Real Analysis
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Lecture - 02
Functions and Relations

Let me begin by repeating what I said in the last lecture namely that this real analysis is a very important and very basic course. And the concepts that you will learn in this course will be quite useful in many other courses in mathematics, like complex analysis and functional analysis and topology differential equations and several other courses. Not only in mathematics, but even outside also and also this is one of the courses, which gives importance to proofs, proofs of varies theorems will be discussed in the course.

And so the logical reasoning that you will learn while going giving those proofs will be quite useful in mathematics and also outside mathematics in life. So, we were discussing in the last class certain accepts about the functions, we were we were revising the some concepts of the functions. So, let me again briefly say a few things now to begin with let us say that…

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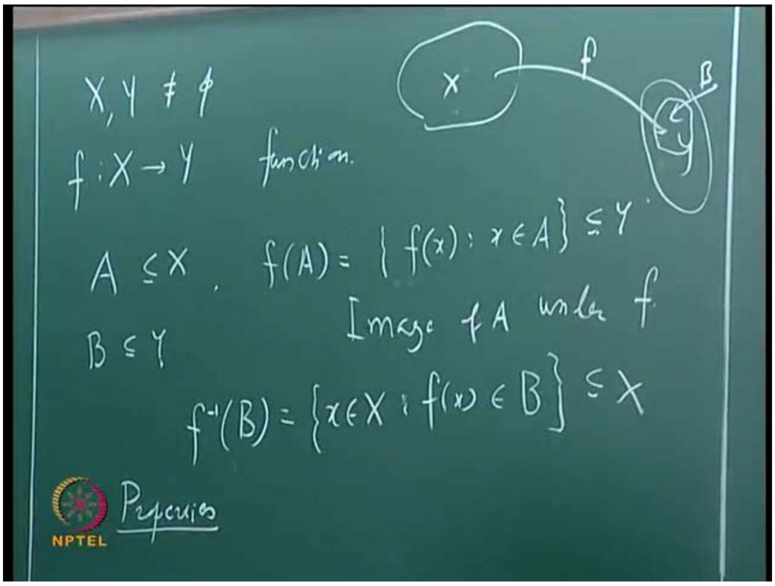

x and y are 2 non empty sets and suppose f from x to y is a function, then given any subset A of x given any sub sets A of x we can talk of what is meant by image of A that is f of A. This will be a subset of y and so this is nothing but set of all elements of the form f x for x in A. And alternate way of describing this will be set of all elements y in b such that there exists x in a such that f x is equal to y. Something like this we have seen.

Similarly, if we take a set say B in y we can talk of what is meant by inverse image that is let us just take suppose this the set x and let us say this is the set y, and you have this function f. Now, if we a take a set some set a here it is image is in y so this is this we will call image of f image of A under f. Similarly, if we take some set let us B here suppose this is B then we can talk of what is meant by its inverse image here it is inverse image and so that we shall denote by f inverse of B. So, what is f inverse of B it is the set of all those x in x such that f x is in b. So, set of all x in x such that f of x belongs to B.

So, this is a subset of x just does this is a subset of y, set operations like unions intersections etcetera inverse images behave in a better way, than this direct images. So, since these are elementary set theoretic properties I shall not go in to the proofs of those, I shall leave those to things to you as an as exercise. So, what are the properties that I am talking about they are this, this is the properties. Like for example, we want to ask some very elementary questions like suppose I take say instead of just taking one set A here, suppose I take say 2 sets a 1 and a 2 and then take a 1 union a 2 then the corresponding images will be f a 1 and f a 2.

Then how is f a 1 union a 2 related to f of a 1 union a 2 so that thus is a kind of question and as you have seen yesterday, then why just restrict two or three or any finite family we can talk of any family of sets. And so what we will do is that I shall let us say that A i is that suffix small i belonging to big I let us say this is a family of sub sets of x family of sub sets of x. So, where this big I is an indexing set something that we discussed yesterday. And similarly, let us say b suffix small j small j belonging to big j this is family of sub sets of y, then properties that I have in mind are the following let me just take summarized all this so let us say first properties.

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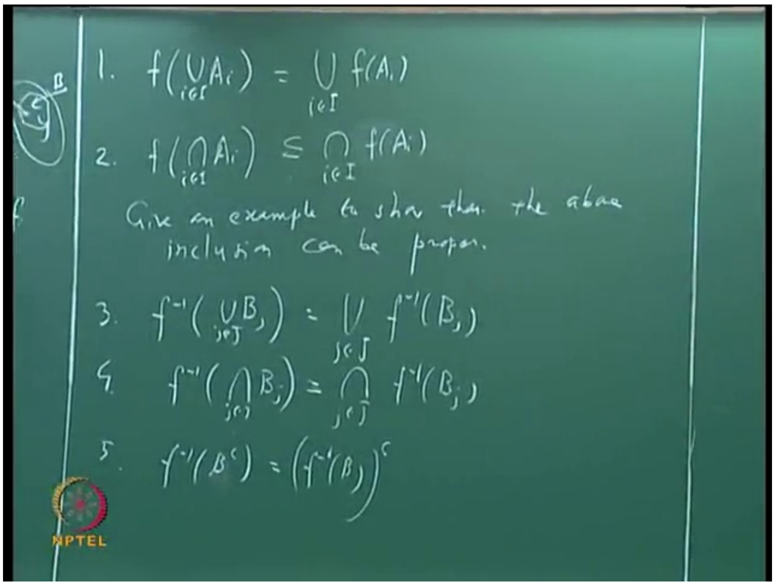

Image of the union f of union A and i, i belonging to i this is same as union f of A i i belonging to i this is an elementary set theory. So, I will give this to you as an exercise try to prove it on your own all right. When it comes to intersection we do not have such a good relationship, there is f of intersection A i i belonging to i you might expect that that is also equal to intersection of f A i, but that is not true so this is only we can only say that this is contained in intersection of f of A i intersection of f of A i i belonging to i right.

And this inclusion can be proper you can find examples of a function and this sets so, that f of intersection of this set is not same as intersection of the images. So, I will give that also to you as an exercise, so give an example give an example to show that the above inclusion can be proper, what does that mean? That is you find the family of just find the example of x and y and f and the family of sub sets of x such that this is properly contained in this, this properly contain this. And I will give you an hint though here I have said arbitrary family you can just get an example of 2 sub sets, then though I do not state here as an exercise, I will ask you one more thing to do.

Under certain conditions on f this becomes equality under certain conditions of f and we have discussed some of the extra properties that f may or may not here, like 1 1 on 2 etcetera. I will not say which exactly the condition you try to find out on your own that under an extra condition on f this becomes an equality, what is that condition? Try to try to find out on your own all right.

Now, coming back to what I said earlier that inverse images be here a better way in this respect and there you do not have these kind of problem. So, third property is that suppose I take this family, family of sub sets of five and take their inverse images then f inverse of union B j, j in j this is same as union f inverse B j small j belonging to big j and same thing is true about the intersections also. Inverse image of intersection of any family, this is same as intersection over small j belong there f inverse B j. So, compare this property four with property two and that is the meaning of saying that inverse images behave in a in a better way, all right.

And there is also one more property that inverse image of compliment is also same as the compliment of the inverse image, that is if inverse of A compliment. Remember A compliments means x minus A x minus A this is same as in. Sorry not A compliment it should be because inverse image for that it should be a sub set of y, A is our sub sets of s. So, let us say B compliment this is same as f inverse B is compliment of this whole thing. So, that is about the elementary properties of functions and how the sets under universe behaves to expect to direct images and the inverse images. Now, let me go to another important concept in the in this review of the elementary set theory namely relations.

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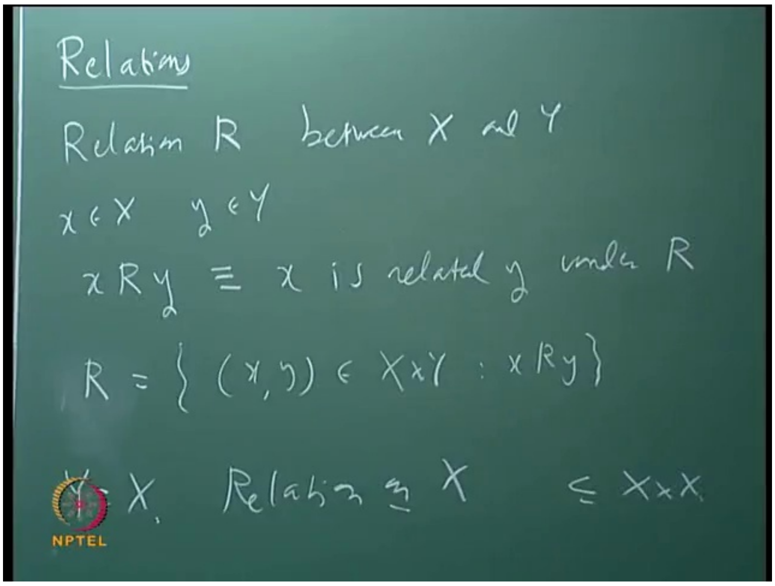

We can similarly, take two non empty sets x and y and talk about let us say a relation are between x and y, just this function is a rule which assigns every element in x and element in y. Similarly, relation is also a rule which assigns elements in x to elements in y, but there is a difference, what is the difference? In the functions every element has to have some image every element has to have some element, where as in the relation that is not the case some elements may have some image or some elements may not have image all right.

One more thing in case of function every element has a unique image it cannot happen that the same element goes to two different elements, but again that is also not required for the relations any number of elements can be attached to any number of elements. So, usual way of denoting that is that we say that we use this symbol let suppose small x is in x and small y is in y we write this x R x x R y x R y this is read as saying that x is related to y under the relation R. So, this we write this as saying that x is related to y under R.

And you have seen so many examples of relations in your several of the undergraduate courses. So, again we will not go into very detailed properties of relations and all that, but let us we need a few things in our portion we shall quickly revise those. Among various relations that we are going to consider one of the important class relations comes when x is same as y, that is relations on the same set. So, such a thing is called relation on x. Now, before going to that let me also mention one more thing with each such relation you can associate a sub set of x cross y. And usually that set is also denoted as by the same symbol R so what I can say is that suppose you collect set of all x y in this x cross y, such that x is the x R y that is x is related to y under the relation R.

Suppose you collect the all such pairs all such pairs is x y, which are related to each other under the relation R, then that gives us a sub set of x cross y that gives the subset of x cross y. Similarly, if you are given any subset of x cross y you can define the relation just pick up those pairs and say that those pairs are related under the relation, and that is why one can say that there is no difference between a relation and a subset of x cross y right every relation is a subset of x cross y. And that is why that is usually taken as definition of a relation a relation is a subset of x. So, subset of x cross y and that is why we denote this also as R we denote this also as R.

So, what are the ways of defining a relation is just give a subset of x cross y that defines the relation from x to y. So, when let me go to the next guess when y is equal to x we call this as the relation on x instead of saying relation between x, y and where relation between x and x which are relation on x and so this will be a subset of x cross x. Now, such a relation on a set x had some very interesting proper as in the sense, we can discuss some very interesting properties of such relations few relations may or may not have this property, but the property themselves are very quite interesting. So, let me just begin with those properties of course, you may have come across this properties earlier, but since we need this properties quite often, we will take a quick review of this.

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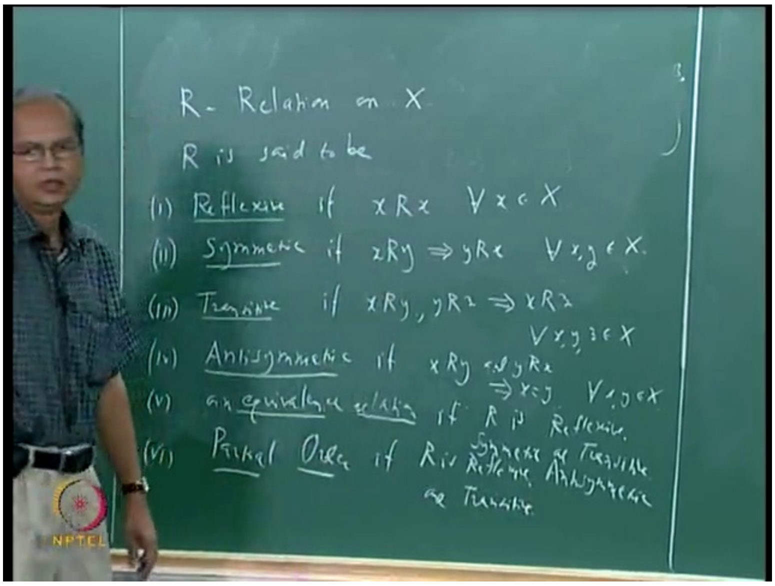

So, let us say that R is a relation on x then we shall first define what these properties are and what is meant by this, we will say that R is said to be the first property as you would offer of unsure reflexive, if what is said to be reflexive. Now, tell me what is meant by that?

Student: ((Refer Time: 15:35))

Yes, each element is related to itself and under the relation so R is said to be reflexive if we say x R x for every x in x, x R x for every x in x. Then second property is yeah symmetric if

Student: ((Refer Time: 16:05))

Repeat it x related to y and this happens for every x and y right if x is related to y, then y is related to x right neither things may be true A x may not may be related to y, but what we want is if x is related to y, then y must be related to x. So, if we should say that x R y implies right you understand this implies, implies means if then ok this is read as if x R y then y R x if x R y then y R x for x y in the x right. Then the third property which transitive R is said to be transitive if.

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Then fine so if x you take x related to y and then y is related to z, this should imply x related to z for whatever three elements you take x y z in x right. Of course, I am not saying that this x y and z must be distinct two or three of them coincide, but that becomes a trivial case right. Then one more thing let us call it what is called anti symmetric, what is this?

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So, anti symmetric is something that is exactly opposite to symmetric right exactly opposite to symmetric, see what does symmetric say that if x is related to y then y must be related x an anti symmetric means, this can never happen, but of course, if it is a if it is the resolution then x is related to itself. So, this the only way in which it can happen x related to x. So, what it means is that if x is related to y, and y is also related to x this means x is equal to y this should happen for every x y in x. This should happen for every x y in x.

Now, since when we are discussing these properties, let us also recall one more very important types of relation. So, let me just remind you that is R is said to be an equivalence relation equivalence relation, if I will say reflexive symmetric and transitive if R is R is reflexive I will simply write reflexive symmetric and transitive. Then one more thing R is said to be partial order, what is partial order? Instead of symmetric that is replaced by anti symmetric right, reflexive anti symmetric and transitive R is said to be reflexive, if R is reflexive anti symmetric and transitive. I am sure you have heard of these terms earlier.

Just to fix the concepts let us see a few examples on the relations and which we come across very often and which may or may not have this properties. Of course, for a relation you need a set you should have to start with that with some non empty set and then define some relation on that. So let us start with the way most familiar set.

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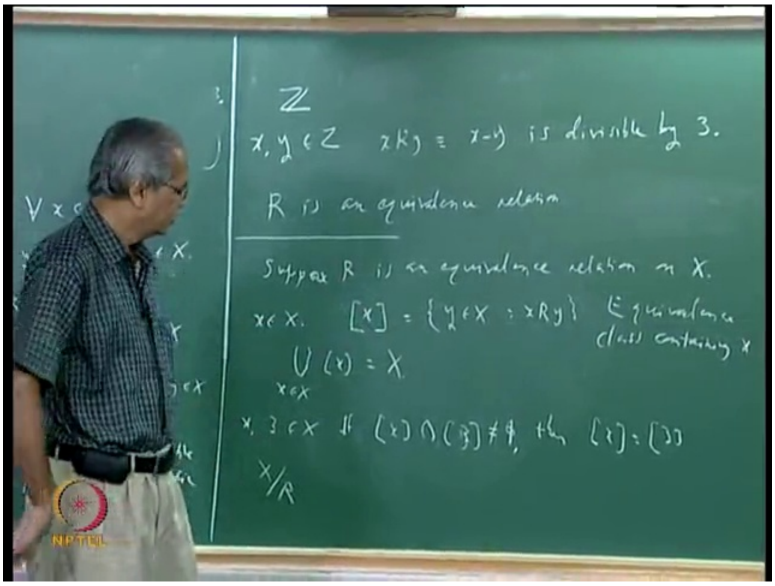

Let us say I take this set Z set of all integers, we have seen this and the relation is I will say that suppose you take 2 elements x y that is 2 integers x y and z I will say that x is related to y because the difference x minus y if that difference is divisible by 3. We say that x is related to y now this is a very well known relation it is called x is equivalent to y on modular three, you must have heard of these terminology we say that x is same as y modular three. But let us not use that term I will simply say this that is say x related to y this means x minus. This means x fine x minus y is divisible by 3 x minus y is divisible by 3.

Now, what are the properties that this has can we say that it is reflexive x, x minus x is 0 its obviously divisible by 3 is it is it symmetric you are x related to y means x minus y is divisible by 3 and y related to x means y minus x is so if x minus y is divisible by 3. Obviously y minus x is also divisible by 3 so it is symmetric also is it transitive?

Student: ((Refer Time: 23:04)) yes, yes, yes.

And what is the argument here suppose, you take three integers x y and z and say x minus y is divisible by 3, and y minus z is divisible by 3 then just add so x minus z is nothing but x minus y plus y minus x. So, you have some of two multiples of 3 right so it is a transitive order also. That means it is an equivalence relation it is an equivalence relation. So, you can say that this R is an equivalence relation, is it also anti symmetric?

Student: ((Refer Time: 24:06))

Why, why yes 3 related to 6 related to three but, they are different elements right so it is not an anti symmetric in fact a relation can cannot be simultaneously symmetric and anti symmetric, right? Thus this clear to you, now you would have heard of this property, but when equal relation that every equivalence relation on a set partition the given set into what are called equivalence classes. And what is this partition? That is if you take any 2 equivalence classes they are disjoint they are disjoint and their union is the whole of x that is what is called partition.

So, let us just recall that that fact suppose, suppose let us say forget about this example what I have written come back to this general. Suppose R is an equivalence relation on x relation on x and you take any element, let us say small x in x then we define what is meant by an equivalence class, containing x an equivalence class containing x, what and we denote usually notation for that is this x placed inside the square bracket. That is called equivalence class containing x.

So, and what is that it is defined as follows it is the set of all y in x which are related to x set of all set of all those elements, which are related to x under this relation R set of all y in x such that x R y x is related to y under the relation R. So, you take all the elements which are related a given element and that forms, what is called an equivalence class all. Now, every element is a every element is in some equivalence class for example, x related to itself. So, if you take union of all equivalence classes that has to be same as x union of all the equivalence classes. So, we can say this by the this is called equivalence class containing x.

So, what we can say is that suppose we take union of all these equivalence classes, union of all these equivalence classes and take this x in x then that union is same as x, but that is trivial, after all you are collecting all those elements which are given related to element and each right element is related to itself. So, you should take the union that is to be whole of x there is nothing great about it, what is important is that if you take 2 equivalence classes.

Suppose, you take 2 elements let us say you take 2 elements x and z let us say x and suppose I take 2 elements in z. And suppose I consider that 2 classes equivalence classes containing A x and equivalence class containing let us say z, the point is this either these 2 classes are the same are they disjoint right either these 2 classes are the same, or there are disjoint or what is the way of saying that it is it is basically same as saying, if you take their intersection if the intersection is non empty then the 2 classes must coincide. That is what we want to say is this if this intersection is non empty, then x is equal to z.

Now, let us just quickly recall the proof of this how does one prove this. Suppose intersection is non empty what does that mean there is some element common to both suppose, we call that element y right. That means, y belongs to the equivalence class containing x also and y belongs to equivalence class containing z also fine. So, what does that means x is related y and z is also related to y, but now use the fact that it is an equivalence relation.

So, if x is related y and first of all 0 related to y implies y is related to z and then x is related to y and y is related to z that will mean x is related to z, if x is related to z it means that this element z itself is in this equivalent class containing x. If that is the case anything related to z is also related to x. So, that should also be here that means all of this whole of this equivalence class containing z must be a subset of the equivalence class containing x its similarly other way.

So, these 2 equivalence classes must coincide right so coming back to what I said earlier, if you should take the set of all equivalence classes that leads to a family of subsets of x which is disjoint family and it its union is whole of x, that such a family is of any set such a family of subsets of any sets is called partition of that set. So, what we have proved is that an equivalence relation on a set x leads to a partition of that set into the family of subsets, and which an each subset in that family is an equivalence class under the relation R.

And usual this is a fare fairly standard notation for this that set of equivalence classes this is usual denoted by this x quotient R, usually that is quotient set that is if we take any equivalence class and consider set of all part equivalence classes ((Refer Time: 31:06)) that is called a quotient set. And this is something you come across very often in mathematics, see you sometimes you want to define something on this quotient set, then you define something using one of the elements. Then the usual problems that you run across is to prove that is the well defined thing.

For example, suppose I want to define something about this equivalence classes x and say as say it I define in the terms of x, but x is just 1 element in the equivalence class suppose you take some other element the definition may change. So, you will have to show at that time that it that does not happen, and that is what you come across many times in mathematics you will come across such situation such groups all right. Let us go back to this example, let us say suppose i consider equivalence class containing 0.

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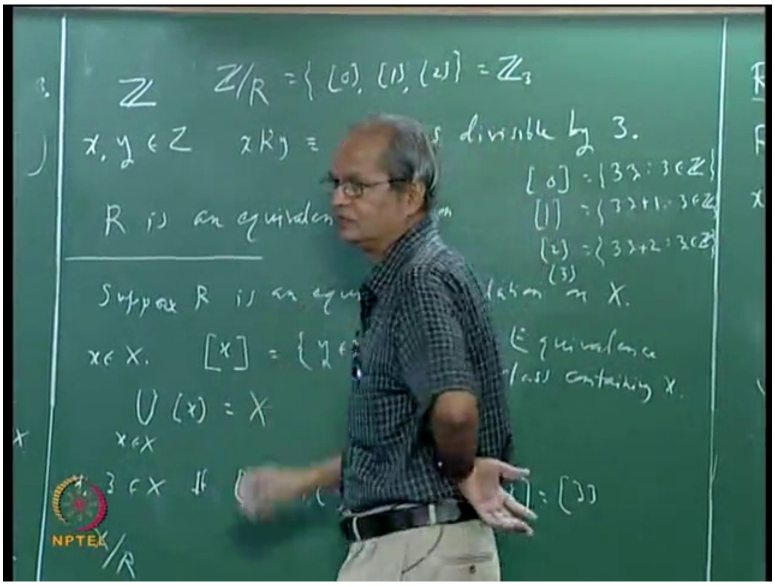

What are the elements in this all multiples of three that is 0 plus or minus 3 plus or minus. So, you can let us say all elements of form 3 z. suppose, I take equivalence class containing one it have 1, 4, 7 minus 2 etcetera. So, can we describe it like this it is 3 z plus 1 the all elements the form three z plus 1, where z is a integer. So, this all elements of the form 3 z plus 1 where small z is an integer all right and what about ((Refer Time: 33:01)) class containing 2. Again I mean 2, 5, 7 again we can say that this is nothing but set of all elements integer of the form 3 z plus 2, where small z is an integer.

What about equivalence class containing 3, that will be same as this so similarly, then if you are take seventh class containing 4, that will be same. So, there are no more equivalence classes right there are only three equivalence classes, there are only 3 equivalence classes. So, if I want to write this set x by R in this case let us say Z by R, this has only three elements equivalence class containing 0.

And you may have across this is usually divided by Z 3, Z suffix 3 have you come across this notation Z, usually it is popular in the group theory Z suffix 3. I will give you another example as an exercise, whatever we have done about this example similarly, check for this.

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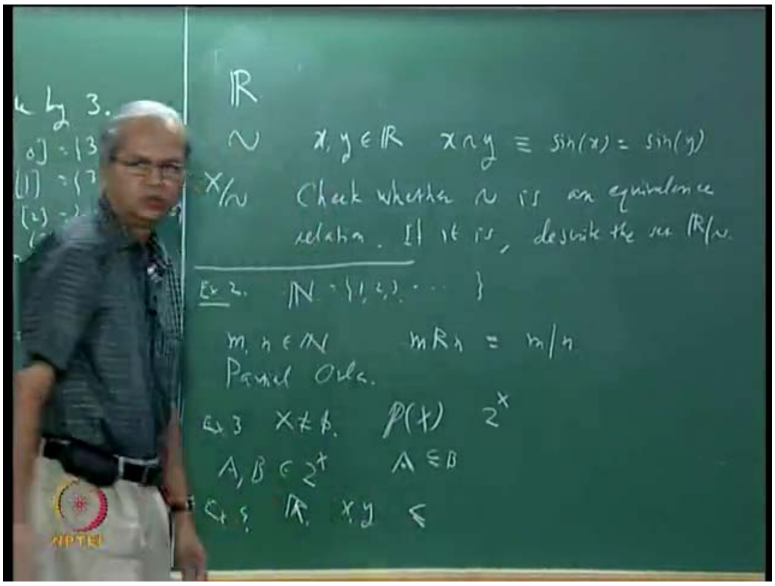

Let us take now the set R and I say that x instead of writing usual letter R see for the equivalence relation this is a usually common used letter. When the relation is a equivalence relation to denote a relation this is the usually used relation. For example, here also we will have something like, so quotient set will be denoted like this x quotient is equivalence to relation all right. So, what I will do is the, what I will define is as follows take 2 numbers x y in R and say that x is related to y this means, suppose this means sine x is equal to sine y.

Now, I will not enter into any further discussion in this example take this as an exercise is that check whether, check whether this is an equivalence relation this is an equivalence relation. And if it is described the set, describe the quotient set just as here we have given the exact description of the quotient set, there are exactly three elements if that is an equivalence relation similarly, you give the similar discussion of all equivalent classes under that relations. How many are there, whether need to identify non set etcetera.

See why we talk about this quotient set in many cases, this quotient set can be identified with some well known set. For example, as I have here this can be identified to the well known object Z suffix 3. Now, let us go to some other examples this time I will change the set and I will call this, I will say suppose you take this is let us say example one, just so that we can refer to it later need then let me take this as example two, I have take this set of all natural numbers and you know natural numbers 1, 2, 3 etcetera.

So, suppose we take say such two natural numbers m and n, I then should say what is meant by saying that n is related to N, I say that m is related to N, if m divides N understood. So, for example, 2 divides 4, so we say that 2 is related to 5, 2 does not divide 5. So, 2 is not related to 5. So, that way so we will say m is related to so I will say that this m R n. This means m divides n you familiar with this notation m divides n, m divides n means what there exist a natural number let us say k, so is that m is equal to m times k that is the meaning of m divides n.

Now, what about this, this is first let us say one by one it is symmetric sorry it is it is reflexive because given any divide itself, set is it symmetric obviously not because if and what and what is the example m, m divides n and that not divided into m fine, it is not symmetric is it transitive. So, m divide n and n divided some p that will imply that m divides right that easy to see, is it anti symmetric, right if m divides n, and n divides m that is if n is equal to some k times m at m is a also called equal to t times n, then the only way the which this acquire that is the number k must be 1. So, in this case m and n must be same this. So, this so this is the example of what we have called partial order.

We shall see some more example because these are the ones which will come across very often and did also very often, let us say this is example three in this case I take x as any non empty set, x as any non empty set and we take this set of all subsets of x. Now, this is something this is something we will need very often, so let me use this standard relation there are 2 notation followed for this is called power set of x, either you use this notation power set of x or another popular notation is this 2 power x.

There is reason for this notation little later, but this 2 notations are used fairly common in for denoting set of all subsets of a given sets x. So, I am going to define the relation not on x, but on this power set of x power set of x, so what are the elements here those are subsets of x. Suppose we take let us say 2 elements suppose I call it as a and b, I will use this notation 2 power x and we use the usual subset relation. So, we will say that A is related to be B if A is subset of B subset or equal to either it I do not say proper sub set it can be equal to so we say A is related to B, if A is contained in B, what about this each set is contained in itself A is a subset reflexive if A is contained B and B is contained in C, A is contained in C that is fine if A is contained in B and B is contained in A, right? It is certainly not symmetrical, so it is also an example for partial order.

I will take 1 more example four and this time I will take set as set of all real numbers R all right. And suppose you take 2 real numbers x and y I will take the usual less nor equal to order, we say that x is related to y if x is less nor equal to y, right? If x is less than or equal to y you can the again this also is reflexive and anti symmetric and also transitive. So, this is an also an example of a partial order. Now, let us proceed further if I say it on which a partial order is defined that is called a partially order set.

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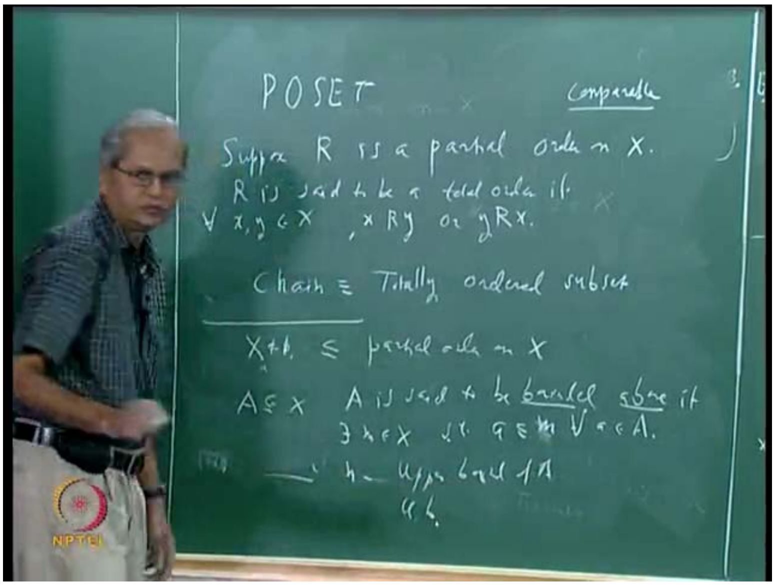

This is the standard terminology for that this is called poset that is partially ordered set. That means a set on which a partial order is defined, now suppose R is a partial order on x. Now, we want to define one more concept related to partial order, suppose you are given two elements x and y in x then for a general partial order or not only for the general partial order. For general relation we are not saying that any two such elements must be related to each other given any 2 arbitrary elements there may be no elements.

It is not necessary either x or y or y or x, but suppose it is a partial order suppose that is true that is true given elements x and y either x is related to y or y is related to x if that happens, then we say that that is the total order. That is the partial order is said to be total order if given any 2 elements, either x is related to y or y is related to x. So, let me just say that so we will say that R is said to be a total order, if for any given x and y in x is related to y or y is related to x. It cannot happen that x and y that is neither of them is related to other.

In general when two elements are given in a partially ordered set if they are related to each other, we say that those 2 elements are comparable given a partial order we say that 2 relation are comparable if they are related to each other. So, this is a term we say so in a general partially ordered set given any 2 elements they may or may not be comparable there will be no relationship between the 2. But in a totally ordered set any 2 elements must be comparable to each other.

Now, look at the examples of the partial order sets which we have discussed here and see what is true, which of them are total orders look at this first example here, that is the last example that is if you are given 2 real numbers, then either x must be less nor equal to y or y must be less nor equal to x. So, that is an example of a the total order, but suppose you look at the first example, suppose we take let us say elements 2 and 5. Suppose we take the elements 2 and 5 then neither 2 divides 5 nor 5 divides 2 those these 2 are not comparable at all these 2 or not comparable at all. So, that is not a total order.

Similarly, these example you can find 2 subsets, 2 sets a and b 2 sub sets A and B of x such that neither is contained in the other. So, total order is something much more extra than the usual partial order, now it can happen that in a partially order set you can find a subset, which is totally ordered that is always possible right. For example, in this itself is not an totally ordered this is not a totally ordered set, but suppose I collect this set say 2, 4 let us say 2 4 8 16 etcetera we can just stop here, 2 4 8 16.

Though the original set is not an totally ordered set this subset is a totally ordered, set right it can happen in case of the partially ordered set you might be able to find subset which are totally ordered. So, such subsets are called totally ordered sets or they are also there is also very popular name for it, they are also called chains, chain is a totally ordered subset. Let me let me again repeat a given partial order may or may not be total order, but that partially ordered set may in fact usually it will content subsets which is totally orders. So, such subsets are called chains.

Now, it is fairly common because popularity of this particular order usually totally order or partial order they are divided by this symbol. Now, we shall use some again some standard terminology, we shall say that let us say that I will say that this let us say that this is set X let X be a set and I will be denote this as a partial order on X, X be some non empty set and this is a partial order on X. Of course I can used you should not confuse this though I am using same symbol it is not that less nor equal to 2, it can be any partial order.

Then suppose if we take a subset let us say A is a subset X, we say that this subset A is bounded above with through this partial order, if there exist some element such that every element in A is less nor equal to that element. We will say that a is set to be bounded above said to be bounded above. Let me give a sub name to that if there exist some element, let us say m in X, such that A is less than nor equals to X for every small a in A. And this is term that we have defined and this elements m is called an upper bound of A, it should be less nor equal to m, thank you it should less nor equal to m.

And m is called upper bound of upper bound of upper bound a standard form uses u, b for upper bound. Then we shall continue with the properties of this upper bounds and similarly, lower bound etcetera, and what happens to this bounded sets in a partially ordered set etcetera in the next class. We will stop with this.