Real Analysis

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Lecture - 17
Open Sets

So, in yesterday's class we discussed what is meant by open balls and the related concept of an interior point which depends on the open ball.

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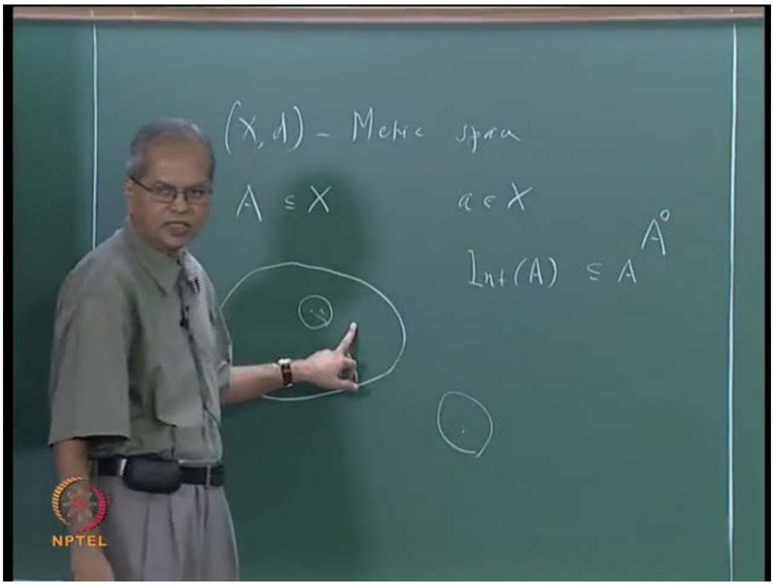

Now, let us start with that again, so suppose X d is any metric space metric space and we took some set a subset a in X and point A in X and we said that a is an interior point of A that is a is an interior point to the A. If there exist some open ball with centre at A which is completely inside a, which is completely inside A we can to understand the concepts, we can give a diagrammatic or representation for this I meant we can draw some pictures to understand the concept. So, suppose this is in the set A, let us say this is a set A, then we will say that the point is interior point, if suppose this is the point then we can find some open ball big centre at A such that that ball is completely inside this A.

So, that is an interior point of course you are, you are used to hearing that one picture is superior to thousand words etcetera, but what can happen is that in since in mathematics are in particular the balls may or may not look like that. So, sometimes pictures may mislead also, you should be careful in using, but of course pictures many times help in understanding what is happening. So, that is about the interior point and then we have said that a set of all interior points of A that we had denoted by interior of A. In some books they also use this notation, a super script this 0 for the interior of A then we also seen that A is always a subset of an interior of, A is always a subset of A.

Now, what is the other possibility suppose you take any point A in x 1 possibility is that there exist, there exists some r such that ball with centre at A and radius r is completely inside A, what is the other possibility. It may also happen that there exists some r such that ball with centre at A and radius r is completely outside A that is also possible. So, it may, it may that ball may be somewhere, here we suppose a point is somewhere, here then there you can find some ball which is completely outside A.

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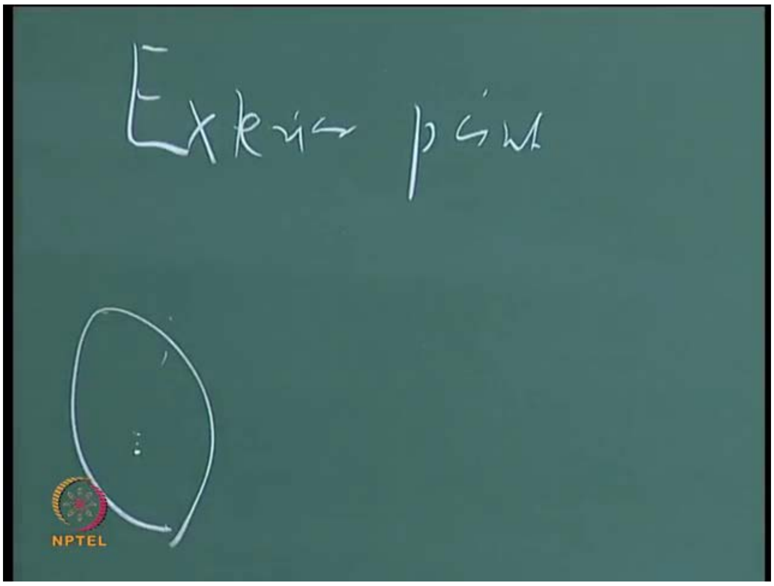

So, such a point is called an exterior point, exterior point what is the meaning of that there exist some r such that.

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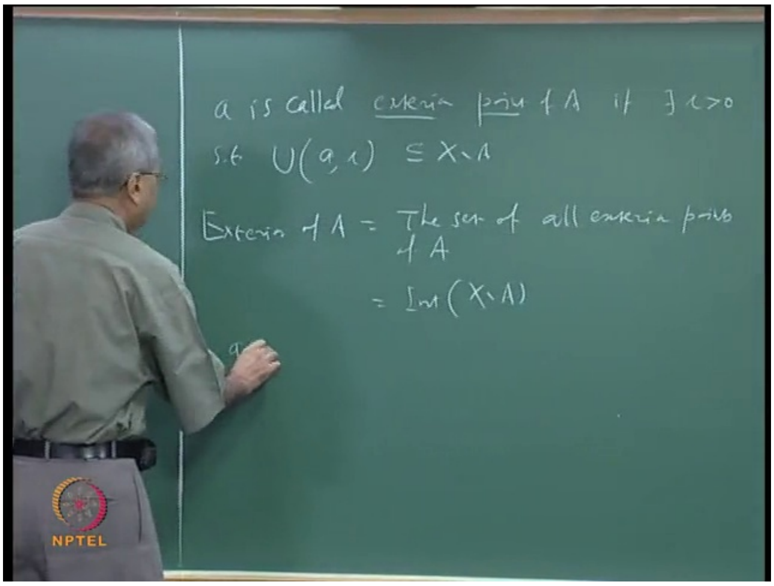

Let, me repeat, here a is called exterior point of A if there exist r bigger than 0 such that open ball with centre at a and radius r is completely outside A that means it is contended X minus A, X minus A and the set of all such points is called exterior of A. So, exterior of A we shall denote that by ext of A or exterior of A the set of all exterior points of A and is it clear to you that a point is an exterior point of A. Same as say that it is an interior point of X minus A because we are saying that there exist r such that that open ball completely lies in the outside a.

That means it lies completely in the X, in X minus, in the compliment of A, so exterior of A is nothing but by definition it is same as interior of X minus A. Now, let us see one more thing, now given a point A in X it can be either interior point of A or it can be an exterior point of A, what is the third possibility. What does it mean that is neither a ball which lies completely inside A nor is there a ball which lies completely outside A, what does it mean.

That is whatever ball you take, that is whatever r you take and if you take a ball with centre at a and radius r inter it will have at least one point in A, and at least one point outside A, that is every ball with centre at a, will have non empty intersection with a as well as X minus A. So, if such a thing happens we say that we call that point as a boundary point for example a point like, here suppose you take any point. Here, then if you take any ball with centre at this, then should we derive it exist at least one point inside A and at least one point outside A, so it will have non empty intersection with a as well as.

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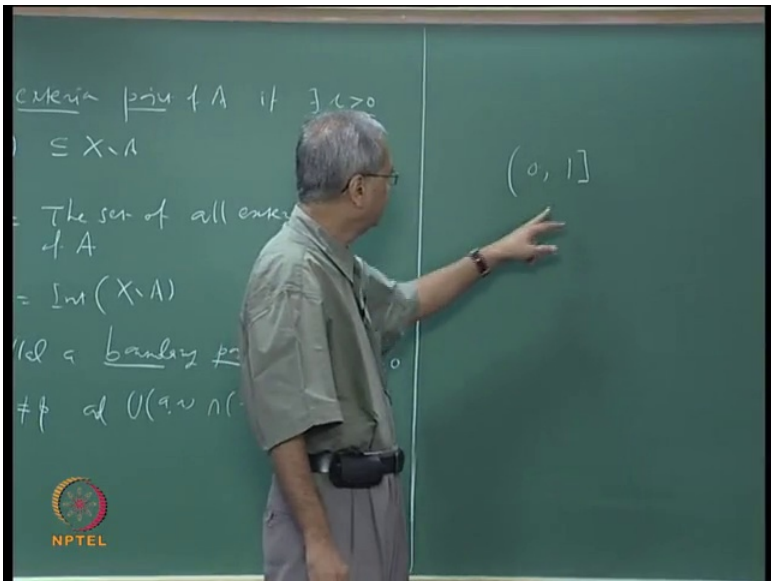

So, let us point that, so we can say that a belonging to X is called a boundary point if for every r bigger than 0. For every r bigger than 0 what should happen U a r intersection A of course boundary point of A, boundary point of A that is important, this is non empty and U a r intersection X minus A is also non empty. That means every ball contains at least one point from A, and at least one point from the compliment of A if such a thing happens it is called a boundary point. You can easily see the examples of interior points, exterior points and boundary points in various examples that we have seen the other day.

For example, if you take an interval like this, for example suppose you take the interval 0 to 1 we have seen that open interval 0 to 1 that is interior of this what is exterior, exterior will be open interval 1 to infinity and then minus infinity to 0. That is exterior, open or both sets and what are bound, what are boundary points it is just these two points 1 and 0, 1 and 0.

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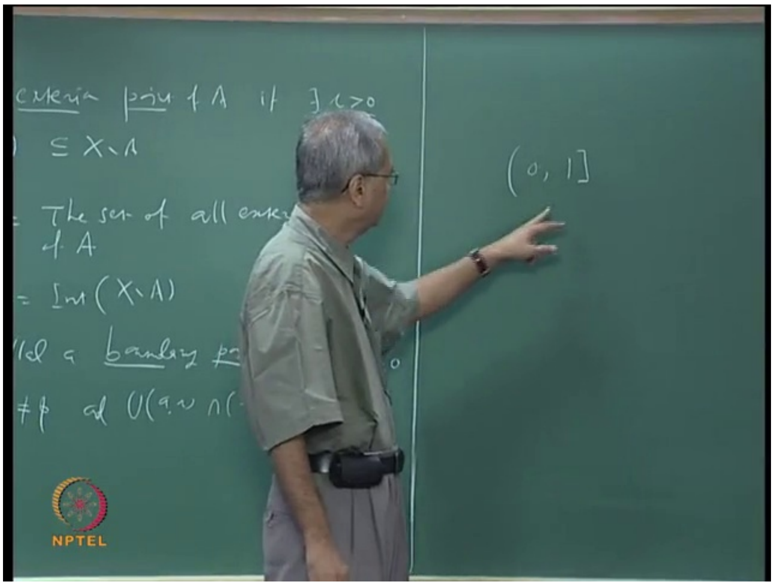

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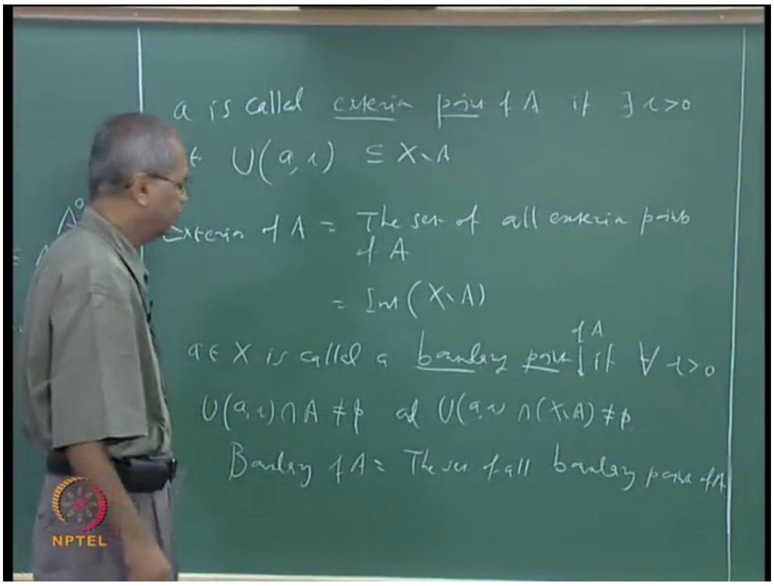

By the way, the set of all boundary points is called boundary of A, set of all boundary points of A is called boundary of A, the set of all boundary points of A. Now, is it, is it clear to you that given any point in X it should be either interior point of A or it should be exterior point of A or it should be if both of these are false it should be boundary point of A. So, it has to be one of these it has, it has to lie in one of these three set interior of A, exterior of A and boundary of A. So, what I can say is that X is always a union whatever, given any non empty set A or given any set A and metric space X, X can be always written as a union of these three sets all of which depend on A.

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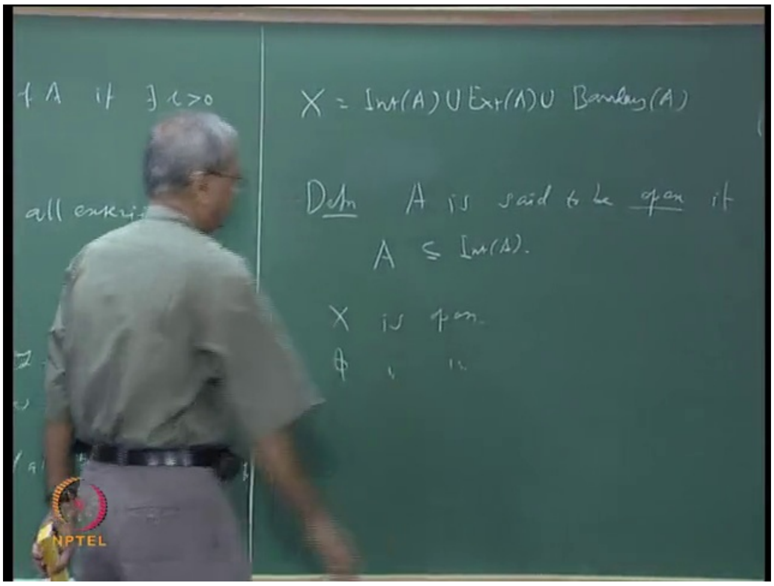

What are these three sets one is interior of A, X is interior of A then union exterior of A, and then union boundary of A. Of course some of these sets may be empty I am not saying that all the thing for A for every A all for example we have seen the examples yesterday where interior is empty, similarly there can be sets where ext exterior is empty. Now, let us go to the next important concept which depends on the concept of an interior point, if it so happens that in a set A every point of A.

See we already we are interior of A is always contending A if the reverse in position is also true that means if every point is an interior point then it is called an open set, then it is called an open set. So, let us go to definition A is said to be an open set or A is said to open if we can say this is the short way of writing A is contained in the interior of A or to put it in a words every point of A is an interior point of A.

To expand it little further given any point A there exist r such that open ball with centre at a and radius r is completely contained in A, all that is said in this A is contained in interior of A. Now, let us see what are the open sets that we come across very frequently when we define for example open ball closed ball etcetera that time I had, we had not yet defined what is meant by open set and, so we had no justification why at all it is called an open ball. But, now we can give that justification, but even before going to that let us dispose of some of the obvious things which are very obviously opened sets.

What about the full metric space X that is, that has to be always because what the whatever ball you take it has to be always inside X. So, X that is always an open set, so X is open, X is open always, now what should happen for set suppose the set is not open then what should happen. Let us, let us take this example again this example 0 to 1 we have all ready seen that this point 1 is a, is a point in the set, but that is not an interior point, so this is not an open set, this is not an open set.

Whereas, open intervals are open sets because there every point is an interior point so if a set is not open what it means say there will exist at least one point, one point in A which is not an interior point. Now, does it follow from this that empty set is an open set because if the, if the set is not open then there should exist at least one point which is not an interior point, so such a thing is not possible for empty set. So, empty set is also an open set, so empty set is open, now let us go to the main proof that we shall, now prove that every open ball is an open set.

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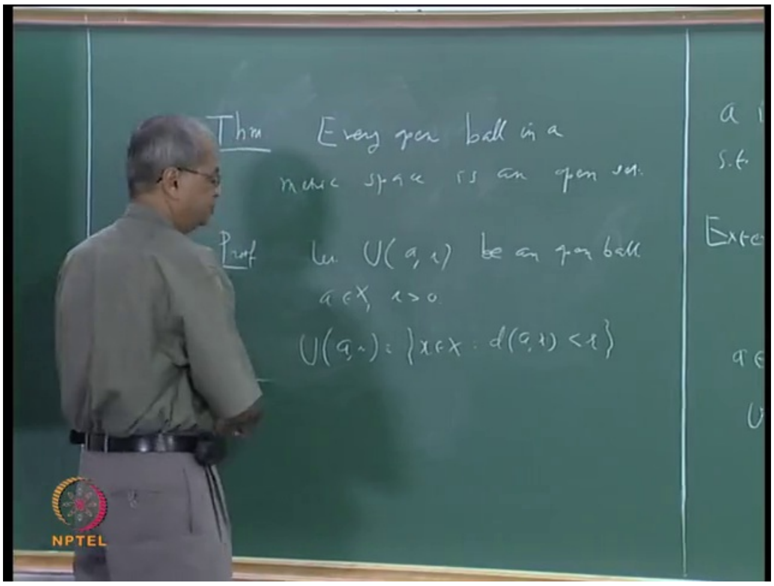

So, let us write it as theorem every open ball in a metric space is an open set, so let us consider some open ball, so let us say that let U a r be an open ball. Then this means what a is in X and r is bigger than 0 and U a r it is recall that it is a set of all points whose distance from a is strictly less than r. So, let us again recall that U a r is the set of all x in X such that distance between a and x is strictly less than r.

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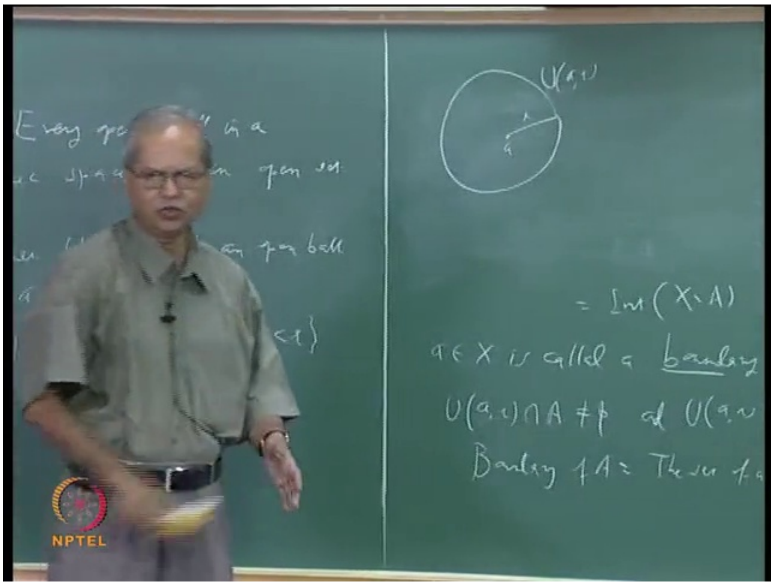

So, suppose this is a and this is r then this is U a r we want to show that this is an open set, show that this is an open set, so what is the way of showing an something is open there is only one way you have to show that every point is an interior point, so take some point.

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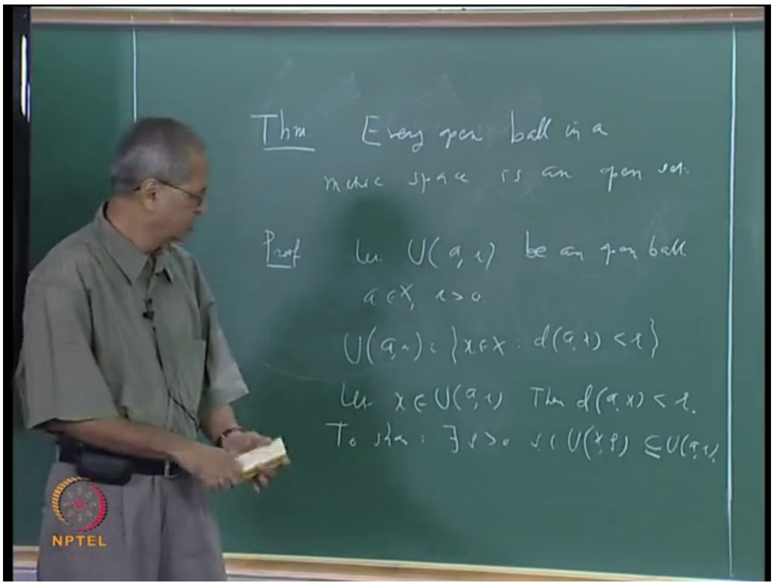

So, let x, let x belong to U a r that is let us say there is some x, so what is a property of this x distance between a and x this distance is strictly less than r, distance between then distance between a and x is strictly less than r. What do you want to prove that such an x is our interior point of this U a r that means there exist some number let us call it true, there exist that we know to should there exist some row such that a ball with centre at x and radius row is completely inside this ball U a r.

That is, that is this is what we want to show to, show there exist true bigger than 0 such that ball be centre at x and radius row is completely inside U a r, so there are two things. Here, first thing is to decide what this row should be, to decide what this row should be and then prove this inclusion, now let us go to the first step what is a choice for row obviously.

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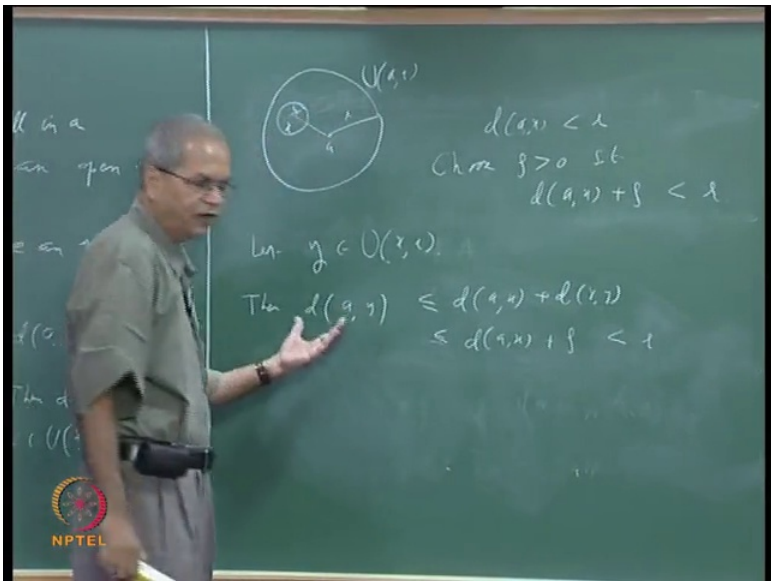

See what we want is that a ball, see this is x what we want is ball with centre at this and radius row that should be completely inside this it should not go somewhere outside. So, how do we choose row for that because what we know is that see remember, here what we know is that d a x is strictly less that r d a x is strictly less than r. Now, when there are two num when one number is strictly less than r you can always find some number some positive number such that if you add that positive number to this, then the sum is still less than r.

For example, you can take r minus d a x divided by 2 that can be one choice r minus d, so what we can say that choose any row bigger than 0, choose row bigger than 0 such that this d a x plus row. In fact we can say that is less than r that such a row exist, that is this number this d a x is strictly less than r, so I can always find some small number such that if you add it to this, the sum is still less than r.

So, that number there will be many one, one choice I already told you, so there will be, there will be many such choices choose any such row satisfying this then we should prove this inclusion that is U x will row is contained in U a r. Anyways, how that one prove that one set is contained in the other this only one, you take a point, here and show that that point lies in this. So, let y belong to U x r then we want to show that, we want to show that y belongs to U a r, we want to show that anything is in U a r you want to show that distance between y and a must be strictly less than r.

So, look at distance between y and a distance between a and y we can always, now here we shall use that triangle equality because we know something about distance between a and x and we know something about distance between x and y. So, distance between a and y should be less not equal to distance between a and x plus distance between x and y plus distance between x and y and, so we know that distance between this, less not equal we know that this is less than row because y is in U x r. So, distance between y and x is less than row, so this is in fact this is strictly less than distance between or less not equal to distance.

We can say less, strictly less than distance between a and x plus row and we have chosen row in such a way that that is less than that is, that is the reason for choosing the row in such a way, so this is less than r. So, that shows this that open ball with centre at x and radius row is completely inside the open ball with centre at a and radius r is it sure. So, every open ball is an open set every open ball in any metric space is an open set and, in yesterdays class we have seen several examples of open balls in various metric spaces.

So, you have several examples of open sets, now we shall see some more examples of course these two are already there the full metric space x and empty set is always an open set. Then we shall see some more properties of this open sets, for example how can we come across or how can we form new open sets from the known open sets. For example, things like what happens if we take unions or intersections whether they are open sets it is what is known in this case is that if you take any family of open sets then their union is always an open set.

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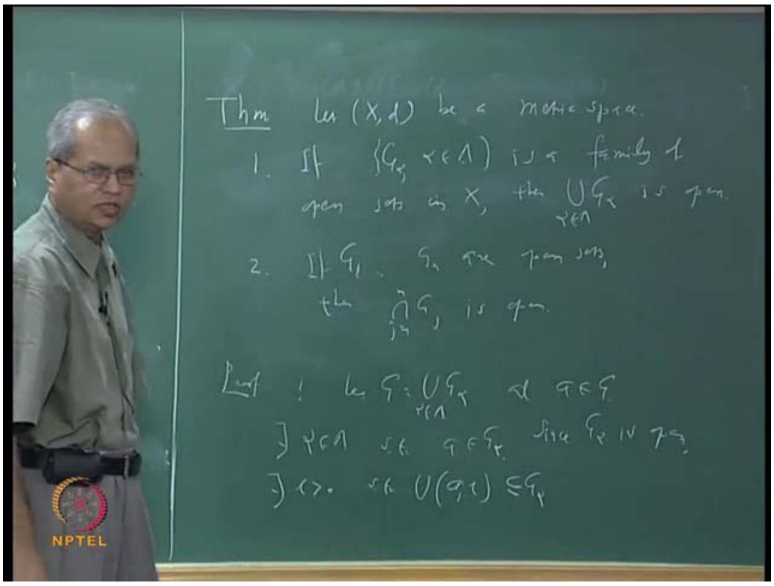

So, let me just write this as a theorem, let x d be a metric space and, so let me write this as a, so first property is that if let us say G alpha, alpha belonging to lambda is a family of open sets, is a family of open sets in X. that union G alpha, alpha G to lambda is open. In other words, you take any arbitrary family of open sets finite, infinite, countable infinite does not matter there union is always open, but similar thing is not true about intersection, in case of intersection we have to take only a finite family.

So, second thing is if G 1, G 2, G n are open sets then intersection G j, j going from 1 to n is open and proof is more or less straight forward. Let us, let us look at this proof first let G alpha be a family of open sets and let us call this union as G, let be equal to union G alpha. Now, we want to show that G is open, we have to show that G is open if G is empty there is nothing to be proved, if G is empty it is all ready open.

So, let us say that it is non empty and some points belongs to G, so and let us say a belongs to G then we want to show that there exist r such that open ball with centre at a and radius r is contained in G. But, again that is trivial, here since a belongs to G and G is union of this G alpha, it is in one of this alphas since G is union. So, we can say that there exist some alpha, there exist some alpha in lambda such that a belongs to G alpha, a belong to G alpha and G alpha is an open set that is how we started G alpha is an open set.

So, there should some r such that ball with centre at a and radius r is inside G alpha, so since g alpha is open. There exist r bigger than 0 such that bound with centre at a and radius r is contained in G alpha, and G alpha is contained in G because G is union of all such G alpha. So, G alpha is contained in G and that proves that a is an interior point, so that shows that G is open.

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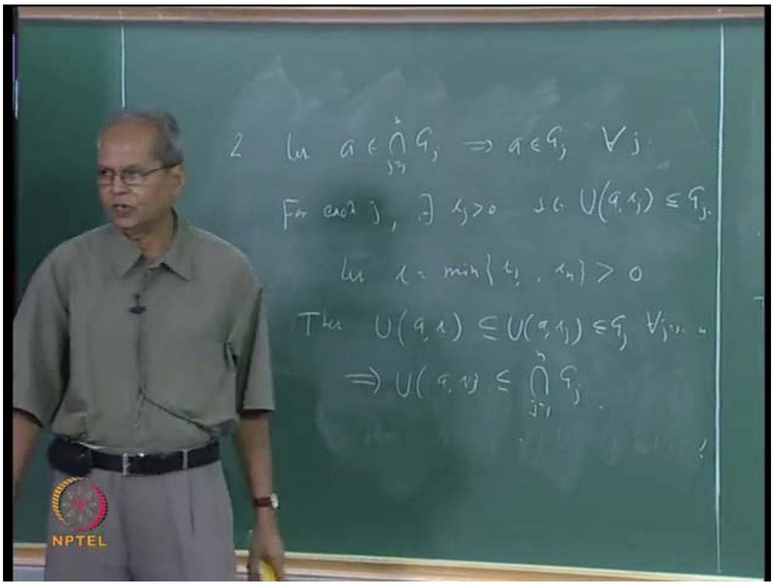

So, let us look 2 now, so we are given that G 1, G 2, G n are open sets and we want to prove that their intersection is open again if the intersection is empty there is nothing to be proved. So, let us assume that there is something in that and then we have to show that that point is an interior point. So, let a belong to intersection G j, j going from 1 to f, but if a belong intersection G j means it belongs to each of these Gs this means, this means a belongs to G j for all j. What follows from this that for each j, that for each j there will exist some positive number suppose I call that number r G such that open ball with centre at a and radius r j is inside G j, so for all j, so we can say that for each j there exist there exist r j bigger than 0.

Such that open ball with centre at a and radius r j is contained in G j what is to be done after that is clear you take the minimum of all these. So, let r be equal to minus of this r 1, r 2, r n there are n such number and since each of this r j is strictly positive that is the most important argument. Here, since each of this r j is strictly positive, r also is strictly positive, this is strictly bigger than 0 then you look at U at centre at with centre at a and radius r take open ball with centre at a and radius r since r is less not equal to r j, U a r is contained in U a r j. So, this is contained in U a r j, U a r j and U a r j is contained in G j and this is for true for each j, for each j this argument will be, it is true for all j equal to 1 to n.

So, and if something is in G j for each j it has to be in the intersection also, this implies u a r is contained in the intersection G j, j going from 1 to n, so we started from a point in this intersection. Then we found an r such that u a r is in this intersection G j, j going from one to n, so this shows that the finite intersection of open sets is open. So, that theorem is usually expressed by saying this that arbitrary union of open sets is open and finite intersections of open sets is open. Now, looking at this proof you will also understand why we cannot take an arbitrary intersection because suppose you follow this same line, here there suppose the, suppose this and in featly many sets.

Here, that will in featly many r js then of course minimum would not exist, we will have to look at infimum and infimum of a set of positive numbers can be 0. So, this proof would not work, but again if you want to show that something is false saying that some proof does not work is not a correct argument. Even if this proof does not work in principle you can say that somebody may be able to give some other proof that is not the only way of proof.

So, in order to say that this, you cannot replace finite by infinite in this are true, here there is only one way of disposing of that question you have to give an example. Example of what, example of an infinite family of open sets such that the intersection is not open intersection is not open, what is an example.

Student: Open interval 1 is minus 1 by n to 1 by n.

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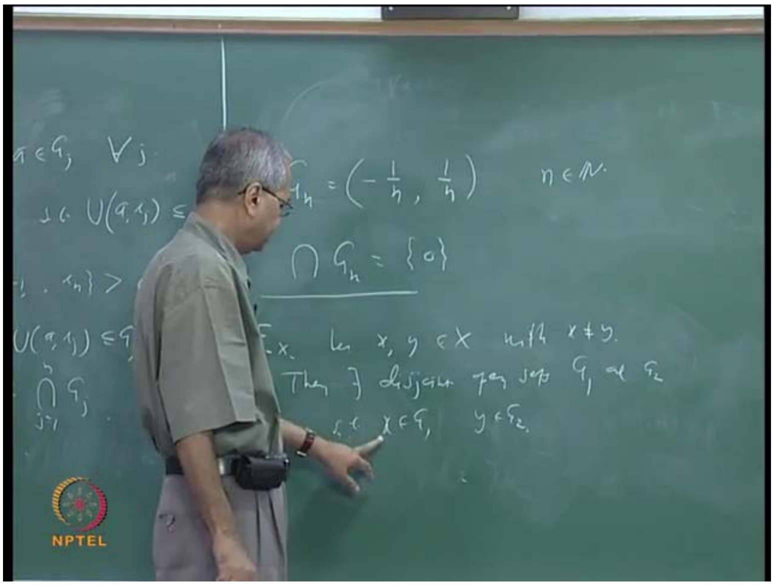

Let us, let we call it G n is minus 1 by n to 1 by n and for n is equal to let us say n belongs to N, so what does the intersection G n intersection G n is just single term 0 and it is clear that a single ton 0 is not an open set. So, you cannot replace a finite family, here by an arbitrary family, now that we have learnt this various properties of this open sets, let us just go to a slightly more general concept I will just introduce that concept here.

But, we will not go more further about that concept than that, but before that let me also make one more point, here or let me use this as an exercise in a metric space the following thing is true. Suppose you are given two different points, suppose you are given two distinct points then you can always find open sets two disjoint open sets containing those two points. So, let me just write, here what is the sum say suppose let x y belong to x with x not equal to y, with x not equal to y then there exist disjoint open sets, open sets.

Suppose I call them G 1 and G 2, G 1 and G 2 such that x belongs to G 1 and y belongs to G 2 that is given two distinct points you can always find two disjoint open sets such that one point is in overall set. Second point is in that, second set is that you can prove it on in fact you can find disjoint open balls. If that two distinct points are given you can find disjoint open balls such that x is contained in G 1 and y is contained in G 2.

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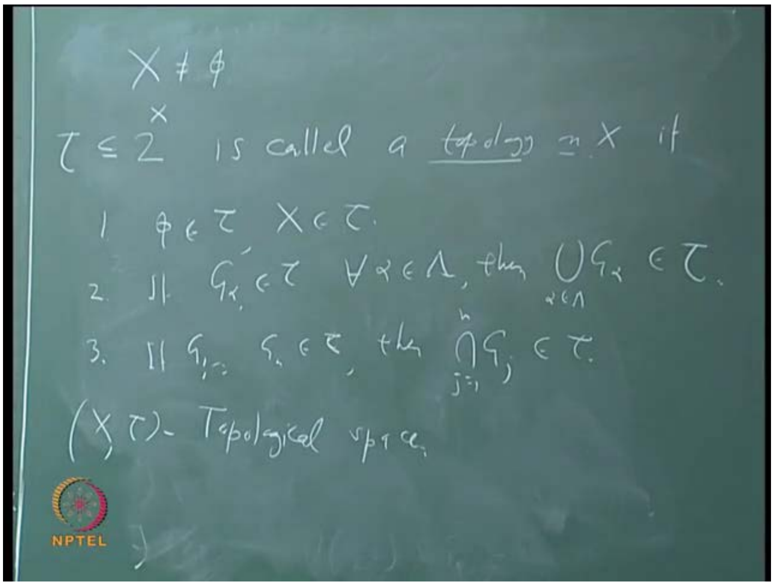

Now, suppose x is any arbitrary set, x is any arbitrary set non empty set and suppose we consider x power z that is we show that 2 power x is nothing but a set of all sub sets of x and I. Consider some subset of this, that means some family of sub sets of x some family of sub sets all. Suppose I call that set as tau, tau is a family of, so tau is a family of sub sets of x with certain properties I would not, I am not taking any arbitrary family. But, some properties so such a family is called topology on x this is what I want to define, it is called topology on x if certain properties as I said if the first property is that empty set belongs to tau and a full space also belongs to tau.

Second property is that if a family of sets is in tau then there a union is also in turn, so if G alpha belongs to tau for all alphas in some indexing set lambda, then union G alpha, alpha belongs to lambda. This also belongs to tau in other words if this family contains any arbitrary if this tau contains any family of sets then it is union of for that family is also a member of tau. Third property is that if it contains a finite, if you take any finite number of sets then their intersection is also in term, so if G 1, G 2, G n belongs to tau then intersection on this that is intersection G j, j going from 1 to n this also belongs to tau.

So, any family of sets that these three properties or these three are again if you write these two things as separate things it will be four properties. It is called a topology on x, is called topology on x and this pair x tau that is called a topological space, so what is a topological space. Topological space is a pair x tau where x is an non empty set and tau is a topology on x topology means family of sub sets that is pairing these properties have we already proved that every metric space is a topological space. See suppose x is a, suppose x has some metric on it, suppose x is some metric on it then take the set of all, then take tau to be the set of all open sets in x.

Then does tau satisfies all this properties this is what we proved just now that is pie and x are open sets and if you take any arbitrary family of open sets then their union is also an open set. Intersection of a finite family of open sets is again open all these things you know in short what we proved was that if you take the family of open sets in a metric space that forms a topology. That forms a topology such that that means every metric induces a topology, every metric induces a topology then, here one can ask a very obvious question whether converse is also true.

Whether every topology is, whether topology is given by or induced by a metric of course the answer is false because if that way true then whatever set is there is no difference between a topological space and metric space. That is not the case topological space is a much bigger class and the study of this topological spaces and functions between the topological spaces that forms a subject on its own called topology. As I have said in the beginning that this real analysis is a very basic subject it has connection to several other subjects for example we have discussed what is norm linear spaces.

So, norm linear spaces and the maps between norm linear spaces that form a topic in what is called functional analysis and, here you have topology. So, basic concepts in analysis are used in practically in all subjects, so let me just before just proceeding further, let me just give an example of a topology which is not induced by any metric. Now, the whole question is this how does one show that a topology is not induced by a metric etcetera we will come to that little later, but let us, before that let us dispose of some trivial examples of topologies.

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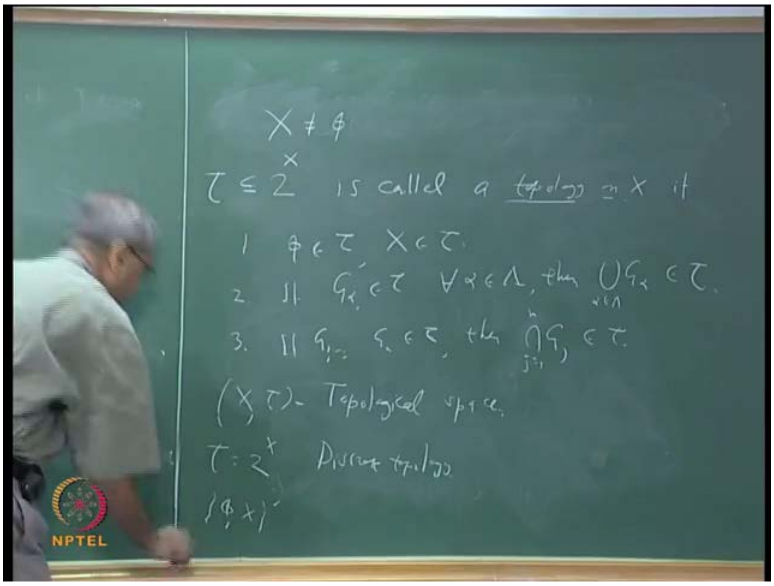

For example I can take tau as this whole thing 2 power x, suppose I take all the sub sets of x then obviously all this properties are said is failed that is called discrete topology and by the way I should mention, here that once you consider topological space members of this set tau are curl open sets open sets with respect to this topology, open sets with respect to this topology because, here basically what we have done. Here, we are taking the abstract concept of open sets and then starting from starting from there and then in topological space what happens is that subsequently you define everything in terms of open sets.

Just as in case of metric spaces you will define all concepts in terms of metric, so similarly in topological spaces you will define all concepts in terms of open sets. So, whichever concepts can we define using only open sets those are called topological concepts, so this is one example of a topology which is called discrete topology you can also see why it is called discrete topology.

We have discussed what is meant by discrete metric space, in discrete metric space what are the open sets let me begin with is a single turn set open in a discrete metric space because we have seen that a ball with centre at x and let us say radius half is single turn x. So, single ton it is an open ball and hence an open set, but once you say that a single set is open, does it follow from there that every subset is open given any subset of x, you can write it as a u union of single term sets. We have already seen that arbitrary union of open sets is open, so in a discrete metric space every subset is open, in a discrete metric space every subset is open.

So, the topology induced by discrete metric is just this discrete topology what we call discrete topology, let us, now take another x stream. Suppose I take just these two sets phi and x, I will just take two sets empty set and x, now that is the minimum these two sets have to be there that is our this the first requirement. Suppose I include nothing else will this also satisfy all this properties because if you take say G alpha, G alpha has to be either empty or x? So its similarly intersection also is that this is also topology, this also a topology that is called indiscrete topology, so in that discrete topology you have all sub sets to be open sets and in the indiscrete topology only empty set and the full space, these are only two open sets.

Now, let us come this question look at these exercise, here what I have said, here is that given any two in a metric space, given any two distinct points you can find two disjoint open sets such that x is contained in G 1 and y is contained in G 2. So, wherever a topological space has this property it is called a housed of topology whenever topology has this property it is called housed of topology, that means two distinct points can be separated by two disjoint open balls. That is called housed of property, so every metric space has this property that is topology into induced by metric has this property thus this space have this property.

Suppose I take this indiscrete topology, suppose you take indiscrete topology then can you say that with respect to that topology also you can find given any two points x and y you can find G 1 and G 2 etcetera. See remember there are only two open sets, empty set and the whole set x, so G 1 and G 2 can be either empty or x, so if G 1 has to contain the point x, G 1 has to be whole of x similarly, G 2 has to be whole of x.

So, there cannot exist any disjoint sets there cannot exist any disjoint sets separating two distinct points, so this indiscrete topology will not satisfy this property in discrete topology will not satisfy this property. So, this topology cannot be obtained by any metric, this topology cannot be obtained by any metric we shall not pursue this topic of topology and topological space any further. This is just to let you know that the concept of open sets is a very basic concept and using that concept we can define several other things and in metric space.

Also we shall show that initially many other things, we shall define using the distance d, but many of those concepts can be, can be defined and discussed using only the concept of metric spaces. Those are the once which are then taken in a topological spaces and those become topological concepts and topological properties etcetera, let us, now come back to the description of open sets in various metric spaces. We already seen that open balls are open sets and just now we have seen that in a discrete metric space every subset is an open set. Let us now go back to our all most familiar space namely r, r with the usual topology r, with the usual topology or usual metric usual topology means topology given by the usual metric.

We already seen that the intervals are open sets, open intervals are open balls and every open ball is an open set, so any arbitrary union of intervals open intervals is an open set, any arbitrary union of open intervals is an open set. But, in the real line we can say something more that every open set can be expressed as a union of open intervals, not only that we can say something more we can say that it is it can be expressed as a union of a countable family of mutually disjoined open intervals. So, let us, let me write by the in any metric space given every open set can be expressed as a union of open balls that should be fairly easy to show in a if you take any open set and if you take a point in that open set.

You can always, you can always find a ball with the centre at that point and that ball is completely contained in that that open set, so suppose you take collect all such balls their union has to be the same as that whole set G. So, in every metric space every open set can be expressed as some union of, some union of open balls, only thing is that those open balls which form that union they may not be disjoint, they may not be disjoint their number may not be countable. But, in case of real line we can say this you can always express every open set as a disjoint union of a countable family of intervals, so theorems.

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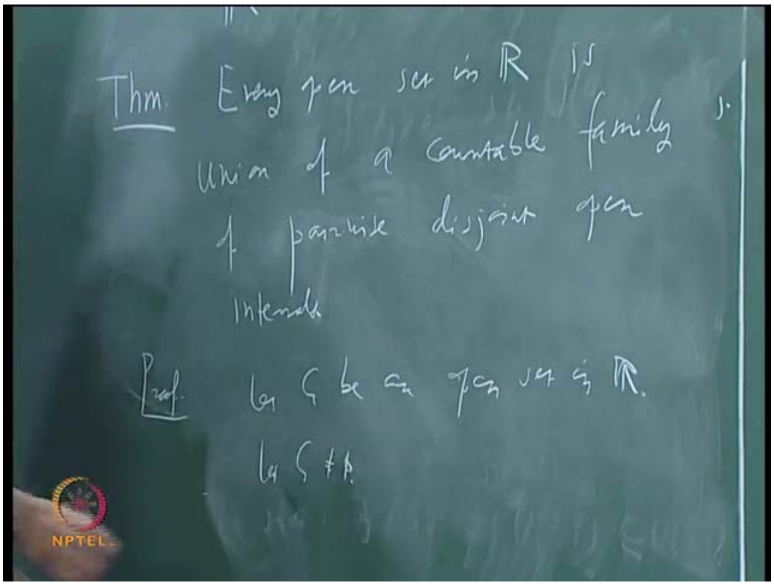

So, every open set in R, R is union of a countable family, union of a countable family of mutually disjoint or pair wise disjoint intervals pair wise disjoint open intervals. So, let us now look at the proof, so let G be an open set in R I will just give some steps in the proof there we will complete the details, again if G is empty there is nothing to be proved. We can say that it is union of empty family of open intervals, so let assume that G is non empty, if G is non empty. So, let G be non empty then to begin with what I will do is that I will define a relation between the points in G.

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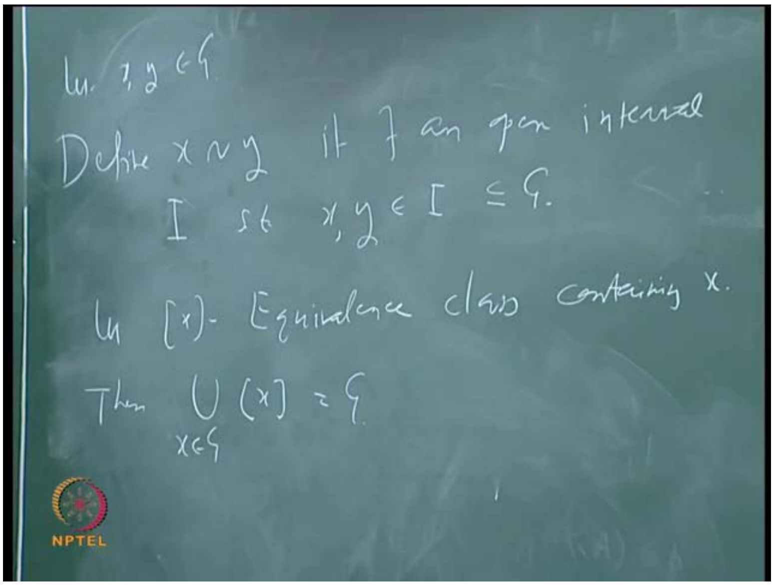

So, suppose we take two points in G let x and y belong to G, so let us say I will define the relation, define x is related to y this means there exist an open interval which contains both this points x and y and it is also contained in G. So, we say that x is related to y if there exist an open interval, there is an open interval suppose I call open interval I such that both of this point x and y belongs to I and I is content in G.

Now, let us strictly see some properties of lesson is this relation reflexive given x can be always find an open interval, so that x is contained in I and I containing G in fact that is that follows because G is open at any point x. We can find always find some R such that ball with centre at x and radius r is completely inside G and that ball is nothing but an open interval x minus R to x plus R it reflexive clearly is it symmetric.

It is also transitive see suppose x related to y that means the exits some interval, suppose you call it I 1, I 1 contains x and y and I 1 is containing G. Suppose y related to z then they exist some other interval G such that j contain y and z and j is contained in G I and z both are contained in G. So, I union G is in G, but is I union j again interval that is where the property that we said yesterday will come into picture because I and j are not disjoined because y is coming to both y is coming to both. So, I j are not disjoined, so this is an equivalence relation, this is an equivalence relation, now what does an equivalence relation do it will partition all the points into what we are called equivalence classes.

So, this equivalence relation will partition the given set G into what are called equivalence classes and the union of those equivalence classes will be same as G, so let us say that let this x. So, denote equivalence class equivalence class containing x then we know that union of these equivalence classes x belonging to G its same as G, now suppose we show that each of this equivalence class is an open interval.

Suppose we show that each of this equivalence class is an open interval and suppose we show that there are total numbers is countable, there number is countable then we would have completed a proof. I think will complete a proof tomorrow, since the time is over, I will stop with this today.