 to 1 mod f X d x. This is not complete in this norm it is not complete. So, it can happen that is saying set with respect to one metric can be a complete metric space and with respect to some other metric, it is not a complete metric space.

Now, how does when show that this is not complete, again will have to give a sequence, which Cauchy, but not convergent. I will not discuss those business, here I will give that to you as a problem. Now, let us go the next task namely we shall discuss some properties of complete metric spaces. The first question is given a complete metric space we have seen that given any metric space every subset can also be regarded as a metric space.

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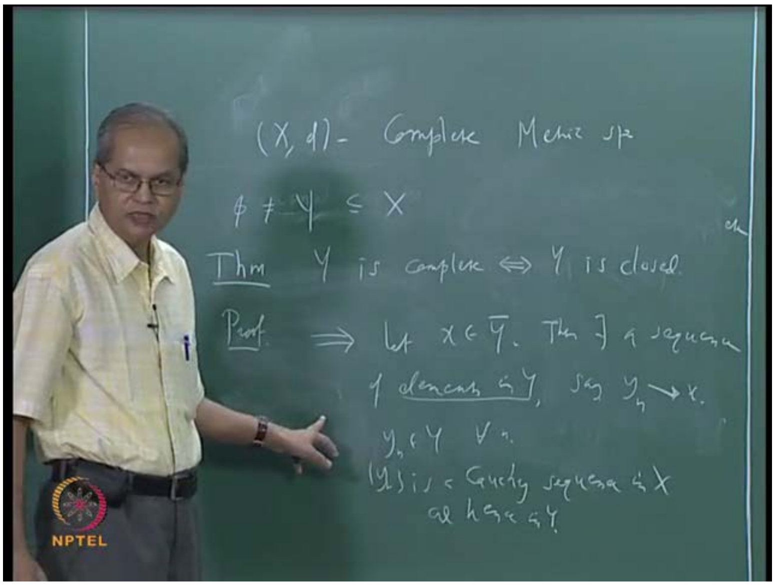

So, suppose X d is complete metric space complete metric space. Let us say Y is a subset of x y is a non empty set that is a subset of x. Then we know that y also can be regarded as a metric space on does a metric induce by the same metric d. Obvious question is when is y itself a complete metric spaces as a subs, as a if y is a subset of x where and if x is complete. When is y complete and the answer to that is very simple y is complete if and only y is closed. If y is a closed subset of x then its complete metric space. On other hand if it is complete metric space it is a closed subset of x. So, that is the first theorem that we shall prove y is complete if an only y is closed. Let us first assume that y is close y is complete.

So, let us look at the proof in this proof is very simple. Now, how does one show that any set is closed. Take a point in the question, so let x belong to y closer and we should we should show that x belongs to y. Now, we have already discuss one characterization of point in the closer in terms of sequences. Obviously, since when whenever we talk of completeness we should look at sequences. So, what do we know? If x is in y closer what should happen? There is sequence in y there is sequence in y, which converges to x. So, this implies that then there exist a sequence of elements in y n. Suppose, I call that sequence y n, say y n converges to x. Now, each y n is in y.

Remember, each y n is in that that is what the meaning of this elements in y x means y n belongs to I, for all n. Now, y n is a convergence sequence does it mean, there is a

Cauchy sequence? It means there is a Cauchy sequence. Does it also mean that it is Cauchy sequence in y? So, let us just say, so y n is a Cauchy sequence in X, hence in y, because the metric is same. Now, we have assume that y is complete then its starting y and this y n is Cauchy sequence. So, y n must commons to some point in y.

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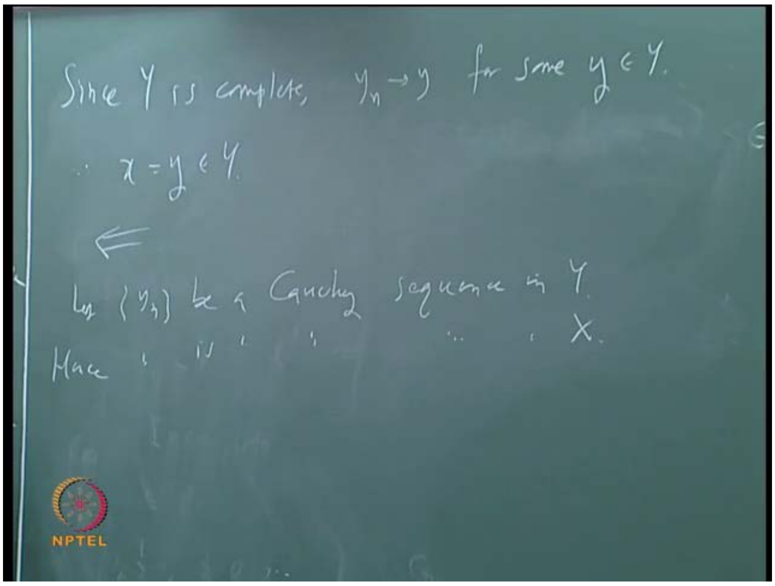

So, there will bounties, since y is complete y n converges to Y for some y in y or some small y in Y. Now, what is the obvious last part of the argument for limit of a? Convergence sequence unique. So, y n is a sequence in X also and X and Y both are points in X. So, if y converges to Y and converges to X, you must have X is equal to Y. so, since limit of a sequence unique we have. Therefore, we have X is equal to Y, which belongs to Y. That is what we wanted to show that x we started with X in Y closer. We show that X belongs Y. So, this shows that Y is closed. Now, let us look at this way assume that Y is closed.

We want to with y is complete, any way how does one start showing that any spaces complete? Take a Cauchy sequences in that sequence an issue of show it converges. So, let y n b Cauchy sequence in Y and what is to be done is clear? It is a Cauchy sequence in Y. Hence, it is also Cauchy sequence in X, because Y is a subset of X. So, hence y n is a Cauchy sequence in X, hence X is complete y n converges to some point in X. So, since X is complete.

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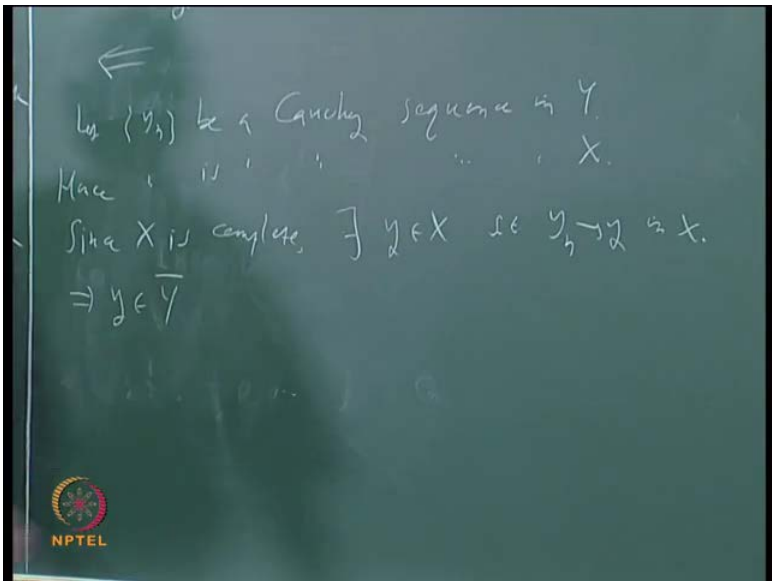

Since, X is complete there exist some Y in X, such that y n converges to Y in X. What is the next part of the argument? y n converges to Y and y n is the sequence of elements in Y. So, its limit must be in y closer, so this implies y belongs to y closer, but we assume that Y is closed. So, y closer is same as Y.

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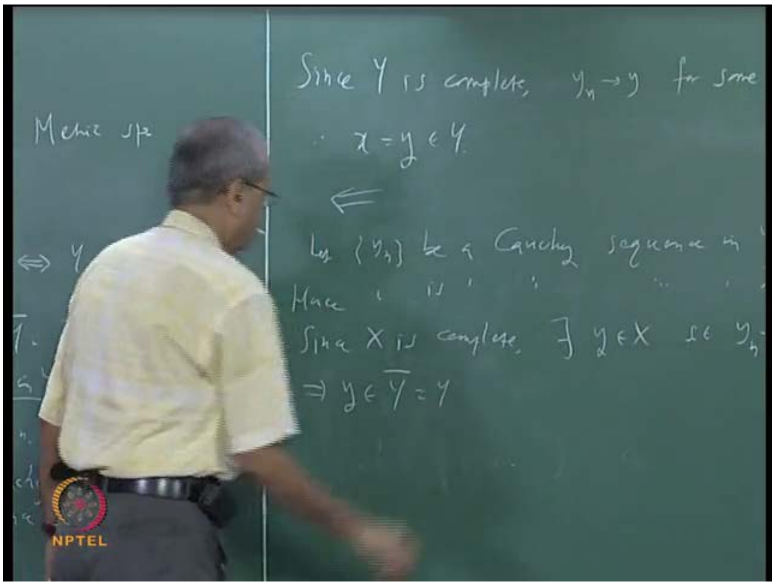

Just recall that we have started with a Cauchy sequence y n in Y. We have shown that that converges to a point in y that converge in a, so for example, in the real line any closed subset of a real line, will be a complete metric space in particular any closed interval. A complete metric space cantor set is a complete metric space. So, take any close set in a any complete metric space that self will be a complete metric space. Then the next important theorem.

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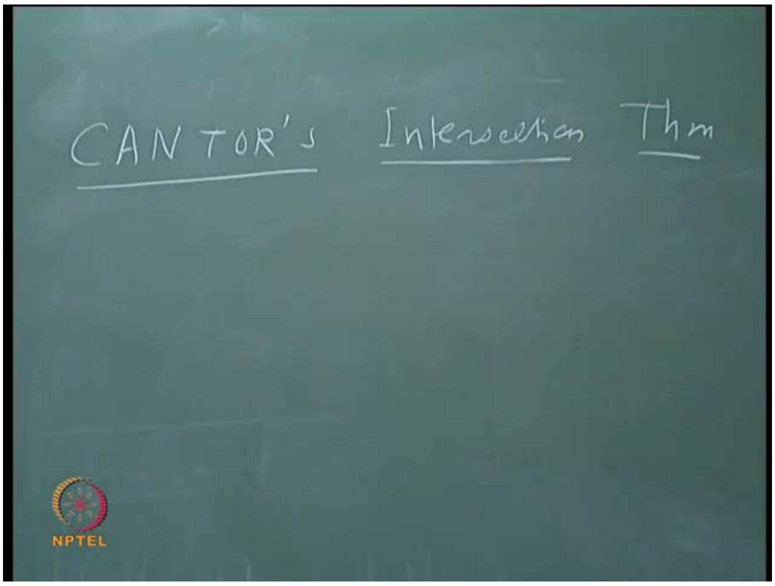

In the complete metric space is what is called Cantor's intersection theorem. Cantor's intersection theorem are in some books it is also called intersection principle Cantor's intersection theorem. Now, to understand what is inwardness intersection theorem.

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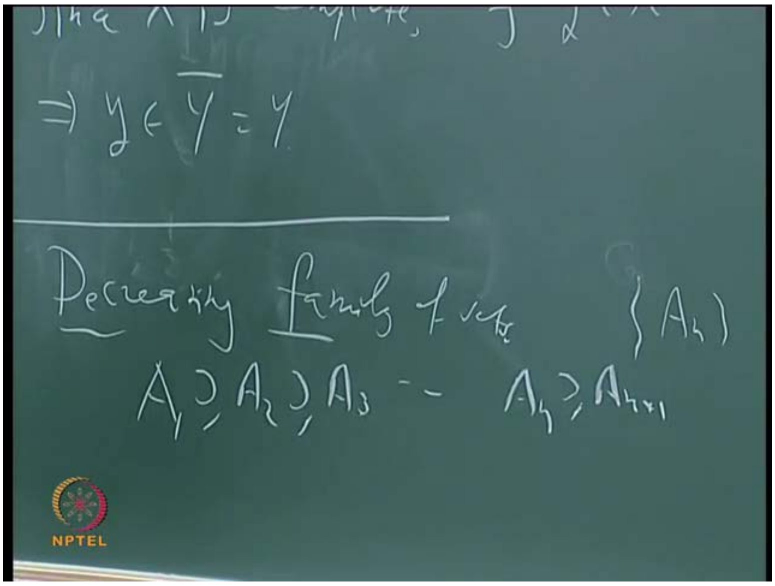

Let me just invest one more term here what is called a decreasing family or decreasing sequence of sets decreasing family of sets, it is clear? What is meant by this decreasing family? That is suppose you have family of sub sets of x, let us these are A n, this is called decreasing family. If A 1 contains A 2, A 2 contains A 3 etcetera. In general and contains A n plus 1 that is, if A 1 contains A 2, A 2 contains A 3 etcetera.

Now, what does Cantor's intersection theorem say that, suppose you have a complete metric space. If you consider a decreasing family or decreasing sequence of closed sets. You assume one more condition that the diameter of F n goes to 0 then intersection of F n contains exactly one point.

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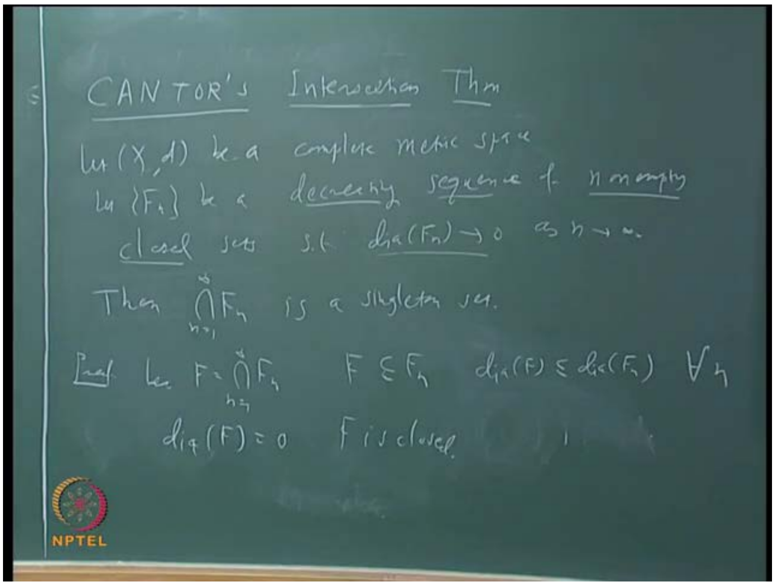

That is what Cantor's intersection theorem says. Let us first look at the statement. So, let X d be a complete metric space. Let F n be a decreasing family or decreasing sequence of closed sets. Such that diameter of F n stands to 0 as n tends to infinity then intersection F n going from 1 to infinity is a single term set intersection. F n is a single term set it contains exactly one point is a single term set. Now, let us look at what are the hypnosis there are three thing, assume that each F n is closed. That is important each F n is closed, what is the second hypnosis? That is a decreasing sequence that is F 1 contains F 2 F 2 contains F 3 etcetera. The third hypnosis is this diameter of F n stands to 0 as n tends to infinity.

So, under these three conditions we have to prove that intersection is consist of c exactly one point it consists showing that. It is consist of exactly one point, means what? First of all it contains some for it is non empty. Secondly, it cannot contain more than one point. The second thing is little easy that we shall prove first that it cannot contain more than one point. That follows from this last assumption here.

So, let some let us give some empty this intersection. So, let us call it f let f be equal to intersection of x n going. Of course, one more thing it is a decreasing sequence of non empty close set that is also important. Because, it is clear, if one of this F n is empty then of always intersection is going to be empty. So, each F n should also be non empty, each F n should be closed it should be a decreasing sequence and diameter should go to 0. These are all the requirements, what can you say about diameter of f, but how is diameter of f and diameter of F n related? Can we at least say this f is content in F n, f content in F n. So, what does it say about the diameters A? So, diameter of f must be less nor equal to diameter of F n.

So, diameter of f is less nor equal to diameter of F n. We should happen for all n. Diameter of f is less nor equal to diameter of F n for all n. We know that this goes to 0 as n tends to unity. So, what does it say about diameter of f? So, diameter of f must be zero If diameter of f 0 then obviously f cannot contain more than one point. So, we have proved this second part that is one part that is required we want shall it is. It contains exactly one point and this type it is proved that it is certainly cannot contain more than one point. Let us now show that it contains at least one point. Now, to show that it contains at least one point. Let us first use the five that each F n is non empty. Since, each F n is non empty we can consider some point belonging to F n.

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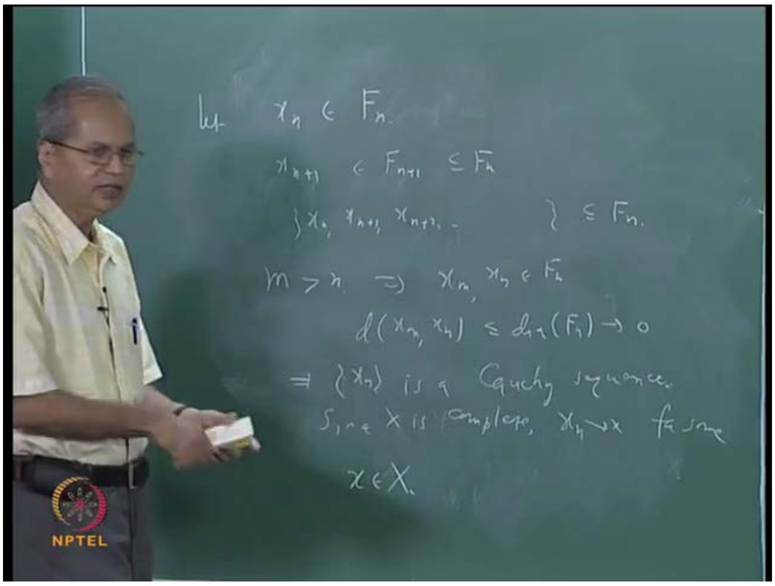

So, we can say that let x n belong to F n. Let x n belong to F n then our idea is to show that. So, this automatically forms a sequence if you choose one x n from F n, for each f for each n. Then it is a sequence we will show that this is a Cauchy sequence. Then will use the assumption of the completeness. Using the completeness assumption it will possible to show that, this sequence converges. They will say show that that limit should belong to f that is the idea. Remember, here you have f is a close set, is that clear? It's intersection of a say intersection of family of closed sets. So, f is closed, so we can if you want to you call here f is closed.

Now, to show that this is Cauchy sequence we use the fact. First of all there are two things used here first thing is that F n is decreasing sequence. Second thing is diameter of F n goes to 0. Now, if x n belongs to F n can you also then see, let us look at x n and x n plus 1. So, suppose you take x n plus one that belongs to F n plus 1. That belongs to F n plus 1, but we know that it is a decreasing family. So, F n plus 1 content in F n, so x n plus 1 belongs to F n also. Similarly, x n plus 2 x all them belong to f, so we can say that this set x n plus 1 x n plus 2 etcetera. All of them are content in F n. Now, if I take let us say any m bigger than n if I take any, so what it means is that for if you take say m bigger than n. Then x m belongs to F n that is what is says, right?

So, suppose now if x m and x n is any way belongs to F n if you take m belong bigger than n then, in fact we can say both that is x m and x n belongs to F n. Hence, what must happen is, it if you look at a distance between x m and x n, then this must be less nor equal to diameter of F n. This must be less nor diameter of F n and diameter if F n tends to 0. Does it mean it x n is a Cauchy sequence? Because, since this tends to 0 given any epsilon. Because, always find some n 0 such that diameter of F n is less than epsilon for n bigger nor equal to n 0. Then the for any m and n bigger nor equal to that n 0 distance between x m and x n will be less than that epsilon.

So, this shows that x n is a Cauchy sequence. Now, we use this first type of this that is how we started. It is x is a complete metric space and this Cauchy sequence in that. So, this must converge this must converge to some point in x. Since, x is complete x n converges to x for some x in x. Now, we want to show that these x belongs to the intersection is x belongs to the intersection. Now, how does want to belong to the intersection?

Then there is only one way we have use that it belongs to each of this happens. Now, x is a limit of the sequence x n is a limit of the sequence x n is also limit of this sequence x n plus 1, x n plus 2 etcetera. Suppose, you take any fixed value of n and take the sequence x n plus 1 x, then also subsequent to the original sequence. So, that is also that should they should also converges to x, but all these points are in F n. We assume that F n is a closed set. So, each of this, so x must belongs to f x, because x is a limit of a sequence of points in F n. So, it must belongs to F n. So, x belongs to F n.

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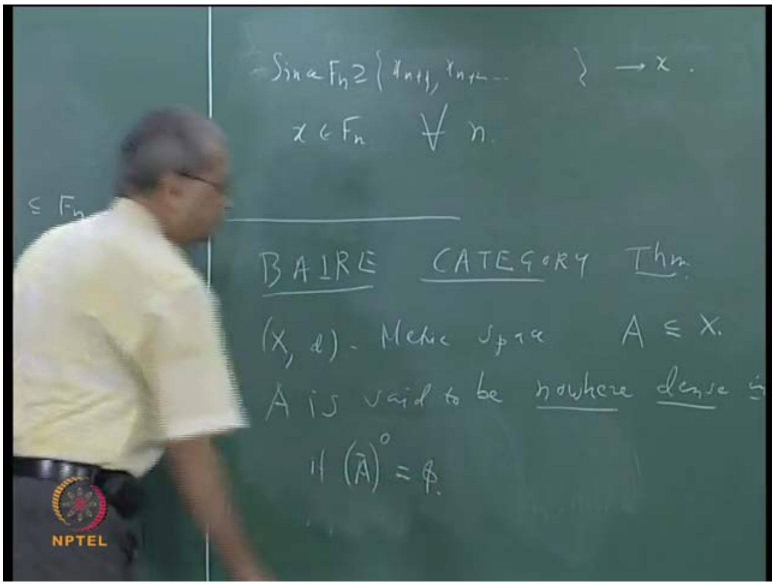

We can say that since this sequence x n plus 1 x 1 plus 2 etcetera. This sequence converges to x and this is this sequence F n contains this and F n is closed x belongs to F n x belongs to F n for all. That is same saying that x belongs to the intersection. So, have you proved everything? We already proved it cannot contain more than one point. Now, we have shown there it contains one point. So, intersection contains exactly one point. So, what are the hypnosis? We are used we use practically origin in that it is a complete metric space, it is each F n is non empty. It is a decreasing family and diameter goes to 0. Now, at this point let me ah recall with I do not remember, whether I made this point earlier or not.

So, paragraph in the introduction of Simian's book, where he says that any proof will any theorem will contain some hypnosis and some conclusion and a proof may contain several steps. Suppose, you understand how n plus 1 step follows from the n steps for each n. That does not really mean that you have understood proof, you should understand proof as whole, as an idea. What is what, are the main ideas involved in the proof. What is the what is the way of really deciding, whether you understood proof or not.

One way is you look at various hypnosis, all the theorem here we have seen. So, many hypnosis and then ask yourself, whether any of this hypnosis can be draw at that means what? Suppose, to begin with suppose assume that x may not be x is not complete, but, all other happen is a true. Then does the conclusion wholes still then you can say that just drop the passionate F n is a family of closed sets instead of close set. Suppose, you take the sets, which they are not necessarily closed will those conclusions still hold. So, do it for each of the hypnosis and see whether the theorem is still true if it is, if theorem is false.

Then for each such case you will should get one example for dropping one hypnosis. You should get one example to show that the conclusion does not whole. If you can do it for all those hypnosis, then only you can be sure that you have understood proof completely. So, do it for this theorem, because this is the first major theorem that we have we have discussed in this course. So, do it for this particular theorem. Now, let us go to one more very famous theorem about the complete metric space. It is called Baire category. Now, to understands what is meant by Baire category theorem? Again, it some terminology we have already defined. What is meant by a dense subset for this? What the meaning of dense subset? Let us again recall.

So, suppose X d is metric space and let us say A is a subset of x. Recall, that we are said that a is dense in x a is dense in x. If a closer is equal to x a closer is now we are going to define something that is exactly opposite to being dense. We want to say and because seen that what is implied by saying that A is dense in A among this. So, many equivalent conditions that we have seen that saying that A is dense means, if you take any open set in x, it as an non empty intersection with A. That is meaning of set mean that follows if A is dense in X, what is the exactly opposite concept of that? It means that if you look at see A closer is x exactly opposite. That means A closer does not have any interior point.

So, such a set is called nowhere dense. So, let us see that definition first A is said to be nowhere dense in x. If look A closer and look at its interior that is empty. Let us again see some examples before proceeding further with this. Suppose, X is a discrete metric space, then we if you take any subset it is going to be closed. Suppose, let us take an non empty subset it will be closed. Hence, it closer will be that same subset also every subset is open. So, it is interior will again with the same subset obviously it will not be a non empty. So, it will not be empty, because we use to started with a non empty subset. So, in discreet metric space non empty subset will be nowhere dense, non empty subset will satisfies this property. Now, let us look at the real line in real line we know. For example, that set of all rational numbers is dense. So, Q closer is R. So, Q closer interior will also be R. So, let us just see Q closer is R.

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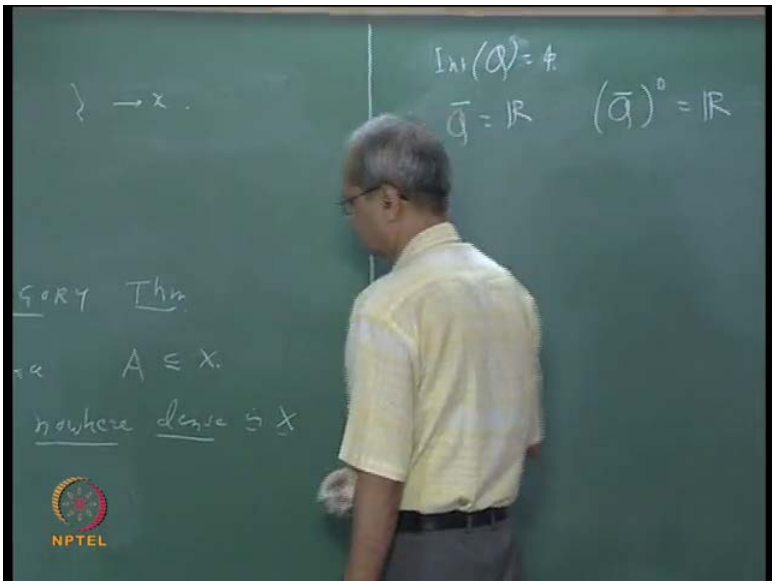

So, if I look at Q closer and it is interior that will be again R. So, Q does not satisfy this property though Q interior is empty, remember this fine q interior is empty. In fact if think you have been using this notation for interior of Q. So, let me do it here also that is interior of A closer. I think I have also mention that this is also fairly common notation used for the interior. Now, it is implore to realize that if a interior is empty that does not mean A is nowhere dense. What must happen is that interior of it is closer must be empty. Suppose, we take let us say n, suppose we take n what is a closer of n, n is closer is n itself and what is the interior of n closer.

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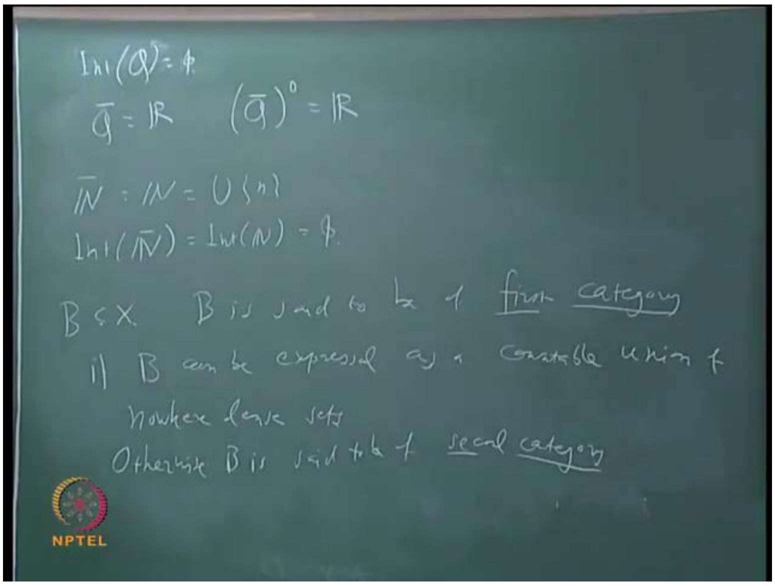

That is same as interior of n and what is that what is interior of n that is an empty set. So, n is an example of a nowhere dense set and once you understand this, now you can constructs some any other example. For example, z is also an example of a no where dense set. So, that is about a about nowhere dense sets dense let us look at the next concept it would depends on this. We say that A set is a first category we said that A set is a first category. If it can be expressed as a countable union of nowhere dense sets, say that suppose I call the set B suppose let us B is content in X B is set to be of first category.

If B can be expressed as a countable union countable union of nowhere dense sets. B can be express a countable union of nowhere dense sets. Again, what is a obvious example of set of first category it in R take this set. Suppose, you take any single tends n can be express countable union of single term x. That is n is you can say n is union single term n. If you take each single term set it is closer is that same single term set and it is interior is empty. So, each single term in R is nowhere dense. So, n is an example of a set first category not only n, but you can say that, since each single term set is count to be nowhere dense. Every countable subset of R is an example of set of first category.

So, in pitiable of an example q is also an example of this set of first category it is not an nowhere dense set, but it can be expressed as a countable union of nowhere dense sets. If A set is not a first category it is called second category. Let us say, so if this not true it is B is said to be a first category. If B can expressed as countable union of nowhere dense. It is otherwise we said B is second category otherwise B said to be of second category.

What does it mean? It means that it cannot be expressed as a countable union of nowhere dense sets. Another way of saying the same thing is that, if at all you write B as countable union of sets. Then at least one all the sets occurring in that countable union should be not nowhere dense, should be that is a meaning of that second category. Now, coming back to Baire category theorem, Baire category theorem is statement is very simple. It is simply says that every complete metric space is a second category. Now, let me just write that theorem here.

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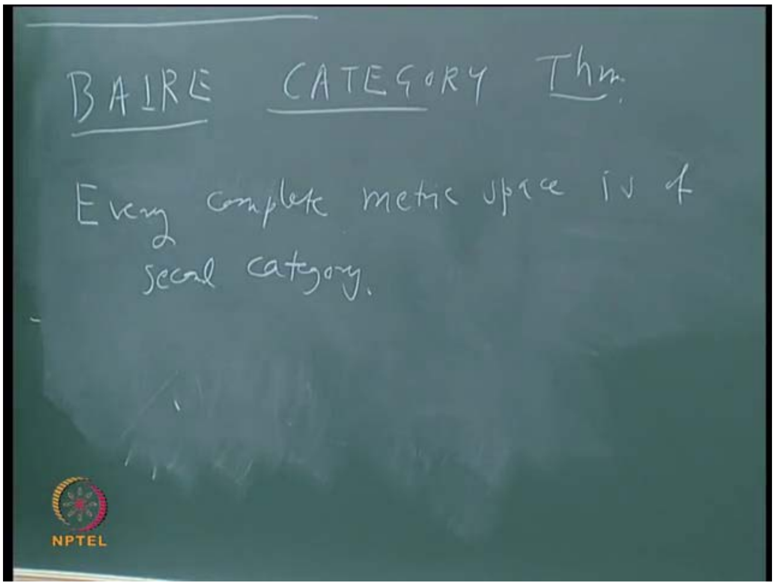

Every complete metric space is of second category. So, what does it mean? You cannot express a complete metric space as a countable union of nowhere dense sets. That is the statement of Baire category theorem it is a very useful theorem. It is used in the proofs of many important theorems, but the only problem is that all those important theorems, whose proofs use Baire category theorem.

You will not learn in the course of real analysis, but most of those theorems you will learn in functional analysis, that is. There are many important theorems in functional analysis known as close graph theorem and open mapping theorem and also, called uniform boundedness principle. They are fairly important theorems from the points of few of applications and proofs of all those theorems use this Baire category theorem. So, we shall consider the proof of this Baire category theorem in the next class.