Real Analysis
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Lecture - 18
Closure Points, limit Points and Isolated Points

We were in the midst of the theorem characterizing the open sets in the real entry, we want to prove that every open set in the real line can be express as union of countable family of pair wise disjoint open interval.

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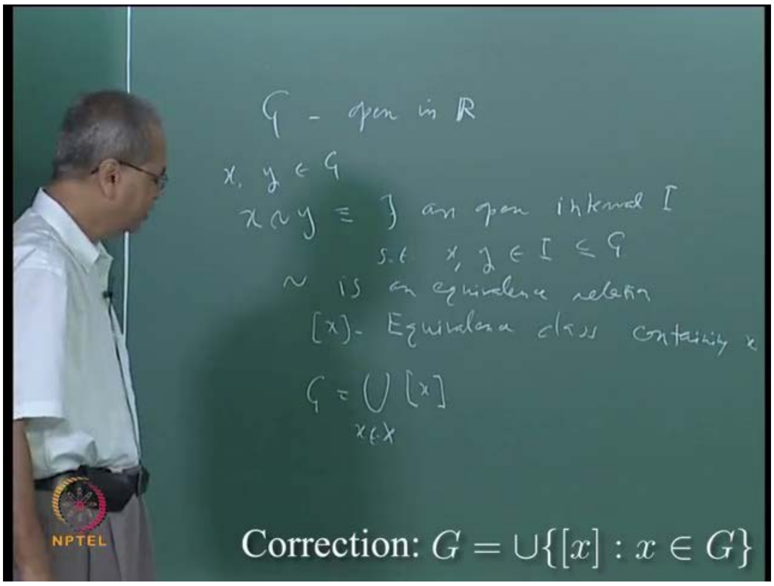

So, let us and start and we had also started proof with we have taken some step, so let us begin with that suppose G is open in R then we have defined one relation on point in G. So, suppose we, suppose we take x y in G then we had say let x is related to y this means there exists n open interval, I open interval such that these two point belong to that open interval x and y lie in that interval and this I is containing G. We have already seen that this is defines and equivalence relation this define equivalence relation on the set on the set of all point in G. We already know that every equivalence relation brings about what is called partition of G that is union g can be union all equivalence class element in G.

So, suppose we denote by this symbol equivalence class containing x, so this is an equivalence relation this is an equivalence relation then, so this x plus in the square bracket that denotes equivalence class canting x then what we already know. Then that G is nothing but union of all this equivalence class and what we also know is that if we take any two equivalence class in this family, then they are either disjoint or they simply coincide.

So, this is disjoint union, so what remains to be shown is following we should show each of the this is an open interval each of the this is open interval and this and it is countable family, its countable family. Let us first show that is an interval or same proof more less interval know how does one show that something is an interval, we have seen that yesterday. That is interval is nothing but you assume that there are two points in the set then all the points lying in between those two should also lie in the set.

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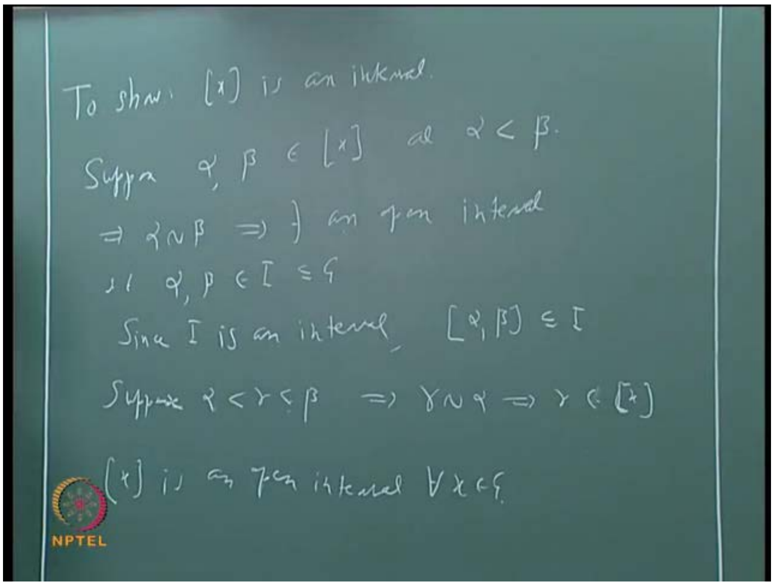

So, first show that this equivalence class is an interval let us say this is what we want to show first this is an interval, now suppose alpha beta belong to this equivalence class, suppose alpha beta belong to this equivalence class and alpha is strictly less than beta. Then what we need to show that any point lying between in alpha and beta is also in that same equivalence class, but alpha and beta are in the same equivalence class.

It means that alpha and beta related by this relation that is clear because both are related to x it is an equivalent relation, so particular thing implies that alpha is related to beta and that means, that means there exists an open interval. That means there exist an open interval which contains both of these points and which is containing this G, so this means there exists an open interval such that alpha and beta both belong to I and I is containing

G. So, does it immediately say that if alpha and beta belong to and this is an interval and, so all the points lying between alpha and beta also line?

So, this since I is an interval we can say that, since I is an interval we know that this close interval alpha to beta that is containing I we can say what we, what is this, whole thing is inside this equivalence class containing x. Now, to do that what we need to show if we take any point in this close interval suppose we take any gamma lying between alpha and beta, so let us consider suppose alpha less then gamma less then beta, of course we can take less than or equal to also. But, as far as alpha and beta are concern they are already in that, but that gamma is ready to alpha you can take same interval, I can take because that is however equivalence relation because it this implies gamma is related to alpha, alpha is in this equivalence class.

So, that means gamma is x also this means gamma is related x also which is same as seeing that gamma belongs to the equivalence class x that shows that, that show this an interval, that shows this an interval. Now, how does it show it is an open interval well you can say that if take any point inside this in fact the same proof for example f any alpha in this we already know that there exist an open interval the open interval which containing G, so which is contain G. So, it will also show this is totally into interval, but it is an open interval, so each of this let me just write this conclusion, so each of this equivalence class is an open interval for x in G.

So, we have prove till now does each of this is open interval we already know that its G can be written as this joint union of open interval family of open interval. Now, what is left we have shown that we have, we have to show that this is countable family we have understood this is countable family. Now, this is where we use property of real number to say that every open interval will contain as its one rational point, every open interval will contain as its one rational point.

Suppose we take disjoint interval then they have contain different rational number cannot be the same, so what we can do we can say each of the intervals you pick up one rational number, from each of this interval you pick up one rational number. So, that will be some sub set of rational number so that is to be countable set because we have already shown set of all rational number is countable. Hence, the cardinality of that should be same as cardinality of this, of this we have pick up exactly one number from, one rational number from each of this interval.

So, this has to be a countable set this is as to be countable family, countable family and it also be clear this or argument will not work some other matrix base. It works in this metric space because you have this set of all rational numbers which has this property then that every interval contain, every open interval contains, every open interval will contain at list one rational number. That is way we can, that is way from whatever family is there, we can show that it is a countable family that completes the proof of that, proof of that theorem. Then we had more or less close to finishing discussion on open set, but there are only one or two small points which let me discuss.

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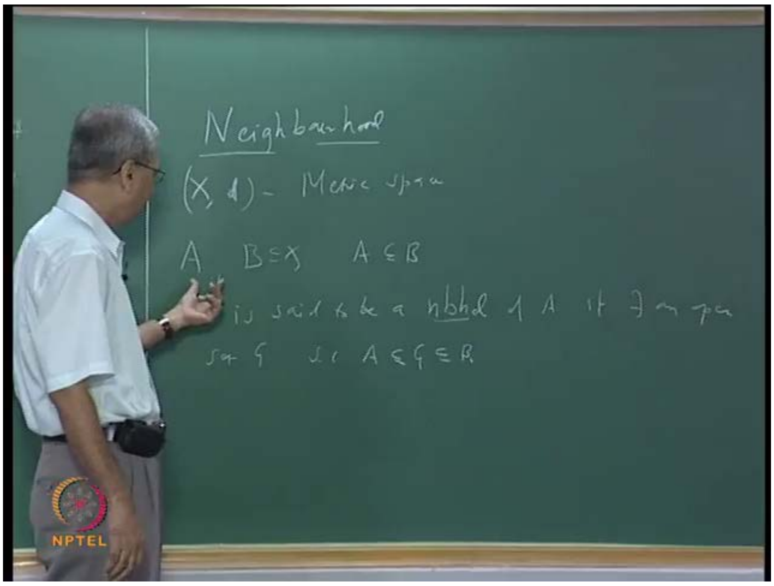

See, there is one terminology which you come across which is called neighborhood it can be neighborhood of a set or a neighborhood of point. Suppose we take say, let us say X B is matrix space and suppose we take two set A and B such that A is contained in B these both are sub set of X and A contained in B. Then we want to say what is meant by saying the B is neighborhood of A, we want to say what is say B is neighborhood of A there is slightly non uniform practice in this terminology.

First let me, let me say what is how is it define in B is set to be neighborhood of A, if B contains an open set which contains A, So, let us, let us take that definition first B is said to be neighborhood and this is standard flat form for neighborhood N B H D, this is standard flat form. We said to be neighborhood how of A if B should be contain open set and that open set should contain A, so if there exists an open set G such that A is contain G and G is contain B off course we have not said that B itself should be open. If in particular B is open then we that is fine you can take that set B as G itself, so in ruding the definition as follows B is called neighborhood of A if B is open and if B contains A it is called neighborhood of A.

So, what Seimans would call open neighborhood, what Seimans would call open neighborhood, so that is slight difference in the it should be clear to you that if B is neighborhood of A it also mean that A is containing if entire of B. It means A is containing in and that will take care both things and in particular A can be just single point, if A is single terms set we say that is neighborhood of point. Neighborhood of point you can see that since G has to open set this is just single point, then neighborhood of point will always contain open ball centered at that point, then there is one more thing is that we want discuss and that following again suppose we start with.

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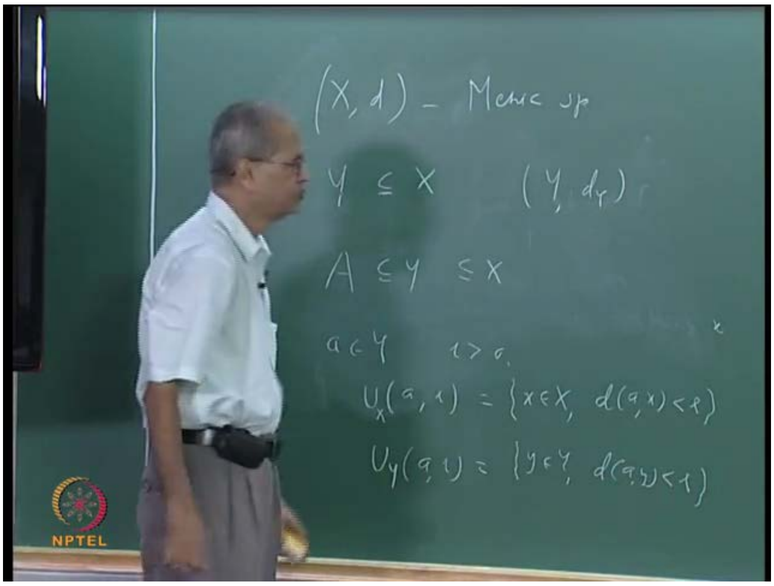

Metric space X d and suppose we take sub set y of X suppose sub set y of X then we, then we also know that is y is itself is also matrix given by the induce matrix which we have, which we can denote by d suffix y. Now, our question as follow suppose I take sub set which is in y suppose A is sub set of y and anyway y is subset of X then A, whether for example we want ask whether A is open or not, whether A is open or not. But, this will have two meaning an open regarded subset X or A is subset of A in general this two things are different for example y itself may not be an open set.

But, y regarded in the subset itself always is open because we have see that full matrix always an open set. So, if you are taking y itself as metric space then in that metric space y is, then in that metric space y is open, so why y is open in y, y is may or may not be open in x. So, let us ask question, in general suppose we take subset f y what is relationship between saying that A is open in y and A is or in general we can say that A regarded as y subset of A of y is open in y.

To do that, best thing to do is look at the relationship between the two different point let us say A is any point y, so let us take r bigger than 0, r bigger than 0 then we want talk of say open in y ball with centre at A and there radius r open. Let us say open boll with centre a and radius and radius r, but now you will see that this will two different meaning whether you regard that an open ball in y or open ball in x right because what will be the meaning of saying that this open ball regarded as open ball n x. By definition this will be set of all x n x and with the property that d a x is strictly less then r, so distinguish between these two thinks suppose I denote this by U suffix x just denoted we are now topic of open ball in x.

So, similarly suppose U suffix y of a r, what will be that, now you cannot take all points in x because we are only in this metric space y, you are only in this metric space y. So, you take only those points which are lie which are lie in y among all this out of all this point you take only those points which are, which lie in y. So, that is the set of all y in y such that d a y is less then r, now how are these two set related.

Student: U y is contained in U x.

Can you say more than that, U y is contained is fine, can you give even better description that is, this is nothing but that intersection of this with y what we can say is that.

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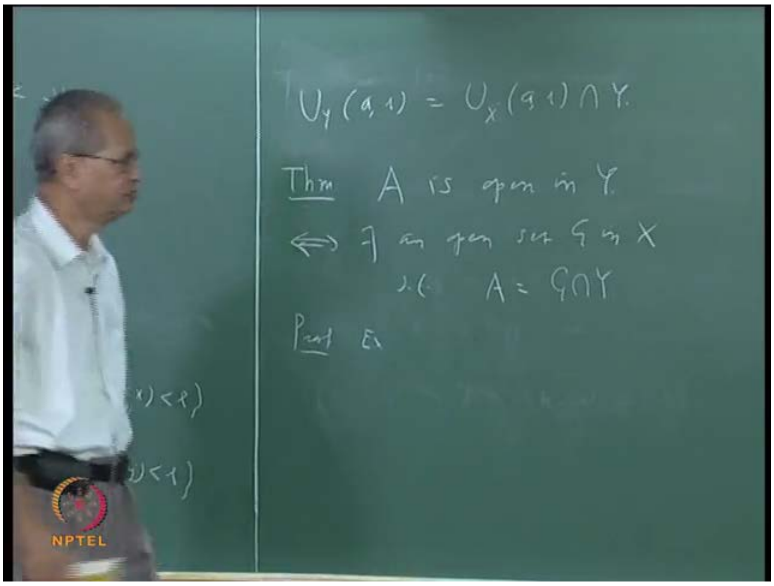

This open ball U suffix y, a r is nothing but U suffix x, a r intersection y, now using this you can show the following. Let me write this as theorem A is open in y we are having this set up A is contained in y, y is contained in x, A is open in y if and only if there exist an open set G in x, there exist an open set G in x such that A is equal to G intersection y. So, this answers the question completely when A is open in y, if you can find some open set G in X and if can write A as y intersection of G then A is open in y.

Proof is fairly simple and I shall give into as next exercise and we are not discussing it there, in fact this part is trivet if you take G is open say x, G and y both are open, so the intersection open. So, that is similarly the other way if A is open in y you have to find open set G such that this is happen, but for that you use this, for that you this fact, now let us also go back to certain types points which we have discussed till now let us say.

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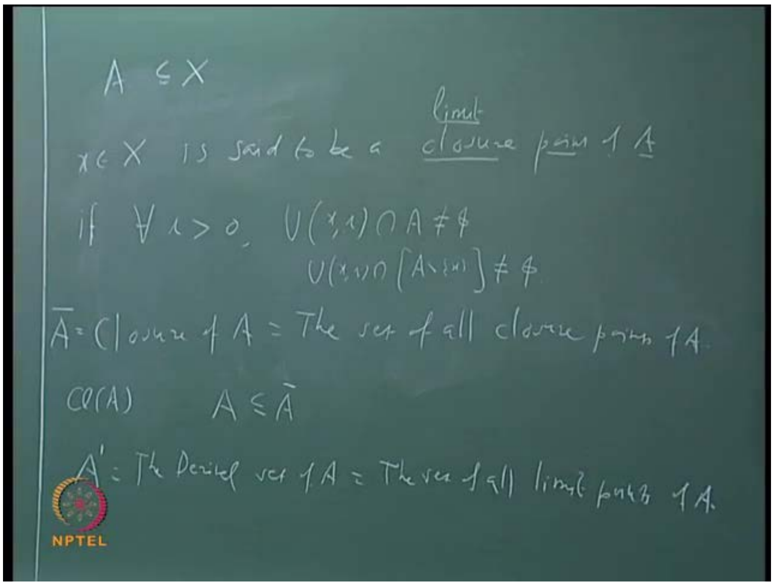

Given metric space X and subset A of X, we have discussed what is meant by interior point we have also discussed what is meant by exterior point and boundary point. Let us, now take five more type of points, so suppose x point, x may or may not be belong to A, now what I want to define what is called a closer point x belong to X is said to be closer point of a, closer point of a if what happen, what happens is the following. If you take any open ball with center at X its intersection with A should non empty, its intersection with A should be non empty, so that means any open ball containing x should have a non empty intersection with A.

So, if for every r bigger than 0 U x r intersection A is non empty that means every open ball with centre at X contains a point from A if that happens it is called A closure point, obviously if the point X is in, it is in A it is always be A closure point, it will be always be A closure point. So, set of all such closure points is called A closure of x, so we shall denote that by A, this is, this is called closure of A, that is, this is set of all closure points of A.

The standard notation for that is A closure, that is A with some bar, A closure or some books also write this C l of A, A closure and what you seen just now is that every point of A is A closure of A, closure point of A. In other, in other words it is this A is contained in A closure, always A is contained in a closure always, now there is another concept which is very closely related to this called closure point, it is called limit point.

Let me just write, here because this is a small change the limit point for limit point what we want is that this intersection U x are intersection just non empty non sufficient it should contain some point which is different from this point x.

Then this is called as limit point that is if you, if every neighborhood of, if every open set or every open ball containing X contains a point from a different from the point x that is called limit point. So, we can say that for limit point if U x r intersection let me write this A minus, this singleton x, A minus singleton x that is remove the, if X does not belong to A, A minus singleton x will be just A. So, U x r intersection this should be non empty, so if x is a closure point of A, but if x is not in A then it has to be a limit point. But, only thing is that if x is inside A then it can be A, it can be closure point without being limit point, without being a limit point.

Just as we said A closure is the set of all closure points of A, similarly if the set of all limit points that is also as a special time which is called a derived set of A. It is known as, it is A prime, A prime the derived set of A, called the derived set of A and this is nothing but the set of all limit points of A and from whatever we were discussed just now we already that A is contained in A closure.

What about this can be also said that this is a prime is also contained in A closure, just now because every limit point is obviously A closure point only converse may not true. If a point is in suppose a point is not a limit point, but it is, if it is still in the closure then it must be point A, then it must point of A because we have seen that if a point is a closure point then it has to be limit point in other words if you summarize this discussion means is this.

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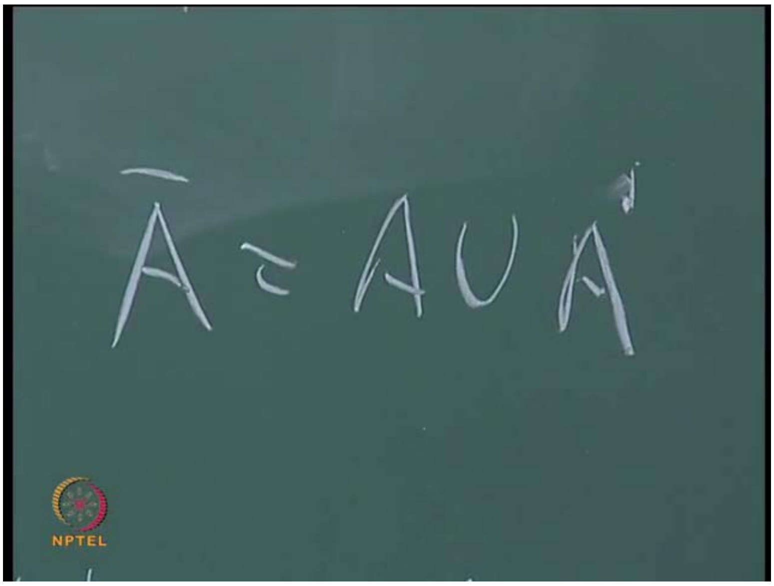

That is A closure is the nothing but A union in pride that is a union that derived set A along will its limit points that forms what is called the closure of A. Now, let us also see what is meant by saying that something is not A closure point or something is not a limit point, let us just take the definition once again. See suppose X is not A closure point what does it mean that is not true for every r saying is not true, that means what, that means there exist some r, there exist some r such that this intersection is empty, so let us, let us say what it means.

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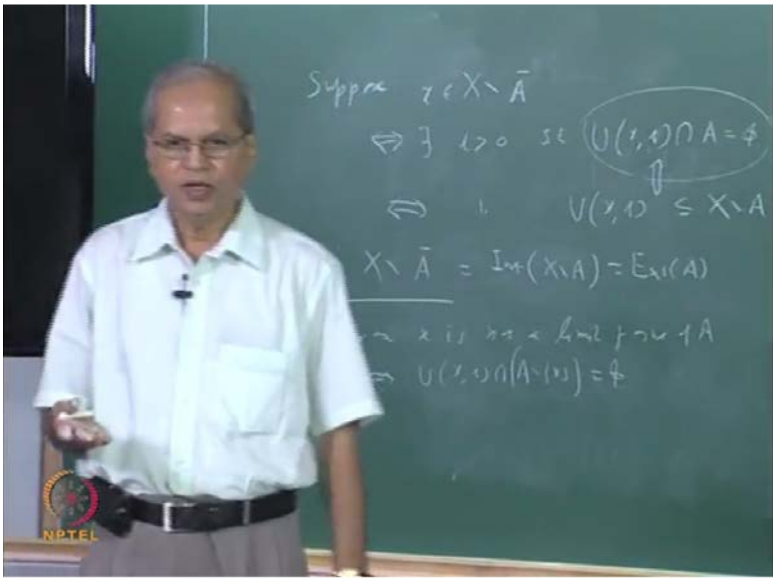

So, suppose x is not A closure point which means what suppose X belongs to X minus A closure, so what does it mean, it mean that there exist some r, it means that there exist r bigger than 0 such that open ball with centre at x and radius r intersection A is empty. Now, this thing open ball with centre at x and radius r intersection A is empty, is it same as saying that U x r is contained in X minus A, its intersection with A means it is contendent is complementary, so these two things are equivalent. So, in fact, here also I can say these two things are equivalent X is not in A closure is exactly same as saying that there exist some r such that U x r intersection A is empty.

But, what is the meaning of this that there exist some r such that open ball of centre r inside X minus A, it is interior point of X minus A, it is interior point of X minus A or which is same as what we have called exterior point of A. So, if something is not in A closure of A there it has to be an exterior point of A, so in other words suppose I want to write this argument involve this that is X minus A closure, A closure compliment it is same as interior of X minus A or which is called exterior.

Now, let us also say similar thing about what is meant by say that something is not a limit, so suppose X is not a limit point of this what we have done X is not in the closure means it is in the exterior. So, it means compliment of A closure is same as the exterior next is suppose X is not a limit point of A it means what again only thing intersection of x with r, sorry you x r limit X intersection A this is minus. That is empty, that is U x r intersection A this contains no point of A other than X itself, other than X itself, so this is same as sine that U x r intersection A minus single term X is empty.

That means what that is open ball with a centriod in a radius r if at all it has to contain any points from a then that point must be X itself. That can happen if X is itself a point of A, that can happen if X is point itself of A, in other words suppose X is a point of A suppose X lies in A and X is not a limit point, X lies in A, X is not a limit point. That means what should happen that should exist some neighborhood that is exist some open ball such that containing X that open ball contains no other points of A only that point itself if such a thing happens such a point is called an isolated point. So, such an x, suppose if x belongs to and x is not a point of A then x is called isolated point of if X.

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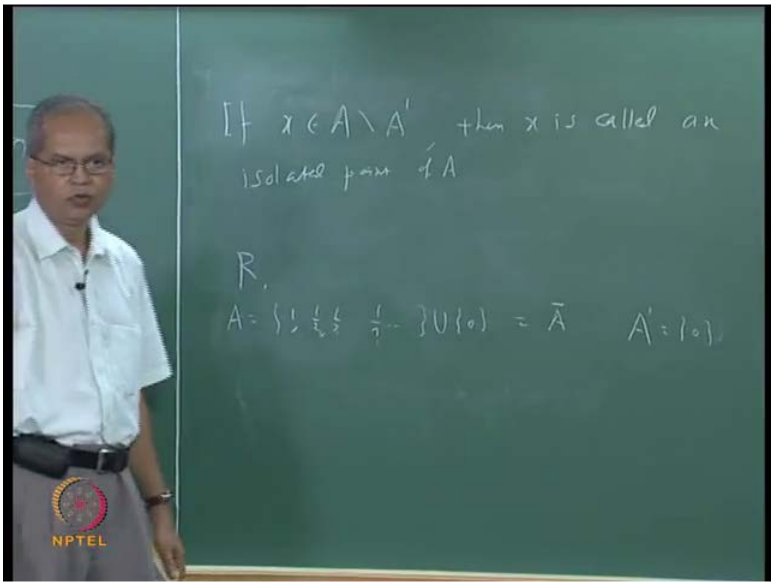

That is let us say if x belongs to A and if it is not a limit that is if x belongs to A minus A prime then x is called an isolated point. So, what are differences we have seen closure point limit point and isolated point if something is not a closure point it is a x to 0 point and something is not a limit point and if it is a point of A. Then it is an isolated point I think we better take some example let us from R, so suppose I take this matrix usual space, suppose A is this space say 1, 1 by 2, 1 by 3 etcetera in general 1 by n and suppose I will add 0 to this.

Now, each of these points are in obviously, each of these points are in the closure each of these points, are in closure is there any other point outside this is, this closure 0 is in the closure and suppose I take any other point than this. Then for example let us say, let us say take a point 2, point 2 is none of these but it is also in the closure I can take if you take let us say R as half then opening trouble with the centriod 2, radius of its intersection of A B will be empty.

So, in fact this A is same as A closure no point A is in the closure of A, now out of all these points in etcetera which are limit points, this is only point because 0 is a point which has A. If you take any interval containing 0 it will contain at least one of these in fact it will, but at least one of these points other than 0 itself, 0 is a limit point, so if A derived set, here a prime is single term 0 and all these other points.

But, 1 by 2 etcetera all of these are isolated points that I should any, for example if you take say 1 by 4 that is you can easily find the open R such that interval with centriod 1 by 4 and r radius r will not contain any parts. Here, it is very easy to construct and you can do it for any n, so all these are isolated points, there is one more thing we should relate concept of closure and that is the following.

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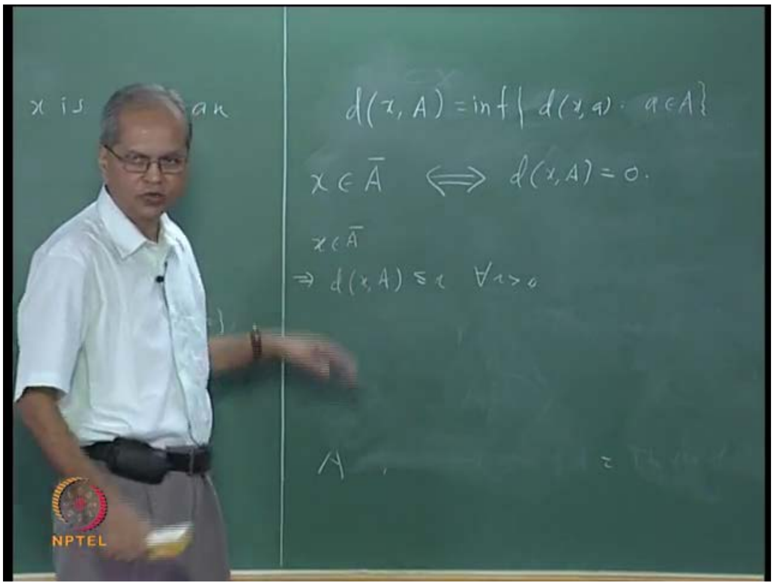

We have already defined what is distance between x and A, we already defined what is meant is distance between x and A, so what I want to say next is x belongs to A closure, x belongs to A closure if and only distance between x and A is equal to 0. Let us, let us see how this can be proved suppose x belongs to A closure, suppose x belongs to A closure that what we have known this for every R, for every R open centriod x in radius r intersection A is not empty.

Which means, that distance between x and A is less than or equal to r, suppose x belongs to closure that implies distance between x and A is less than or equal to r for every r bigger than 0, for every r bigger than 0. But, if it less than or equal to r every r bigger than 0 and has to be anyway this is a non zero number, anyway non negative number. So, this will A if it is less than or equal to every positive number it has to be 0, so x belongs to A closure distance A is 0 on the other hand suppose distances between x and A is 0 then look at it how did we define. It is infimum of distance between x and A where interval in A and A from the infimum you will say that any strictly positive R then that is not a lower bound because 0 is infimum.

So, there should exist some A such that distance between x and A is less than R which is same as saying that intersection of U x r with A that non empty. So, try to write this on proof on our own, this is the good character closure that is to decide whether a point belongs to closure or not it equivalent to say that distance from the set A is 0. Now, having since such many properties of closure, now let us come to the important definition.

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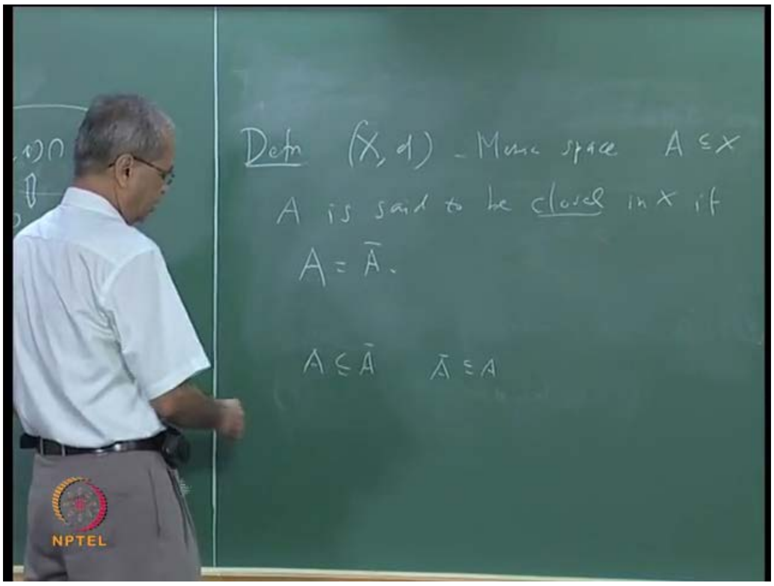

Namely what is meant by a closed set, so let us again say x d in the metric space and A is contendent in x, A is said to be closed in x if A coincide with A closure, if A coincide with closure if A it is A closure, of course we know that A is always contending in A closure. That is this is always true, A is always contending in A closure, so addition we require is set or these sets are the same which means what very point in that what is that requirement is A closure is contending.

This is what that means every closure, every closure point should also be in A, is it same as A contains all its limit points because it is A closure if it is not in A, A has to be limit point. So, such a point should also be there, so saying that A is a case is same as that A contains all exposure points that is same as saying that A contains all it limit points.

Student: We can say that all the boundary points are limit points, all the boundary points are limit points because if we include the boundary in open set, it is a closed set.

We can say that, so that is, that is how the close sets are defined and compare the definition with the open sets in a open sets what we have to say, what is meant by an interior. We had said that the set is open if A coincides with its integer at in case of interior we know that interior A is always contended in A, it is the other way interior of A is always containing A and when these two are equal we say that set is open.

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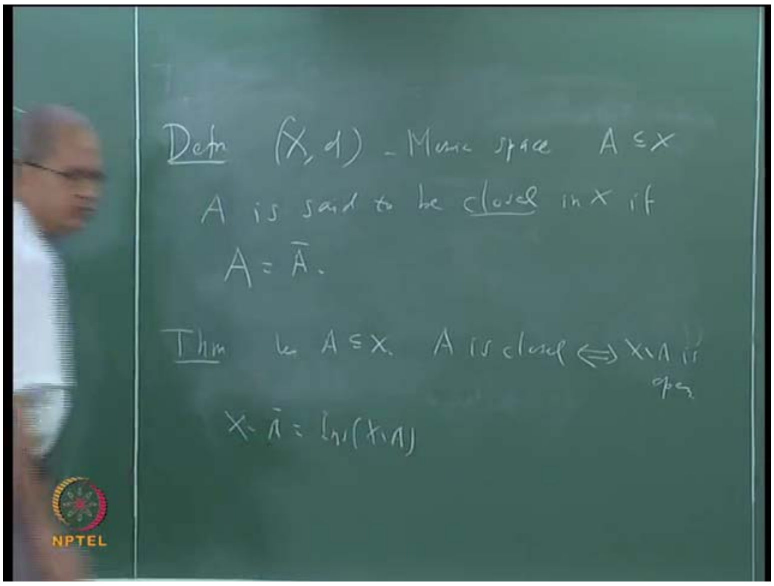

So, which means every point should be arranged in the interior point that definition of closure is the other way A is always contending A that A is close if every point in the closure is also point of A. As somebody observed this term, it also means that A should contain its boundary because every point in the boundary is also in the closure, every point is in the closure. Now, let us proceed further we can, now look at some examples of pro sets and some properties on the closed sets. But, what we should do is that instead of going to get we should prove one theorem which will exactly give you a several big class of examples of close sets and that determines the following.

Let A be a sub set of x then A is closed if and only it is compliment is open, A is close if and only it is compliment is open, now I would say that the proof is more or less, here already. Here, we have proved that x minus A closure is interior of x minus A of course you have to observe, here that is it, is it close to the interior of any set is open set interior of any set is an open set, it is element is that you can prove. Now, suppose A is a close set that means A is equal to A closure , that means A is equal to A closure , so that means x minus A closure is same as x minus A.

If x minus A is same as interior of x minus A, x minus A must be open, now on the other hand suppose x minus A is open, suppose x minus A is open then interior of x minus A should be same as x minus A. This means that x minus A is closure as x minus A and if x minus A closure is same as x minus A using that you can A is same A as closure, so this x minus, so you can say that proof follows by just observing this x minus equation it is same as interior of x minus A. So, this is a good characterization of close set A, set is closed if and only it is compliment is open, now does it immediately give us a large number of examples of close sets all that you do is it.

Whatever example of open sets you have seen, take their compliments those are examples of closed sets, in addition to that we have to justify a terminology. Now, we can say that why we had called certain set of balls as closed balls, just as after defining open set we proved that every open ball is an open set, in the similar way we shall, now prove that every close ball is a close set.

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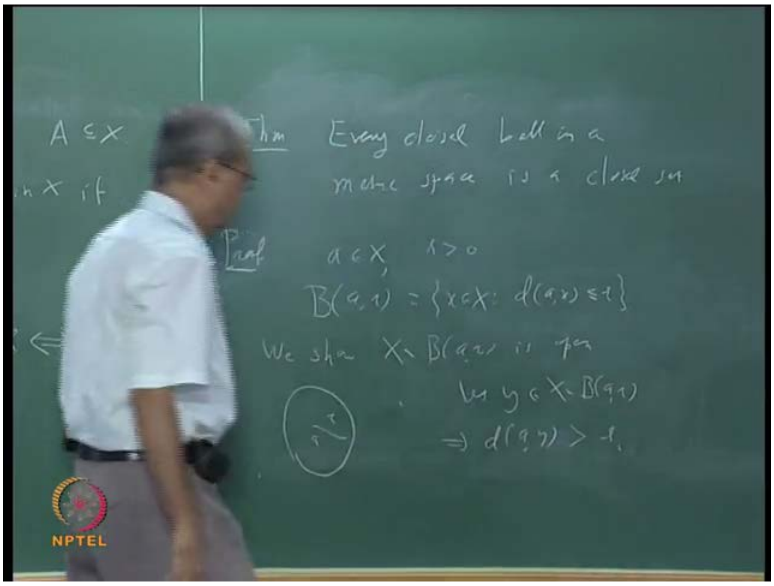

So, every closed ball is a metric space is a closed set look at the proof, so we need to possible some close ball, so close ball means for it take some point a as the centre and some positive number. So, close ball is B a r this is nothing but the set of all x and x with the property that distance between a and x is less nor equal to r and using this we can say that to show that this is closed. We can show that it is compliment is open, we will show that x minus this is open, we should shown x minus B a r is open once you understand this what is to be done after this.

So, in fact in this case certain diagram may help suppose this is B a r this is a and this is r what you need to show that x minus B a r is open you know it, suppose you take some point outside this. So, suppose I take some point y then you should construct find some rows such that such that open ball is centre at y and radius row is completely outside this, that is what we need to show.

So, let us say that y belong to this X minus B a r what is that mean, it means distance between a and y must be strictly bigger not greater than equal to, it is closed ball remember it, so distance between a and y must be strictly bigger than r. So, this implies distance between a and y is strictly bigger than r, now we do the same thing what we did in the last proof of similar type see if this number is strictly because of r, I can always find some number row such that it is bigger than r plus row, so let me continue here.

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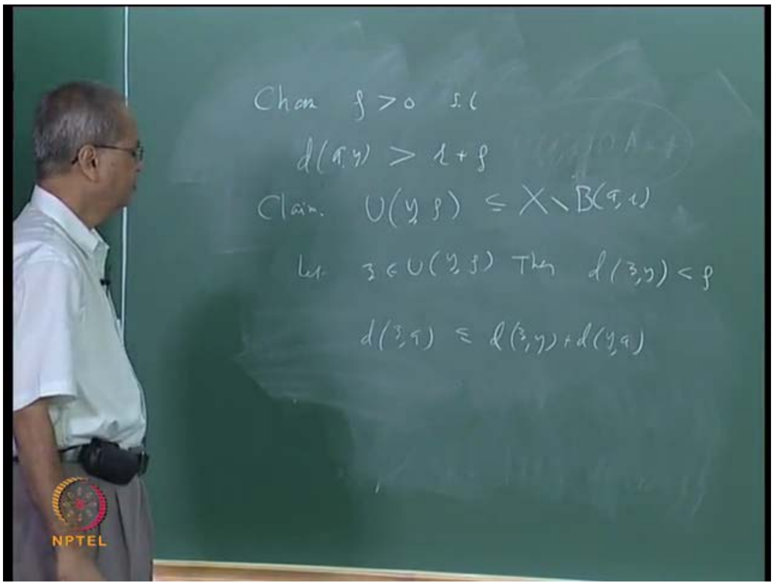

Choose row bigger than 0 such that distance between a and y is bigger than r plus row that is, this is distance between a and y this is r, so choose rows such that if you add this r and row then it is the distance will still be bigger than that. So, then what we need to show is that if you take this y and row as the radius then this open ball is completely opposite there is no common point between this open ball this close ball. So, then I will write this as a claim that it open ball with centre at 5 and radius row is upside, this that is in this x minus B a r and to prove this we shall just use this performance is very simple just take some point here with this.

So, suppose, so let suppose I call a point z, let z belong to U y row then the z y is strictly less than row, the z y strictly less than row then what is it that we need to show that the z is strictly bigger than r that is what we need to show. Does it, does it follow from this let us, now let us look at d z a let us look at d z a this is less nor equal to d z y plus d y a and using this it will follow that d z a must be strictly bigger than half d z a must be strictly bigger than half. So, this is a close sets, so every close ball is a close set, is a close set then using the several properties of the open sets which we have discussed. Using this particular theorem we can prove several theorems about the closed sets for example, let me just write those theorems.

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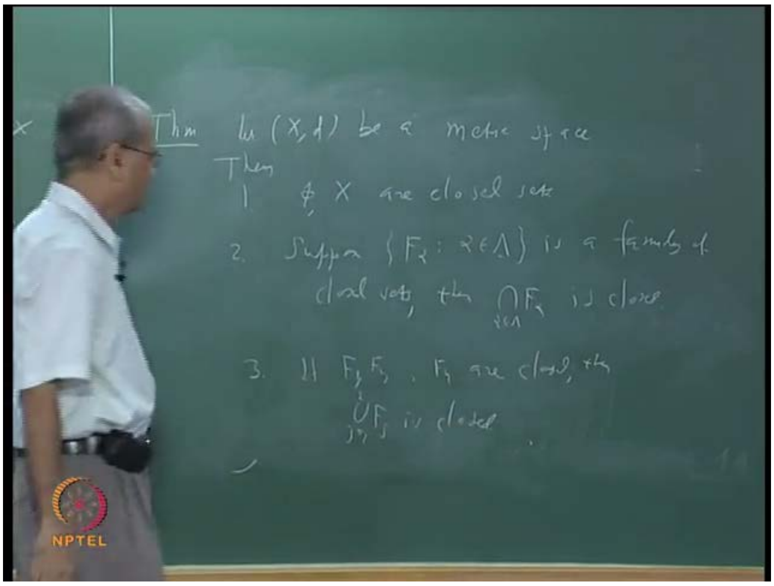

So, let x d be a metric space and since the proof is very straight forward I will not spend much time in discussing this then first thing is empty set and x are close sets and about unions and intersections the rows are interchanging. In case of open sets we know that the arbitrary union of open sets was open, here we can say that arbitrary intersects on of close sets is close, so if you take any family of closed sets.

So, suppose f alpha belonging to lambda alpha, alpha belonging to lambda is a family of closed sets then intersection of this f alpha, alpha in lambda is closed. When it comes to union of only finite family is closed if F 1, F 2, F n are closed F 1, F 2, F n are closed then union F j, j going from 1 to n is closed in short arbitrary intersection of closed sets are closed and finite union of closed sets is closed. It is completely opposite to what happens in case of open sets and in the proofs also you can prove that by using the corresponding property of open sets for example if since we know that phi is open.

So, that means x is closed and similarly phi and x is, x is open it means pie is closed and you can do the same thing, here for example if each f alpha is closed it means its complement is open. So, you take union of those complements that will be nothing but the complement of this that will be open, so this is close and in a similar way you prove this. In other words you prove all these things using this theorem and the corresponding properties of open sets and, so called the Morgan's laws, I think we will stop with that for today.