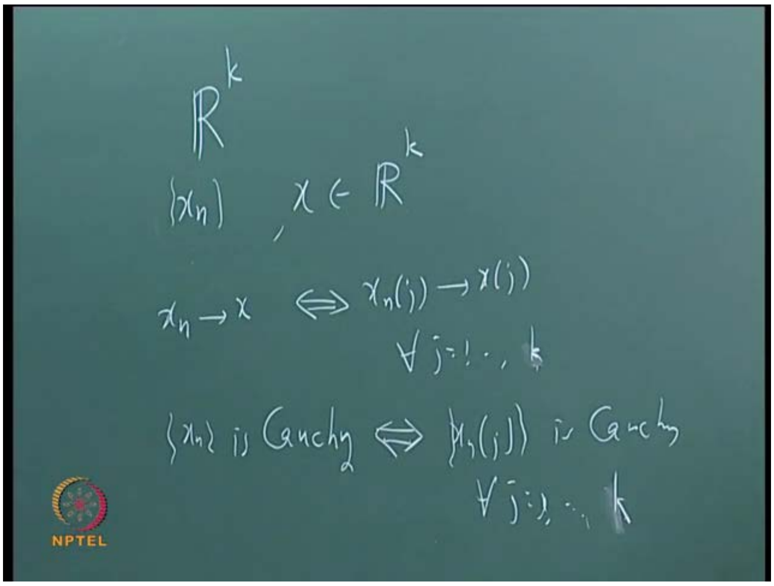
Real Analysis
Prof. S.H. Kulkarni
Department of Mathematics
Indian Institute of Technology, Madras

Lecture - 20
Sequences in Metric Spaces

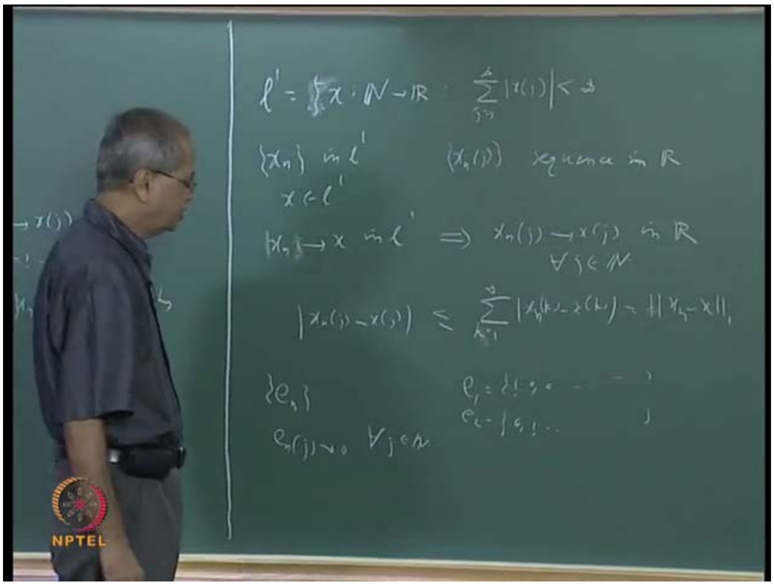
In yesterday's lecture we discussed what is meant by convergence of a sequence in a metric space. And also another concept of what is meant by Cauchy sequence in a metric space and we also discuss few things about these spaces.

(Refer Slide Time: 00:28)



Let us this space R k and in this space we said that if we take a sequence x n and let us say that a x in R k then x n converges to x if and only if x n j converges to x j for each j equal to 1 to n and also if x n is Cauchy if and only if each x n j is Cauchy for all j equal to 1 to n. So, any sequence in R k will give rise to k sequences of real numbers and if the original sequence is a convergent sequence each of those k j, j goes from 1 to k. Actually each of those k sequences are convergent and similarly if the given sequences Cauchy each of those k sequences are also Cauchy then will it comes to l 1.

(Refer Slide Time: 01:57)



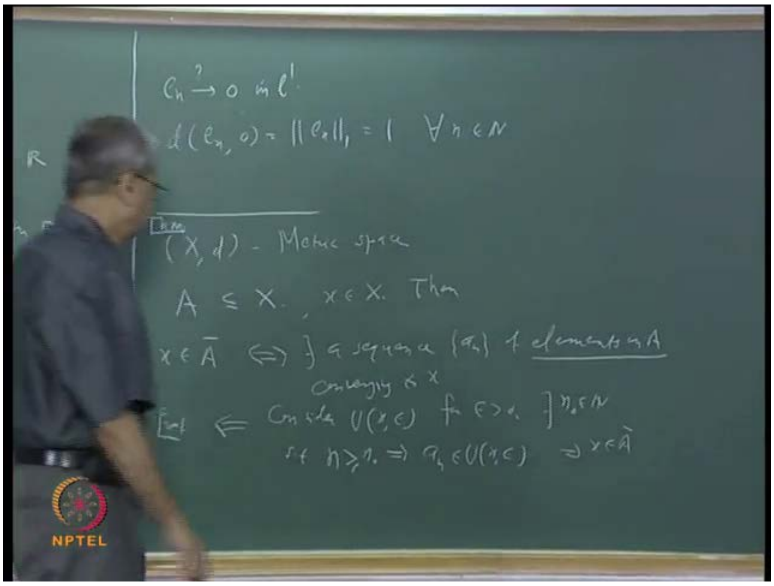
Let us again recall l 1 was this space it is the, it is the set of all sequences x from n to R satisfying that sigma mod x j is convergent. Now, in this case what will happen is that if we take some sequence x n in x, sorry x n in this l 1 then this will lead to infinitely many sequences of real numbers for each because infinitely many for each j j belongs to n. There will be one sequence of real numbers, so x n j is sequence of sequence in R, now here what happens is that we can say that if x n let us say suppose we take some x n l 1 and suppose x n let us just this x n converges to x in l 1.

Then this will imply that x n j converges to x j in R and this is easy to prove, this is easy to prove because if you look at distance between x and j that is what we do is mod of x n j minus x j this is certainly less not equal to sigma j equal from 1 to infinity. Let us say sigma k going from 1 to infinity mod x n k minus x k and with this is what we have call norms of x 1 of x n minus x. So, if this is small for a, if at all if this is less than epsilon this will also be less than epsilon, so what we can say that x n converges to x and l 1 that implies x n j converges to x j in R for each j in n.

But, in this case the converse is false in this case the converse is false it can happen that x n j converges to x j for each j, but x n may not converge to x in l 1 and as you know to show that anything is false there is only one way. We need to give a counter example and to give a counter example let us take this sequence e n, this is a very well known sequence e n what is e n, e n is the sequence whose n-th term is 1 and all other terms are 0.

So, for example e 1, e 1 is this sequence e 1 is 1, 0, 0 etc e 2 is 0, 1 etc that is e n, now in this case for example if you look at e n of j look at the sequence e n for example suppose I take e n of 1, what is that sequence e n of 1 is 1, 0, 0, 0 etc what is e n of 2 it is 0, 1, 0, 0 etc. So, in general suppose we take any en of j any en of j it is 0, 0 js term will be 1, I will again 0, 0, 0, so is it clear that en of j tends to 0 for each j in n, e n of j tends to 0 for each j in n, now the question is does e n tend to 0, does e n tend to 0.

(Refer Slide Time: 05:40)



This is the question does e n tend to 0 in l 1, what is the meaning of saying that e n tend to 0, 0 means the sequence 0, 0, 0 constant sequence 0, 0, 0, now to do that we need to look at distance between e n and 0. Look at distance between e n and 0, look at distance between e n and 0 that is distance between e n and 0 that is nothing but norm of e n, that is nothing but norm suffix e n and what is that.

Student: Summation.

Summation of, so what is that, what is the value for example what is norm of e 1, so norm of any e n is 1, norm of any e n is 1, so that is equal to 1 for every n. So, can they converge to 0 for every n distance between e n and 0 is 1, so certainly e n cannot converge to 0 in l 1 even though e n, j converges to 0 for all j. So, in spaces like l 1, l p etc we have to be little more careful this though x, whenever x n converges to x in l 1, x n j converges to x j in R that is true, but the converse is not true and similar thing can be said about the Cauchy sequences.

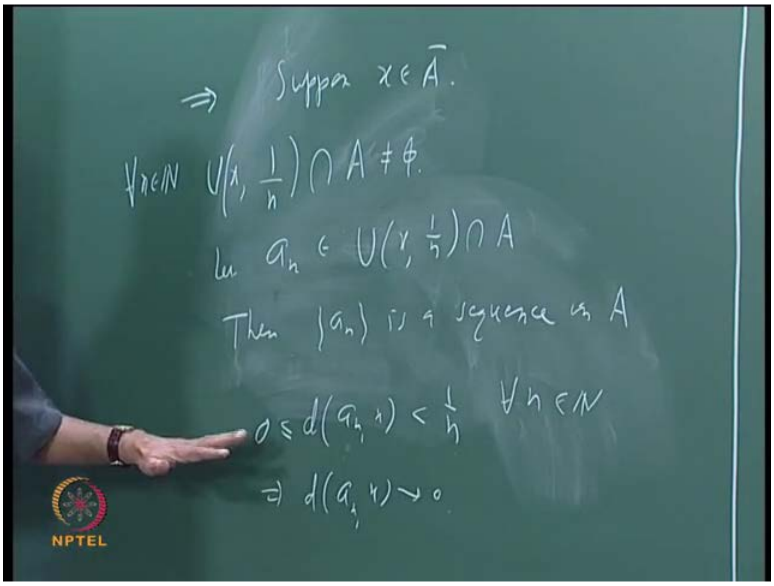
If x n is a Cauchy sequence we can say that each x n j is a Cauchy sequence, but again the converse is false, again you can take the same example. Now, we should also relate to this concept of the convergence of a sequence and the concept of A closure that we have seen earlier, so to do that let us again take a general metric space X d is a metric space and suppose A is a subset of x. Then what we want to say is that if you take any sequence of elements in A, if you take any sequence of elements in A and suppose that sequence converges to some element not necessary in it some element in x. Then that limit must be in A closure, then that limit must be in A closure conversely if you take any element in A closure it must be a limit of some sequence of elements in A.

So, what we what we want to say this suppose A is con sub set of x then suppose you take x as any point of x then x belongs to A closure, x belongs to A closure if and only there exist a sequence. Let us say the suppose I call it sequence a n suppose there is a sequence a n of elements in A, there exists a sequence a n of l this in point that an that all elements are in A that is important sequence of element in A converging to x. So, you can call this as a theorem, so this expresses the points in the closure of a set as limits of sequences of elements in the given set, now to look at the proof let us look at this way first.

Suppose there exists a sequence of elements n in A such that a n converges to x, now to show that x is in the A closure what is it that we want to show, you take any open ball with centre at x and show that open ball will centre at x its intersection with A is non empty. So, let us let us take an open ball with let us say centre at x n radius epsilon, so consider U x epsilon, consider U x epsilon, so what we know is that the sequence a n converges to x.

So, for this epsilon there exists some n 0, there exists some n 0 such that whenever n is bigger not equal to n 0 all those ns are in this. So, now let us consider for some epsilon bigger than 0 then there exists n 0 in n such that n bigger not equal to n 0, that implies a n belongs to U x epsilon and a n is in A and this in U x. So, we have shown that if you take any open ball with centre at x it its intersection with A is non empty, so that shows that x belongs to A closure, so that show that x belongs to A closure.

(Refer Slide Time: 11:16)



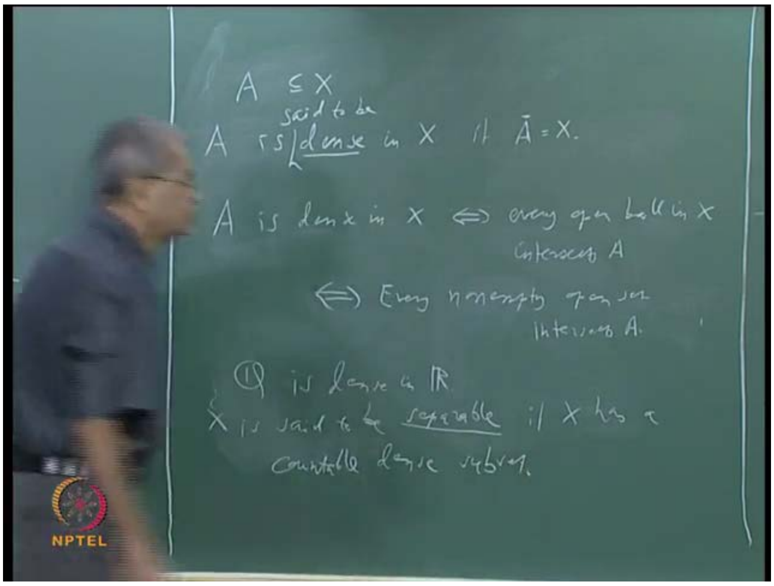
Now, let us look at the other way that is suppose x belongs to A closure then we want to show that there exist a sequence of elements in A converging to x, converging to x. Again let us look at it, x belongs to A closure means what you should take any R positive then open ball with centre at x 1 radius R that should, that should contain a point from A. So, I will take R as 1 by n, I will take R as 1 by n. So, what you can say is that open ball with centre at x at radius 1 by n for every n, for every n in open ball with centre x at radius 1 by n its intersection with a must be non empty.

So, take some point in this, take some point in this and I will call that point a n, so let us say let a n belong to this U x 1 by n intersection A then a n is a sequence in A, then an is a sequence in A because each n belongs to A, each n belongs to A. So, then a n is a sequence in A, in A, so what reminds to be shown that an converges to x, what reminds to be shown that an converges to x.

But, is it, is it clear what can we say for distance between a n x since a n belongs to U x 1 by we know the distance between a n and x, since a n belongs to U x 1 by we know the distance between a n and x is strictly less than 1 by n and for all n. This is always a non negative number, it should distance, so we can say that 0 less not equal to this and this is less than 1 by n for all n and, so we know this goes to 0 as n goes to infinity.

So, this also must go to 0 as the, so called the sandwich theorem, so this also must go to 0 as, so this means distance between a n and x that is, that is this implies that distance between an and x tends to 0, which is same as a that an converges to x. Now, that we are and the points of discussing some properties of the closure, let me also introduce couple of terms that we shall require very often.

(Refer Slide Time: 14:06)



We say that a suppose we take any metric space X and A as a subset of X, we use this terminology we say that A is dense in X, A is, A is called dense in X this means A closure is equal to X. So, A is said to be dense in X, A said to be dense in X if A closure, A closure is equal to X let us see what is the, what is the meaning of this A closure is equal to A, A closure is always going to be sub set of x.

So, what is important is X is in A closure which means every point of the metric space is in the closure of A, every point of the metric space is in the closure of A, what is the meaning of this. If every point is the closure of A if you take any ball with this centre at that point its intersection with A must be non empty, but if that has to happened to any point let me see must point to any open because every open ball is equal to be open ball with some centre.

So, we can say that we can write this ball also A is dense in X if and only every open ball with X has a non empty intersection with A, every open ball in X has a non empty intersection with A. Usually we simply say X plus that has say every open ball in X intersects A, it intersects A means it has a non empty intersection with that serve five customary usage of terminology. We said every open ball in X intersects A, is it also that if every open ball intersects A then every non empty open set will also intersect it, if you take any non empty set it has to contain a some open ball, some open ball.

So, that will have intersection with not empty intersection with this, so the given open set should also have non empty. So, in case of this is if and only every open set not every open because empty set is also open, so we have to exclude that, so every non empty open set, every non empty open set intersects it what are the well known examples of such sets. For example, we can take this set of all rational numbers in the real life what is the closure of that we have seen that it is, it is R, so we can say that Q is dense in R, Q is dense in R and what will this statement mean every open ball in X intersects.

Same as every open interval contains a rational number, every open interval contains a rational number, something which will already prove what about the set of all rationales that is also dense, that is also dense what about set of all natural numbers, and A it is a closed set and different from all. Now, one more definition we say that a metric space X is separable we say the separable if it has a countable dense subset X is said to be separable, said to be separable if X has countable dense subset what is the obvious example of separable metric space. You already have a Q is dense in R, Q is countable, Q is countable, so R is a separable metric space, what about R 2.

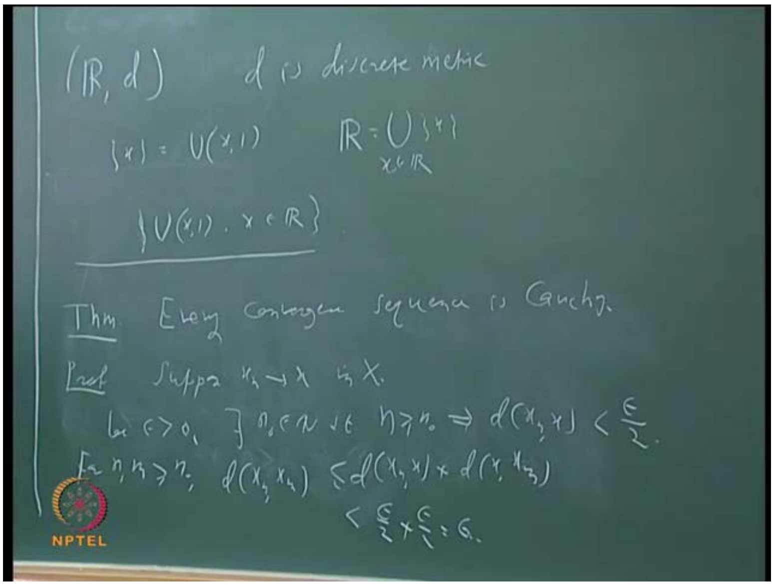
Student: Q cross Q.

You can say Q cross Q is dense in R 2 in, Q cross Q is dense and Q cross Q is countable, so that is also an example of a countable dense subsets, so it is an example of a separable metric space, what is the example of a metric space which is not separable.

Student: l, l1.

No, l 1 is separable, but only thing is it, is it is somewhat difficult to show that we shall discuss it little later, now let us look at this a is dense in X means every non empty set should intersect A. So, if you take every non empty or every open ball n, X should intersect it, but suppose a metric space contains an uncountable family of pair wise disjoint open balls then such a metric space cannot have a countable dense subset because every open ball has to contain for. If any set is dense in that metric space then every open ball has to contain at least one point of that set, so and if the family is uncountable that set will be automatically uncountable, so no countable set can be dense in such a, such a metric space, now suppose I take this set R.

(Refer Slide Time: 20:50)



But, instead of taking usual metric suppose I give discrete metric, suppose I take R d where d is a discrete metric, d is a discrete metric. Then we have seen that discrete metric even this single term x is in open single term x is nothing but, open ball with centre at x and various one and R is, so R is union of this single term x for x in R. In another words suppose we take this U x 1 with x in R that, suppose I take this set U x 1 with x in R that is in uncountable family of open balls and any two open balls in this are disjoint, any two open balls in this are disjoint.

So, this cannot have any countable dense subset, this cannot have any countable dense subjects, so R with discrete matrix is a simplest example of metric space which is not separable will proceed further. We shall, now go back to the theorem which we were about to prove yesterday that is every convergent sequence is Cauchy, every convergent sequence is Cauchy is Cauchy. But, proof of this is again very similar to the proof of the corresponding theorem in about sequence of real numbers the idea is basically the same, that is if a sequence converges to some point x.

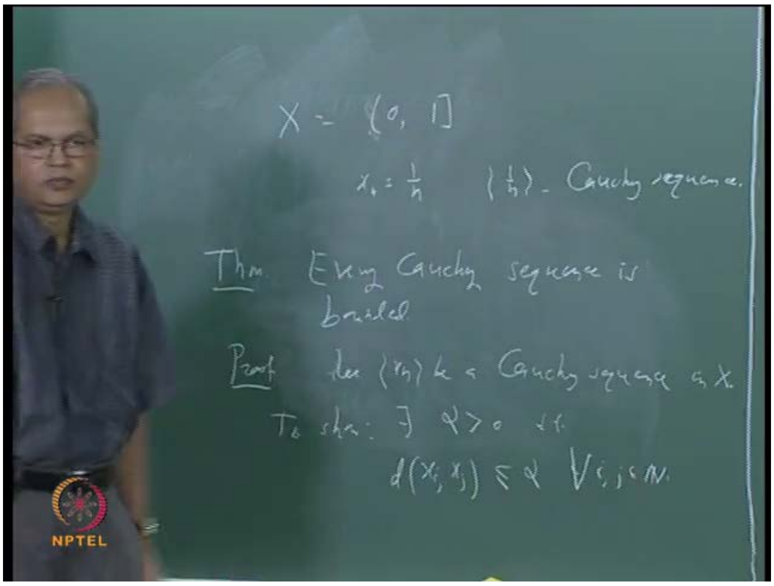
It means all the points with the sequence come very close to the, arbitrary close to the point and hence any two points on a sequence should come close to each other does the, does the whole idea of this proof. So, suppose x n converges to x in x then we want to show that x n is a Cauchy sequence, so let us say let epsilon be bigger than 0 and what we want to show is that given for this epsilon there exists some n 0. So, that whenever m and n both are bigger not equal to n 0 distance between x and m is less that epsilon, but what we do, now is it is a convergent sequence.

So, for every epsilon we can always find some n 0 such that whenever n is bigger not equal to n 0, distance between x n and x is less than epsilon. So, we can say that there exists n 0 in n such that n 0 in n bigger not equal to n 0 this implies distance between x n and x is less than I can take epsilon to instead of epsilon because this is true for every positive number. So, instead of epsilon make all as per take epsilon may two which is what you have done in the in that proof also, then consider any n and m bigger not equal to n 0 then for n and m bigger not equal to n 0 distance between x n and x m what is to be done is clear, use triangle equality.

It is this is less not equal to distance between x n and x plus distance between x and x m and each of that is less than epsilon by 2, so what is observed. Here, is that proof is more or less similar only thing what is happened during that wherever there was mod x n minus x n or mod x n minus x that kind of terms are replaced by distance between x n and x. So, essentially the same idea is used, here in this proof also and, now this raises very important question, here what about the converse in case of real line we had proved that the converse is also true.

We had proved that using several so many things limit superior limit inferior and l u b actions and all those things in a general metric space this cannot be true in arbitrary metric space, this cannot be true what are the obvious examples. We have seen that given any metric space and if you take any sub set of that metric space then that itself is a metric space with the metric induced by the original metric, so what we can say is that for example.

(Refer Slide Time: 26:08)



Suppose I take instead of taking X as real line, suppose I take X as let us say 0 to 1 by all that say 0 to 1 that is open at 0 at of course we can take open at both ends also. But, does not matter open interval semi open semi close interval open at 0 and closed at 1, now this in, this is also metric space with the usual metric coming from R.

Now, in this metric space suppose I consider sequence x n as 1 by n we already know that this is, this is a Cauchy sequence, 1 by n is a Cauchy sequence, 1 by n is a Cauchy sequence. But, is it also clear that this does not converse in X because what will happen is suppose it converges to some point in x what is the requirement then that point, that point must be strictly bigger than 0.

But, we also know that this sequence this is also sequence in R and in r it converges to 0 that will be in that the convergent sequence in R has two different elements that is not possible, that is not possible that means this is the Cauchy sequence. But, it has no limit in x it has no limit in x, so in arbitrary metric space it is not true that every Cauchy sequence converges, so those metric spaces which have the selectional property they had given some special name. But, we will come to that little later before that I want to also discuss one more property of the Cauchy sequences and that is the following and let us take this also as a theorem every Cauchy sequence is bounded, every Cauchy sequence is bounded.

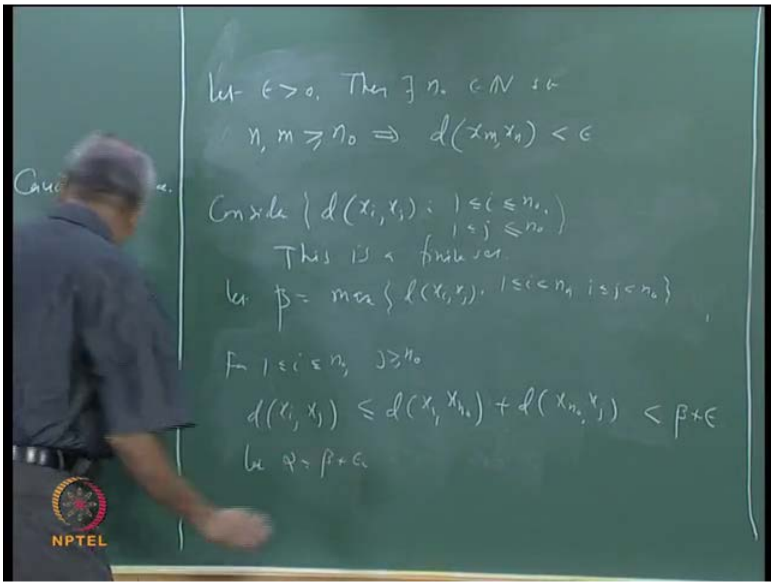
So, again a Cauchy sequence is bounded means what yes the closest thing is lower bound and upper bound it is a sequence in any arbitrary metric space we can talk of lower bound or upper bound only of sequences of real numbers when you talk of sequences in a arbitrary metric space. Then we would know all that, but we have defined what is meant by bounded set in a metric spaces what is that it its diameter is finite, it for any set we say that it is a bounded then if it is diameter if finite what is meant by a sequence is bounded.

You take the set which is the sequence that is in other words in general what when will we say that a function is bounded it is again this bounded, it is again this bounded. So, sequence is a function, so saying that sequence is bounded it is same as saying that it is range of that, what is the range of the sequence it is the set of all points which make that sequence we will have to show that is bounded. So, what is that we think of that to suppose x n is Cauchy sequence, let us say suppose let us say let x n be a Cauchy sequence in x then we want to show that x n is bounded in.

Now, in view of all this discussion what it means that if you take any two points on the sequence x n and x m distance between take distance between x n and x m, x n and x m. Then that should be bounded above regardless of what n and m you take that should be bounded above because the diameter of this will be supremum of x n and am for n and m both vary that is what we should show. Let me just say what is what is, what is to be shown to show this there exists some real number alpha bigger than 0 such that some distance between suppose I take any two points. Here, distance between let us x i and x j this is less not equal to alpha for all i and j that is the meaning, that is the meaning.

Now, what is the obvious way of proceeding with this we have to make use of the fact that it is a Cauchy sequence, you have to make use of the fact that it is a Cauchy sequence. Let us, let us make this let us what is the what is the meaning of Cauchy sequence, if you are given any epsilon bigger than 0 there exists n 0 etcetera let us, let us start with that let us take any arbitrary epsilon.

(Refer Slide Time: 31:27)



So, let epsilon bigger than 0 then there exists n 0 in n such that n and m for all n and m bigger not equal to n 0 this implies distance between x m and x n is less than epsilon, what is that we want to show. If we take any i and j there exists some alpha such that if you take any i and j distance between x i and x j is less not equal to alpha what we already got here that if these i and j is both of them are bigger than or equal to n 0. Then this number is less than or equal to epsilon, now what is left on if those i and j which are less not equal to n 0 those i and j which are less.

But, how will such things will be there will be only finite in this might actually how (( )) it is they does a finite number, so consider this set consider distance between x i and x j where 1 less not equal to i less not equal or less than n 0 and 1 less not equal to j less than n 0 this is a finite set, this is a finite set. So, since it is a finite set we can take maximum of this we can easily find some number which is bigger not equal to all this numbers suppose I call that number beta this is a this is a finite set.

So, let us say suppose I call that number beta let beta be equal to maximum of distance between x i and x j 1 less not equal to i less than n 0 and 1 less not equal to j less than n 0. Now, what is the situation we know that if i and j both are bigger not equal to n 0 then distance between x i and x j is less than epsilon if i and j both are less not equal to n 0 or less than n 0, then the distance between x i and x j is less than beta less not equal to beta, so what is left out.

Student: Maximum of beta and epsilon, maximum of beta and epsilon.

Maximum of beta and epsilon why do u say that if this distance between x i and x j should be less not equal to maximum of beta and epsilon in all such cases

Student: Because if it is monotonically increasing sequence, then all terms would be less and that last supremum will be maximum in that case.

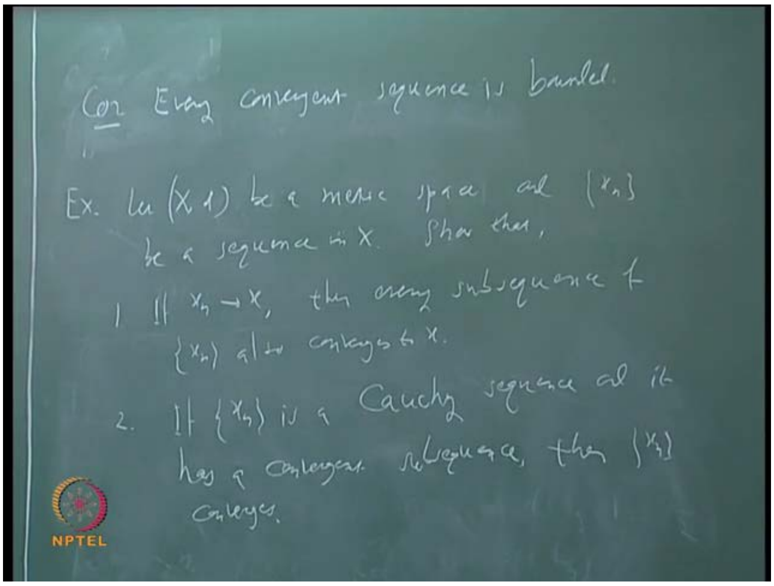
But, this is not monotonically increasing sequence, see again look at this what are the cases you have dispersed of already, if i and j both are bigger not equal to n 0 and if i and j both are less than n 0 what is the case left, equal to equal here, equal to this is equal here.

See can it not happen that one of them is less than n 0 and other is bigger than 0 it can happen that one of the for example that i is less than n 0, but j is bigger not equal to n 0 that is not covered by any of this, that is not covered by any of this. In fact, here for to take into account that is let me first take this equal to, here this equal to and, now let us say suppose one of them is less than x suppose and for 1 less not equal to i less not equal to n 0 and say j bigger not equal to n 0, j bigger not equal to n 0. Then what we can say about distance between x i and x j, what we can do is that use the again the usual primary inequality we can say the distance between x i and x j this is less not equal to distance between x i and x n 0 plus distance between x n 0 and x.

Now, this is less than beta or less not equal to beta and this is less than epsilon, so the whole thing is less than beta plus epsilon. Now, is it clear to you that whatever i and j you take distance between x i and x j will always be less than beta plus epsilon, if both are less not equal to n 0 if i and j both are less not equal to n 0 distance is less than beta less not equal to beta. If all of them, if all of them, if both of them are bigger not equal to n 0 the distance is less than epsilon and if 1 is less not equal to n 0 other is bigger not equal to 0 it is less than beta plus epsilon.

So, that means you just take alpha in equal to, take alpha is equal to beta plus epsilon and this will be true distance between x i and x j is less not equal to alpha for all i and j maximum of beta and epsilon will not work, maximum of beta and epsilon will not work. It is because it does not cover this case of course we know that the converse of this is false there can exist a bounded sequence which is not Cauchy. You have seen a example of that can real numbers the standard sequence 1 minus 1, 1 minus 1 it is a bounded sequence. But, it is not Cauchy, it is not Cauchy does it follow from here that every convergent sequence is bounded, we have shown that every convergent, every Cauchy and, now we show that every convergent sequence, every Cauchy sequence is bounded.

(Refer Slide Time: 38:13)

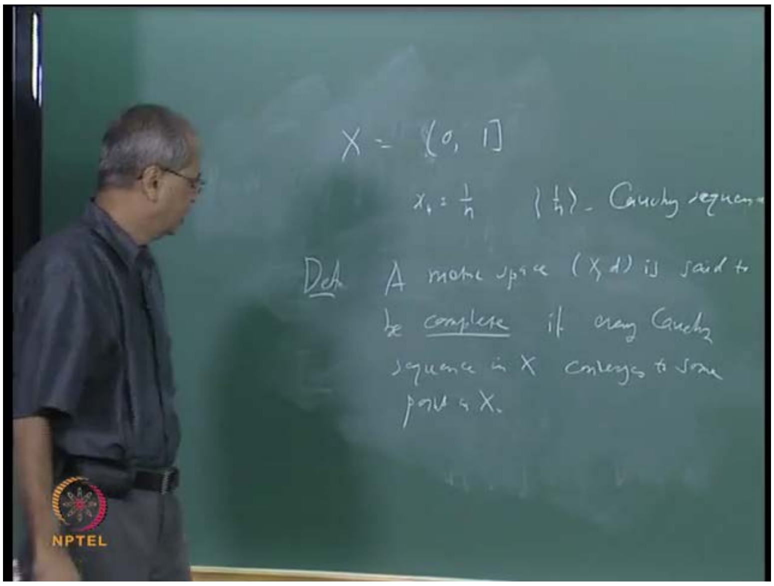


So, we can say as a corollary every convergent sequence is bounded convergent sequence is bonded in all this proofs as I said earlier you, now seen that we have imitated the proofs of the corresponding theorems about the sequences of real numbers. But, of course while imitating those proofs we have to take theorem not to use those properties which are peculiar to real numbers. For example, there is like upper bound lower bounds are there protonically increasing sequences or decreasing sequences those are the properties which are peculiar to real numbers.

Those are not properties shared by arbitrary metric spaces because an arbitrary metric space there may not be any order, now before proceeding further let me give you a couple of exercises here and that is about the subsequences show. Suppose let us say that let X be a metric space metric space and let us say that x n be a sequence in X, be a sequence in X then show that 1 if x and converges to x, if x n converges to x that is if x n is a convergent sequence converging to x.

Then every subsequence of x n also converges to this, every subsequence of x n also converges to x and the second property is that if x n is a Cauchy sequence and if it has a convergent sequence then the original sequence itself converges. It is if x n is a Cauchy sequence, Cauchy sequence and it has a convergent subsequence, convergent subsequence then x n itself is a convergent sequence then x n converges.

(Refer Slide Time: 41:41)

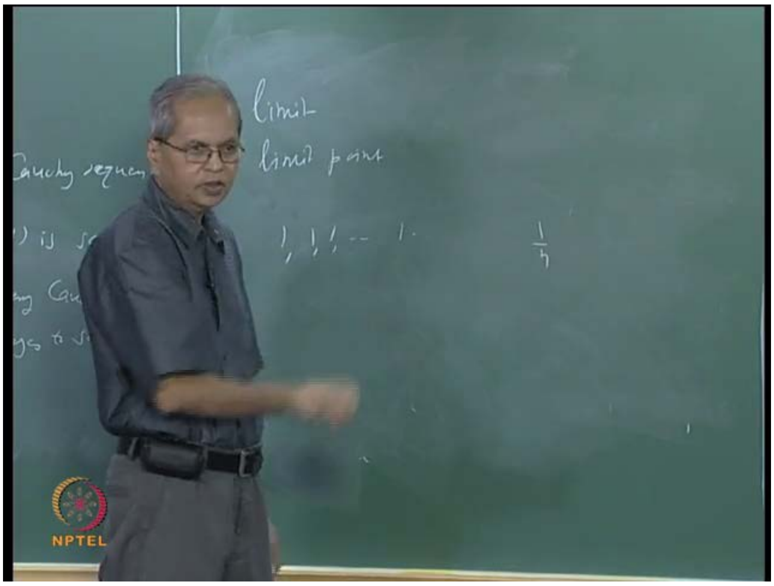


Now, let us again come back to this our discussion of relationship between convergent sequences and Cauchy sequences and Cauchy sequences we have seen that every convergent sequences is Cauchy. But, in arbitrary metric space the converse is false, so those metric spaces in which the converse is true that is in those metric spaces in which every Cauchy sequence is convergent. Those are a given special names, those metric spaces are called complete metric space and the completeness is a very important property of that a metric space may or may not have.

So, let us start with this definition a metric space X d is said to be complete if every Cauchy sequence in x, if every Cauchy sequence in X converges to some point in X. Of course, it should happen enough to simplify every Cauchy sequence in X converges means converges just to some point in x is understood. But, this last words are written just to make it clear because in this case a Cauchy sequences one by another Cauchy sequence in x, but it does not converge in X it converges to some point which is outside X that is not what is intended.

Here, it should converge to some point in X. then before proceeding further let me also make one comment about the couple of points which sometimes create confusion in the minds of the students, there are two concepts which we have seen while looking at the sequences.

(Refer Slide Time: 43:50)



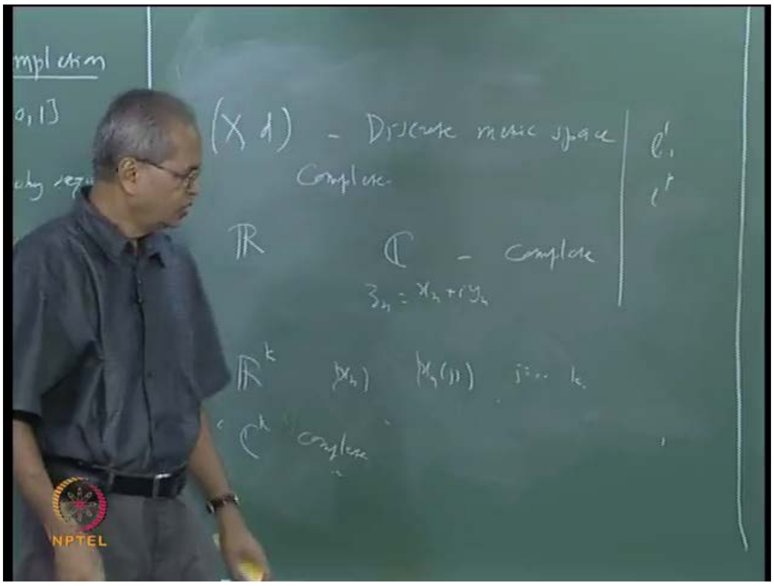
One is a limit of a sequence and other is limit point and quiet frequently it is found that there is a confusion about these two concepts, you should be very clear that these two are totally different concepts. We talk of limit of a sequence and we talk of limit point of a set, we right we talk of a limit of a sequence and we talk of limit point of a set of course we can talk of limit point of a set which forms the sequence that is we can talk of the limit point of the range of the sequence. But, that is not the same a sequence may or may not have a limit a sequence, may or may not have a limit point these two things are a quiet different, it may have a limit it may not have a limit point and vice versa.

We can take examples of all sorts for example let us think a sequence 1 minus 1, 1, 1 minus 1 etcetera this has neither limit and it is it does not have a limit point also because its range contains only two points, its range contains only two points 1 and minus 1. So, it will not have any limit for on the other hand suppose we take the constant sequence 1, 1, 1 etcetera then it has a limit, then it has a limit, but it does not have any limit point because what is the definition for limit point. If you take any open set containing 1 it should contain a point which is different from 1 that is not the case, here that is not the case here.

So, it is, it has a limit, but it does not have a limit point if you take the sequence 1 by n then this has both limit as whereas a limit 1 by n cannot be 0, 0 is a limit of 1 by n it is also a limit of 1 by n, it is also limit point of the set 1 by n. Now, coming back to this definition complete metric space as I said earlier completeness is a very important property that a metric space may or may not have and those metric spaces which are complete are quiet useful in practice.

We can say several things, the convergence of various sequences and those are quiet useful in various approximation schemes etcetera, those things we shall see little later. But, to begin with let us first see among the various examples, we have seen about metric spaces which ones are complete and which ones are not complete, first of all.

(Refer Slide Time: 46:15)



Let us start with a discrete metric space, let us let X be non empty set and d is a discrete metric space, now we have seen that what are, what are the Cauchy set sequences in discrete metric space, we have seen that a discrete metric space.

Student: Constant sequence.

If a sequence is Cauchy, it must become constant after some stage, so sequence Cauchy if and all if it is eventually constant which are the sequence always we convergent because if it is anyway it is becoming constant eventually. So, it will be a convergent sequence, so discrete metric is always a complete metric space, so this is complete then

R is an example of a complete metric space R is a, because that is, that is what we have proved that in R every Cauchy sequence converges.

This is an example of metric space which is not complete, this is an example of a metric space which is not complete of course, here I can make a small comment that is this is not complete because point 0 is not there. Suppose you add a point 0 to this and then the new space will be this 0, 1 then this will be a complete metric space, then this will be a complete metric space and usually this thing can be done that is given any metric space. If required, you can its space to the slightly bigger space to make it complete that is possible in case of all metric spaces and such a thing is called a completion of a metric space.

So, we shall not right now discuss this concept of how does one completed, one metric space etcetera, but that is, that is something a different topic all together that we might see after sometime. Now, coming back to the examples is it clear that R is complete the next set for example also let us, also look at C, in C the idea is simply this that is suppose you take Cauchy sequence suppose z n is equal to x n plus i times y n. Suppose this is a Cauchy sequence then what we show is that x n and y n are also Cauchy sequences, x n and y n are also Cauchy sequences and we use the same thing that is z n minus z m will be x n minus x m plus i times y n minus y m.

So, mod x n minus x m will be less not equal to mod z n minus z m and similarly for mod y n minus y m and, so x n and y n both are Cauchy sequences, so since they are Cauchy sequences of real numbers and below that R is complete. So, x n converges to x, y n converges to y then we can show that z n converges to x plus i y, so this will be a proof to show that every Cauchy sequence in C also converges. So, R and C both are again the examples of complete metric space and whatever proof I have said, here essentially the same proof that will work for R 2, R 2 or for that matter any of the R ks, so let us again recall.

Here, that this in suppose we take R k, we have seen that incase of R k if you take x n it leads to this k sequences x n j for each j equal to 1 to k and, so what we have seen here is that if this x n is a Cauchy sequence each of this x n j is also Cauchy sequence. It is a Cauchy sequence of real numbers and we already know that R is complete, so each of the sequence converges, so x n j converges to some let us say x j.

So, there are j such real numbers x 1, x 2 etcetera there, sorry there are cases such real numbers x 1, x k, so that will be an element of R k and then this x n will converge to that particular x, so each of this R k is complete similarly, each of this C k is also complete. Now, the next important types of spaces in which we need to discuss the completeness are the spaces like l 1 and l p, l infinity etcetera and the spaces of functions. As we have seen this in the beginning of the class this kind of requirement will not work immediately for the spaces like l 1, l p it will, it will need some extra work. But, it is true that all of this l p spaces are complete we shall just see how that can be proved and then we will talk about the function spaces, we shall, we shall do this in the next class.