Real Analysis
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Lecture - 16
Balls and Spheres

We have seen definition and several examples of metric spaces so far. Today, we will begin with some methods of constructing new metric spaces from the non metric spaces.

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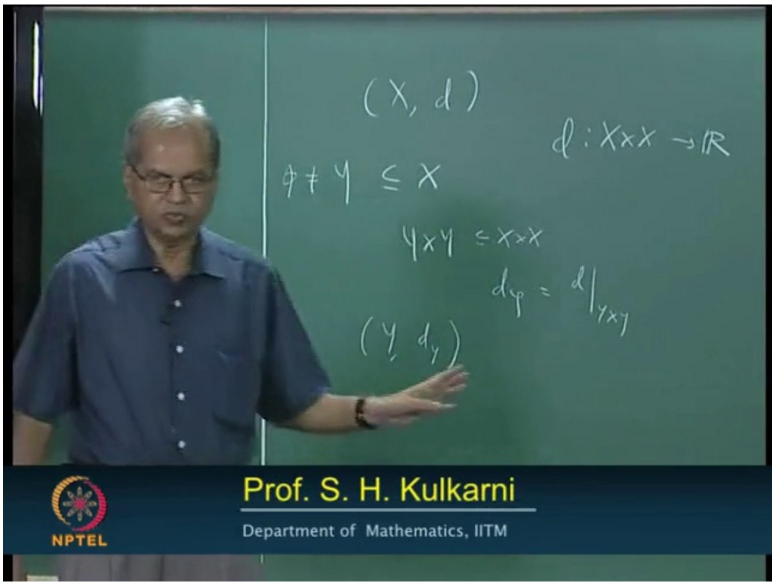

To start with, let us take a metric space, suppose X, d and suppose Y is some subset of X. As usual to a word that trivialities, let us take a non empty subset. Then, what we know is that d is a function from X cross X to R. Now, since Y is subset of X, Y cross Y is a subset of X cross X. So, this function d can be restricted to this subset. That is in other words that if you can talk of this distance with any 2 points in X, you can take the same definition for a distance between any 2 points in Y. So, suppose I define that function as d is of X, Y is nothing but we can say d restricted to Y cross Y, d restricted to Y cross Y.

Then, with this, we can say that Y and d is of X, Y. This will be the new metric space. In another words, every non empty subset of a metric space is also a metric space with respect to the metric, which comes from X as a restriction metric. So, this metric, this metric d suffix Y, which is restriction of d, it is called induce metric, metric on Y induced by this metric on X. So, this is one way of constructing new metric spaces. There is you can regard a subset of given metric space as a metric space in its own right.

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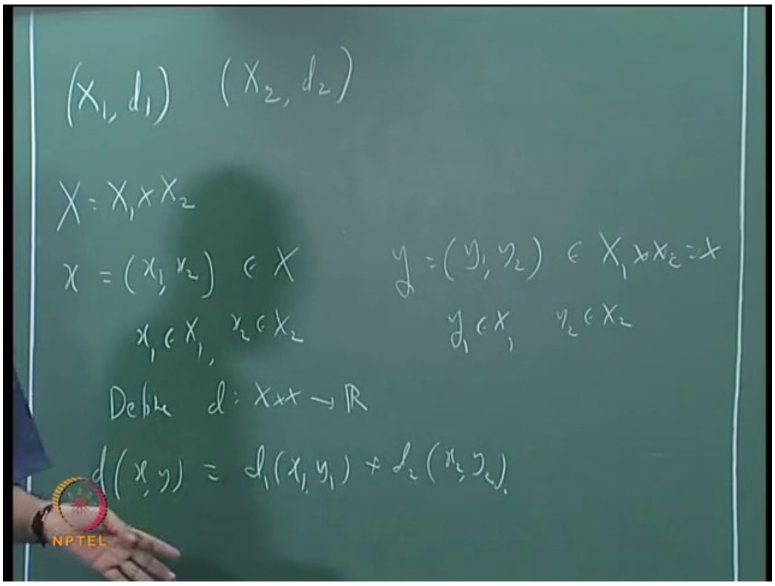

Another method is that suppose we can take metric spaces X1, d1 and X2, d2. I mean as a particular, I am not saying X 1 and X 2 must be distinct. They can be same also as a as a particular case. So, take X as X1 cross X2 that is Cartesian product. Then, as you know that elements of X are nothing but the order pairs of elements that is first elements comes from X 1, second from X 2. So, suppose you take any element x in X. It will be of the form, let us say small x 1, small x 2 that is in x. So, this means x 1 is in big X 1 and x 2 is in big X 2.

Suppose I take another element, let us say y as y 1, y 2 that is also be X 1 cross X 2, which is what is X. So, y 1 will be X 1, y 2 will be X 2. Now, suppose I want to define what is meant by distance between x and y. So, suppose I want to define what is meant by distance between x and y. Then, since x 1 and y 1 are already in X 1 and on X 1, there is already a metric d. One can talk of d 1 between x 1 and y 1, and similarly d 2 from x 2 to y 2. I can combine these two functions in some way so that this new function also satisfies all the properties of a metric.

Now, there are several ways of doing this. Let me just take illustration. So, d x y suppose that is same as d1. Suppose I define it like this, define this d from X cross X to R is given by d1 between x1, y1 and plus d2 between x2, y2. Now, I will leave it to you as an exercise to check that this is a metric that is straight forth. You will have to use that the given and the two are metric that is d1 is a metric on x1 and d2 is a metric from x2. Using that, you will be able to d is a metric on x, on this x and again this is not the only way. I could have also taken d as maximum of these two numbers d1 x1, y1 and d2 x2, y2 that will also be a metric and not only that, several other ways of combining are also there.

For example, I can take 1 square plus d2 square and take the square root of the whole thing. That will also give rest to metric. So, there are several ways of introducing metric on the cost product of two metric spaces. Each of these will make this x as a metric, as a metric space. I think to begin with these examples are sufficient all the units to illustrate. There are several ways of constructing new metric spaces from the known already known metric spaces. One is you can regard every subset as a metric space and also cost product of cost metric spaces can be converted to metric spaces. Now, let us again go some general facts about metric spaces.

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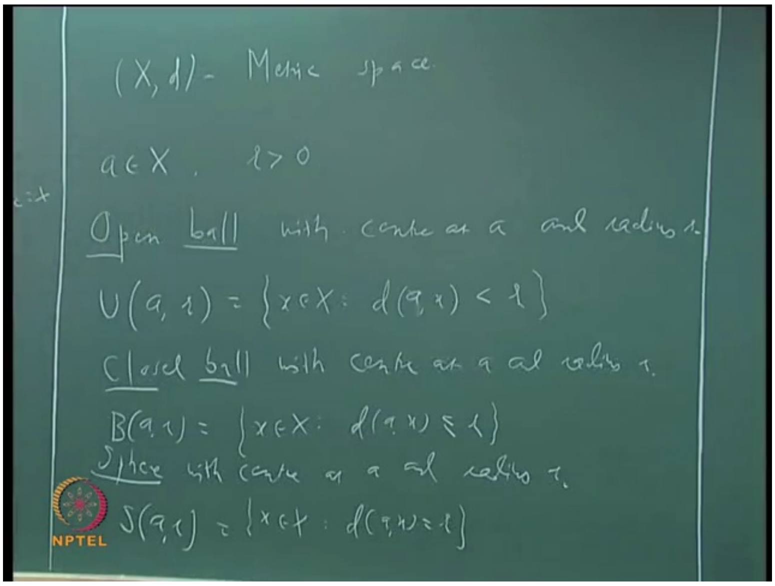

So, suppose X, d is a metric space. We do not want to consider some special sets. We begin with an arbitrary metric space. Then, we will look at examples in some concrete cases. So, suppose let us say a is any point in X and suppose r is any positive real number. We want to talk about what is meant by various sets, which depend on this point a and this number r. First of them is what we will call open ball, open ball with centre at a and radius r, so open ball with centre at a and radius r. Let us fix a notation for this because this something which should be using quite often, so suppose I call that U a, r.

Now, it is clear how such a thing should be defined because ball is a set up of all points we are taking a at the centre. So, you take all those points to centre from a is strictly less than r. So, that is what will we called open ball. So, this will be set of all x in X with the property that distance between a and x r a and xn x r a, whichever way you write, it makes no difference. So, distance between a and x is strictly less than r. That is what is called open ball. Similarly, we will define what is meant by closed ball, again closed ball with centre at a and radius r.

Suppose that there is B a, r. What should be that? Instead of this strict equality, we will take less than or equal to r. So, this S will be set of all x in X with the property that distance between a and x is less than or equal to r. The last set in this category is suppose I take all those points whose distance from a is exactly to r, a is exactly to r. Then, that will cause sphere with centre at a and radius r. So, we will say sphere with centre at a and radius r. That is a sphere. So, let us go give the notation. I will call, denoted by S a, r. That is set of all x in X, set of all x in X. So, in that, distance between a and x is equal to r.

Now, though we have called these things, open balls, closed balls etcetera. By these words, balls and sphere may form some image in your mind, but these sets may not look like that it various metric spaces. They will look like that in the standard metric spaces, but the arbitrary, but these open balls, closed balls can have any arbitrary shape. Look one more thing I should say here. For example, you know definitely is clear that these two sets are non empty because a certainly belongs to this of course because we taken r bigger than 0. It belongs to here also, but you cannot say anything about this because there may not be any point at a distance exactly equal to r from a.

So, sphere can be empty also. Sphere can be empty also, but open and closed balls will be always non empty. Now, of course, you might ask why we call that open ball and why we call that closed ball. There is reason for it. We shall come to that reason little later. There is one more thing I should mention here. It is that unfortunately, this terminology is not very standard. Different books allow different nomenclature for this because what we called open ball in Siemens books, you will see the same thing is called opens sphere and this called closed sphere etcetera. In ruling, you will see that the same thing is called a neighborhood of a and in some books, you may also called this is called open disk and it is called closed disk etcetera.

So, when you are reading any books, you just write in the beginning, you see what those definitions are and then go to those terminologies. They are followed for the sake of consistency. We shall follow this terminology throughout, this same terminology and this notation U for open balls, B for closed balls and S for the sphere. Now, let us see some examples how these various balls look like.

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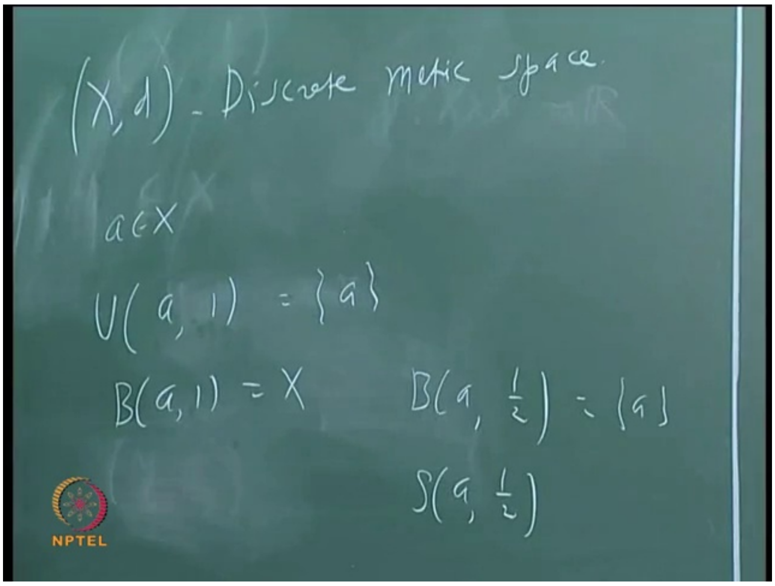

Let us begin with simplest case. Then, we suppose X is a discrete metric space. Suppose it is a discrete metric space. As I said discrete metric space is a metric space whose main use is to understand the concepts. What is the distance here that is distance between X and Y is 1 if X is different from Y and 0 is X is equals to Y. So, suppose now I take any a in X. Suppose I consider what is U a, 1. What will be that? It is the set of all points whose distance from a is strictly less than 1. What are those points? It is only a because distance from a to a is 0, distance of any other point is 1. So, this is open ball with centre at 1 is nothing but singleton a.

If I take numbers smaller than 1 suppose half or 1 by 4, it will be same. Suppose instead of 1, I take half. Then, it must be, it will be the same. It will be the same. There is no change, but if I take a is bigger, some something bigger than 1, then it will be whole of

X. Then, it will be whole of X. Similarly, you can say about this B a, 1. Why is X minus a? Distance between a and a is 0 that is less than or equal to 1. So, this is X. This is x all.

If you take any numbers smaller than 1, some any number smaller than 1, for example, what is, what is let us say B a of half. It is just a singleton. It is just singleton a. So, if you take any numbers strictly less than 1, this closed ball with centre at a and radius that numbers, it is just singleton a where there is no other point in between.

Now, since that we are considering this number, what is this S a half? It is a sphere with the centre at a and radius half. It will be empty set because there is no point with this time from a is exactly half. So, as I was saying in this, sphere can be empty. Sphere can be empty. Now, let us take little more familiar space, take usual R with the usual metric, usual metric in the space. So, what is usual metric?

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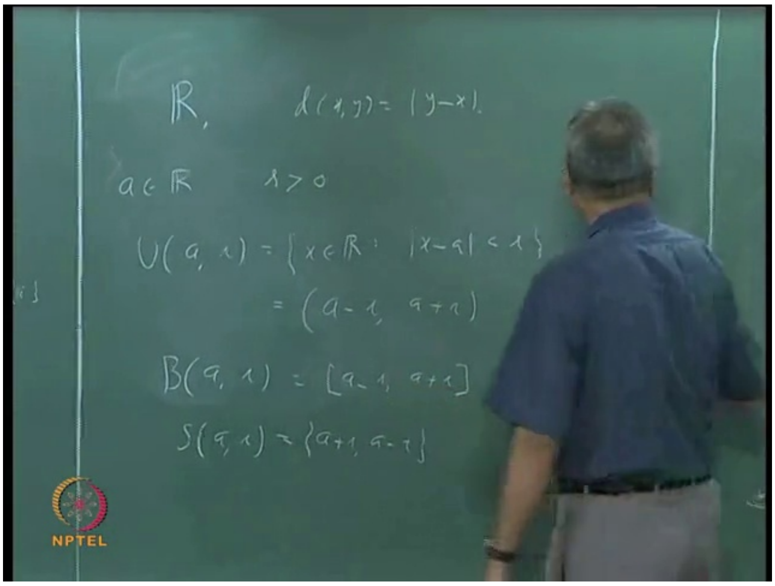

It is d x y is small y minus x d x minus and suppose you take a as any real number and r is bigger than 0, r is bigger than 0, then what is U a, r? If you look at definition, it is the set of all those real numbers, it is the set of all x in R such that distance between a and x should be strictly less than r. That is mod x minus a is strictly less than r.

What is this set? It is a set, opening interval, opening interval. Which is this opening interval? There is a minus r to a plus r that opening interval centre at a and lying this 2 r. So, it is this opening interval a minus r to a plus r. What about this B a, r? It will be the same interval, but the close interval, so it because n points are also because mod x minus less than or equal to r. So, it is a minus r to a plus r closed interval. What is S a, r? Yes, there is only because then you take only those points for which mod x minus a is equal to r. There are only two such points namely a plus r and a minus r.

So, it is a set of these two points here. So, it is all set of these points a plus r and a minus r. So, let me again remind you about my rework. Though we have called them open balls, closed balls, sphere etcetera and these words may create some image in your minds; you should just forget that because the balls may look like any. For example, this open ball is just an interval, whereas here it is just a point. So, the appearances can be quite different from whatever image these words may form in your mind.

Now, since here we have come across some discussion of intervals, let me also say of few things about the intervals because some of these things we shall require in our subsequent discussion. Let me just say what all the kinds of intervals that you are familiar with. We already saw, talked about these, open intervals, closed intervals.

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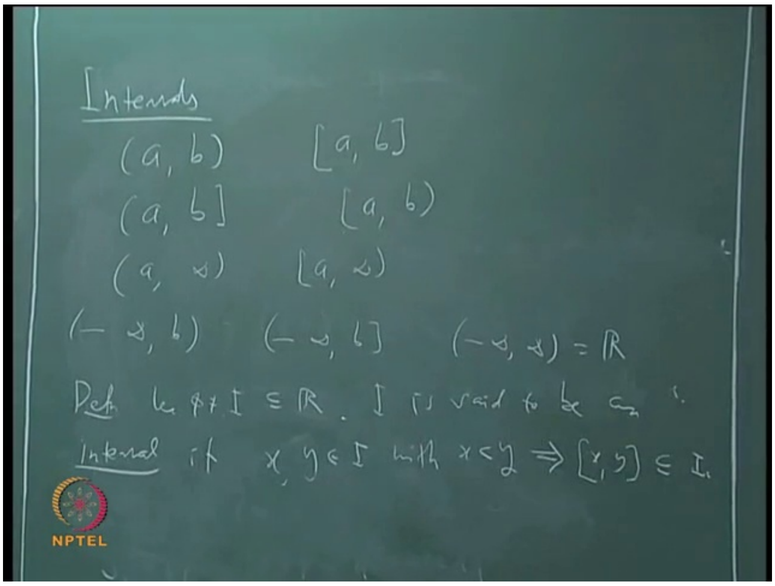

So, let me just say of few things about intervals. You know that this is what we have called open interval. Let us have a set of all x for which a less than x less than b. then, this is what you called closed interval where what they n points are also include, we can also think of including 1 and excluding the other. For example, it can be open at this side and close at this side that is also a interval. Similarly, the other way that is include a and exclude b, all these are intervals. What about something like this a to infinity? It is a set of all points, which is strictly bigger than that interval.

That is also an interval. What about this interval? This interval, when we say minus infinity to b that shows centre. Similarly, minus infinity to b that is also an interval and finally this minus infinity to infinity is nothing but what we could call r, what you call r is that an often interval. So, all these are examples of intervals, but what is it definition of interval? How do you define an interval which is of two points? For example, this we will use. You will say that this is not an interval, this is no n points, R is an interval or not. It has no sequence.

So, we have come across a pressurized definition of interval. In fact; that was the reason for going into, going for this discussion. See which subset of a real line is called interval that is a question. So, that question should include all the possibilities. When do you say that the subset of a real line is an interval? Now, let me give that the definition to you. Now, listen that you should have the following property that is if you take two real numbers say x and y and suppose x less than y, if these two numbers lie in the set, then all the numbers lying between x and y should also be there in that set. If what I said satisfies this property, it is called an interval.

So, let us just write on the definition. Let I be a subset of R. Suppose I call subset non empty subset R, since there are I is said to be an interval if x and y belongs to I with x less than y with x less than y. This should imply I is the set of, all the points lying between x and y should also be in I, which is same as saying that closed interval with n points x n y should also be in I. So, if x less than, if x and y belongs to I with x less than y, then this interval x to y that should be containing I.

So, this is the definition of an interval that is interval is a set if it contains any two points, it should contain all the points lying between those two points. Do all the sets satisfy this property? Are there any other types that can that an interval can take? Does it include all the possibilities? Can an interval be of any, something else other than whatever types we have written there? Of course, a and b can take different values. No, but is it obvious or does it need proof? I will not going to the proof. Of course, you have to prove it rigorously and I will give that proof to you as an exercise. I will just give you a hint about how to go about this.

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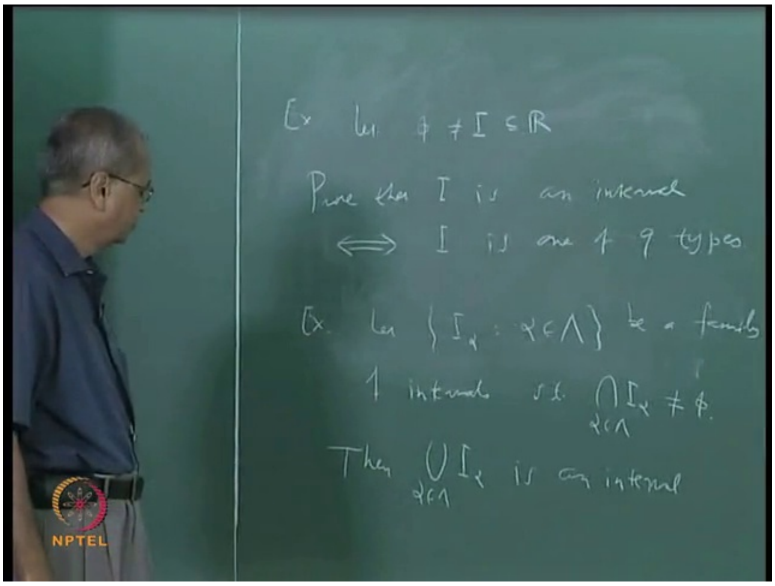

So, this is an exercise. What is the exercise? Let I be a subset of R, non empty subset. Prove that I is an interval if and only I is of how many types are there, 6 plus there are we have listed some 9 different types of interval, so if and only if I is one of the 9 types, open, close, semi open, semi close, one to one, or both, n points with that includes all the possible cases. Of course, this implies that this is trivial. That is what we have seen. If I is one of those sets, then I is an interval. That is clear. The real proof is required for the when I is an interval to show that it as to be one of these types.

Now, my hint is as follows that depends on LUB axiom. Again, consider the various cases. I is bounded above, not bounded above or bounded below, not bounded below and again various cases. That is this will have various combinations bounded above, but not below, below but not above and bounded on etcetera, etcetera. It will cover all these, all these cases. Then, if it bounded above, then that set must have least upper above. That set must have least upper above. Of course, that least upper above may or may not belong to I. Again, that will be decided, what is the type of the interval? Similarly, look at the greatest lower above.

So, everything will depend on the application of this one, least upper bound property and making use of the least upper bound property and various cases, you will be able to show that all the intervals are one of these types, one of these 9 types. The next question that we would like to know is that suppose we are given two intervals. Suppose you are given two intervals and you consider that union will that be an interval in general. For example, you take the interval 0 to 1 and then take the interval at set 2, you could see. If you take their union that will be not an interval, but instead of that, instead of 2 to 3, suppose I take half to 3, then that will be an interval. So, what is the difference now from the earlier two sets?

These two sets, these are not that is their intersection is non empty. Their intersection is non empty and this is something that you can prove about any family of intervals. So, that I am giving you as a test exercise. What is the exercise given in it is not just to intervals, it can be any family of intervals; given a family of intervals whose intersection is non empty. So, that union is again an interval. So, let us write in a usual format.

So, let us say let I alpha, alpha belonging to some indexing set lambda I, alpha belongs to some indexing set, lambda be a family of intervals, be a family of intervals such that intersection of this, all this I alpha, alpha in lambda, this intersection over the whole family is non empty. That is there is at least one point we just come across all that is non empty. Then, union I alpha, alpha in lambda is an interval there of course and I need not prove that. That is understood. That is exercise.

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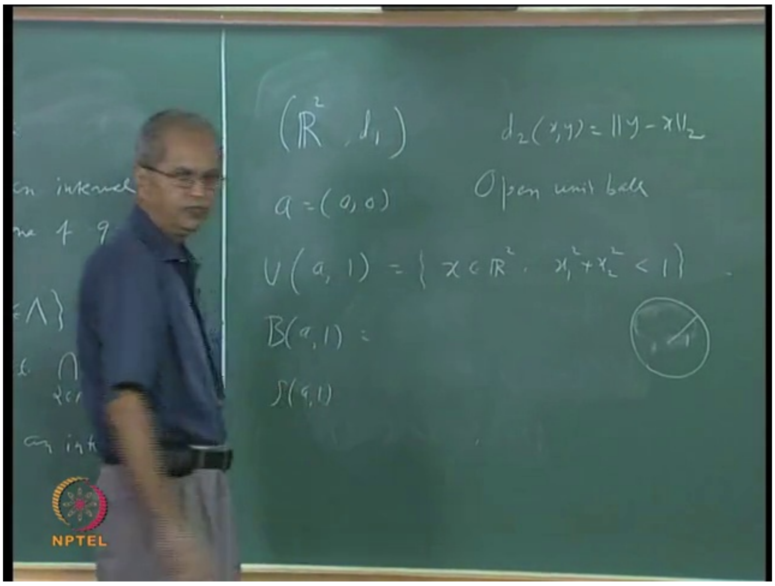

Now, let us go next familiar metric space namely R 2. As you know, R 2, R 3 etcetera, you can have various types of distances. Let us begin with the most familiar case. You take the distance, which comes from, which we had called d 2, so called Euclidean distance. That is given by norm suffix 2 that d 2 x y is norm of y minus x and where this is norm suffix 2. We will take a as the origin to begin with, a as 0, 0. Let us tale r as 1 as we taken in the earlier case. Suppose you take this U a, 1. So, it is a unit ball with centre at a; a is this origin and radius 1. a is origin and radius is 1.

So, what is this? By definition, it is the set of all points such that distance between x and a is less than 1. So, that is let us say, it is a set of all x in R 2 such that distance between x and a is nothing. Somehow if x is x1, x2, then distance between x and a is nothing but x1 square plus x2 square. The square root of that whole thing is less than 1. x1 square plus x2 square is less than 1. Its square root is less than 1 is the same as saying that this itself is less than 1. So, this is nothing but usually what you would call a region enclosed by the circle, region enclosed by the circle with the centre at origin and radius 1 of which is inside and excluding the points which lie on the circle.

That is the reason why some books call this disk, disk with the centre at a and radius 1. It is more popular in complex analysis books; they will call such sets as disk. What is B a, 1? The only thing here is it will be less than or equal to 1. That will be nothing but the region enclosed by the circle including the circle. It is say including the circle. What will be S a, 1? It will though we would not use those points where the distance is exactly equal to 1 that is the nothing but the circle. That is the nothing but the circle. Now, instead of R 2, if you take R 3, which is again the same distance d 2 which we call Euclidean distance, then that will be set of all x for which x1 square plus x2 square plus x3 square is strictly less than 1.

So, that will be the region enclosed by this sphere with this centre at 0, this one which looks like a ball. That is what we normally call it, a ball. So, if you have R 3 with the usual distance given by d 2 that is Euclidean distance, then the balls, all balls looks like ball and sphere looks like sphere. Now, let us make a small change here. For example, in R 3, open ball with centre at original radius 1 will be open ball and this close ball will the will the 1 which will also include the sphere and this sphere will be just the sake of all points which are exactly had a distance 1 from the original but, the way in these spaces r 2 r three etcetera when the centre is 0 and the radius is 1 those kind of balls are called unit balls.

For example, this is called open unit ball. Open unit ball means what the centre is the origin and the radius is 1. Similarly, closed unit ball means it is a closed ball with centre at the origin and radius 1. Similarly, unit sphere, unit sphere means with the centre at the origin and radius 1. Now, what we shall do now is that we will just make a small change here. Instead of taking this distance given by d 2, instead of taking this distance given by d 2, let us take the distance given by this d 1. What is d 1? d 1 will be this suffix 1. That is mod y1 minus x1 plus mod y2 minus x2. So, suppose now I look at U a, 1 it will not be this because this is not the distance.

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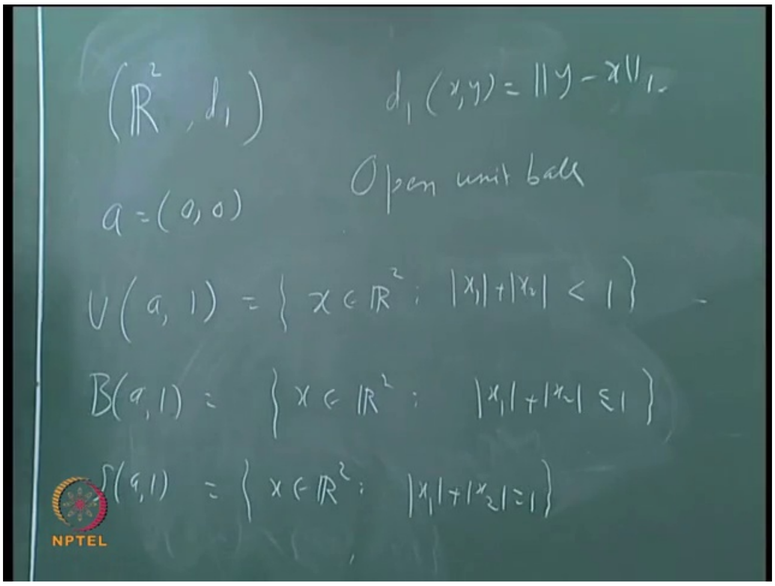

It is mod x1 plus mod x2 mod x1 minus 0 that is same mod x1. So, it will be mod x1 plus mod x2 less than 1. Similarly, this will be set of all x in R 2 with the proper mod x1 plus mod x2 less than or equal to 1. This last thing will be set of all x in R 2 with mod x1 plus mod x2 equal to 1. So, how will these sets look like? I think it is easiest to look this first. Once we fix this, all the others are easy.

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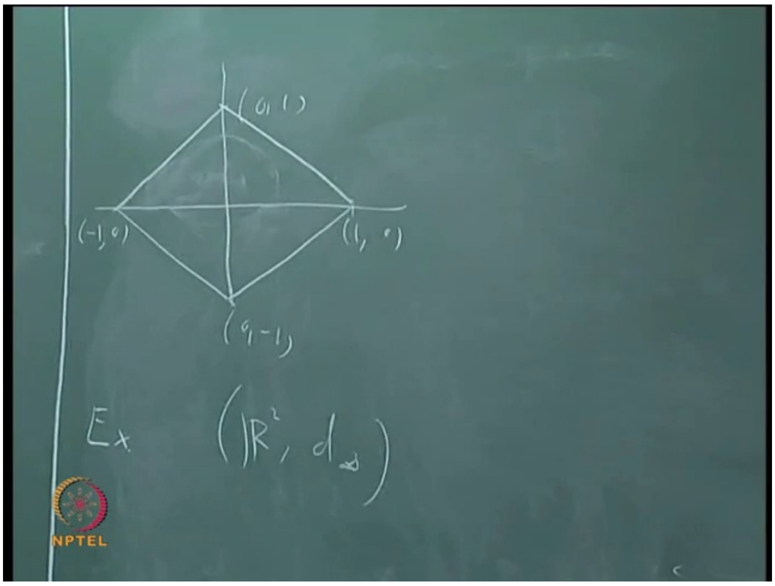

So, let us say this is the origin and look at this mod x1 plus mod x2 equal to 1. You may make various cases x1 bigger nor both or bigger nor equal to 0, both or less than or equal to 0 or one is bigger than 0 and other is less than 0. Let us start from the beginning. Suppose both are bigger than or equal to 0. Then, mod x1 plus mod x2 is nothing but x1 plus x2, so that is nothing but x1 plus x2 equal to 1. So, the set of all those points for which x1 plus x2 is equal to 1, what does that represent? It is a straight line. It is a straight line passing through the points 1, 0 and 0, 1. It is this straight line.

Now, take another case. Suppose x1 is positive and x2 is negative. Then, this will become x1 minus x2 equal to 1, sorry this one, x1 minus x2 equal to 1. What will that represent? It will be line. So, this is minus 1, 0 and 0, minus 1. Similarly, we are looking at the other two cases. For example, if both are negative, it will be x1 plus x2 is equal to minus 1. So, it will be passing through this point, these two points. So, this will be this sphere. This will be this sphere with the centre at 0 and radius 1. Once you decided what a sphere is, then what will be the open ball? It will be nothing but this set of all points which lie inside this excluding the point from the boundary.

Similarly, the closed ball will be all the points lying on this as well as points is same. So, the open balls, closed balls are etcetera, their appearance and shape will depend on what exactly is that metric. Is that clear to you? This is because since you will be using these concepts very often, so I am spending some time on this.

Now, let me give you an exercise. Similarly, draw the picture of these spheres. Instead of taking the distance d 1, take the distance d infinity, do similar things for R 2 with d infinity. What is d suffix infinity? Of course, it is norm y minus x suffix infinity. That is nothing but the maximum of mod y1 minus x1 and mod y2 minus. So, what will be the changes here? Instead of this mod x1 plus mod x2, you will get maximum of mod x1 mod x2. Then, see how the whole thing looks like. Then, see how the whole thing that I will leave it to as an exercise. We will just see one more example and then will go to board.

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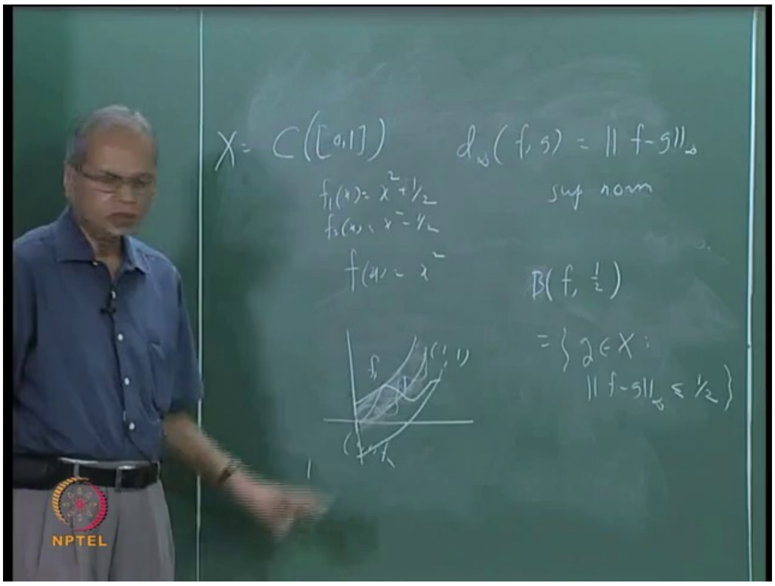

Let us look at this set C 0 to 1. We have that. That is a set of all continuous functions on the interval 0 to 1. We have taken distance here, d suffix infinity. Let us take this distance d suffix infinity. It is nothing but the norm of f minus this suffix infinity of between f and g is norm of f minus g suffix in this. This is what we have called sup norm. Sup norm is supreme of mod fx minus that is norm of f minus g supreme of mod f x minus g is where x is in 0 to 1. Suppose I take some function let us say f. Let us take for example, fx is equal to, let us say fx is equal to x, some function like this. So, that is an element of this. Let us also digraph this.

So, how will that look like? It is a function. So, this is the point. So, this is the point 0, 0 and this is the point 1, 1. It has to pass through these two points. So, it may look something like this. Now, suppose I want ball. Let me take closed ball, closed ball with centre at f and let us say radius let us say half. The radius, I will take half. What is this by definition? It should be this set of all those functions, set of all those functions whose distance from this given function g is less than or equal to half. Let us write it that is this is suppose, this is our metric space x. It is the set of all those functions g in x such that distance is d suffix infinity such that norm of f minus g suffix infinity is less than or equal to half.

Now, can we describe in terms of graphs? Suppose there is a function g and let us say it is a graph. It is something like this. It is a graph is something like this. What is distance between f and g by definition? See how it defined, at each point, you consider the difference fx minus gx. So, suppose this is difference. This is difference. So, you take this supreme. In fact, in this case, it will be maximum because it is a closed bounded interval. So, the maximum difference between fx and gx should be less than or equal to half. The maximum difference between fx and gx should be less than or equal to half.

Now, what will be the geometric way of describing all those functions in terms of graphs? For example, let me just take these two points. Suppose I take f 1 of x as let us say x square plus half. How will the graph of that look like? This at least is shifted above somehow this half. So, let us say this is f 1. Similarly, I will consider f let us say f 2 of x as x square minus half is nothing but at least shifted below. Now ,suppose you take any g whose graph lies in this, whose graph lies in this strip like any g. Then, will it follow the distance between f and g must be less nor equal to half?

So, in this case, closed ball with centre at f and radius half is nothing but all those functions whose graph lie between these two graphs, graphs of f 1 and graphs of f 2. That will be the closed ball, but what about the sphere? Suppose I want to digraph description of a sphere with the centre at f and radius half. What will be that? Yes or let me more specific. Will this sphere consist of only those two functions, f 1, f 2 less than that? Everybody agree with that? Why? It is suppose I take a function like this, yes, at this. See, the distance is a supreme case. So, the distance between such g and f will also be half, so sphere.

So, in the case of a sphere, what we will require is that you take those functions whose graph is at least once needs one of the functions f 1 and f 2; not just these two functions, all those functions whose graph needs the graph of f 1 or f 2 at least once, those will be the functions which are very good in this sphere. So, if you look at this example, you will see that this so called balls and spheres, they can have practically any arbitrary shape in various metric spaces. I think this is enough to begin with as far as examples are concerned. Let us now proceed little further with the theory.

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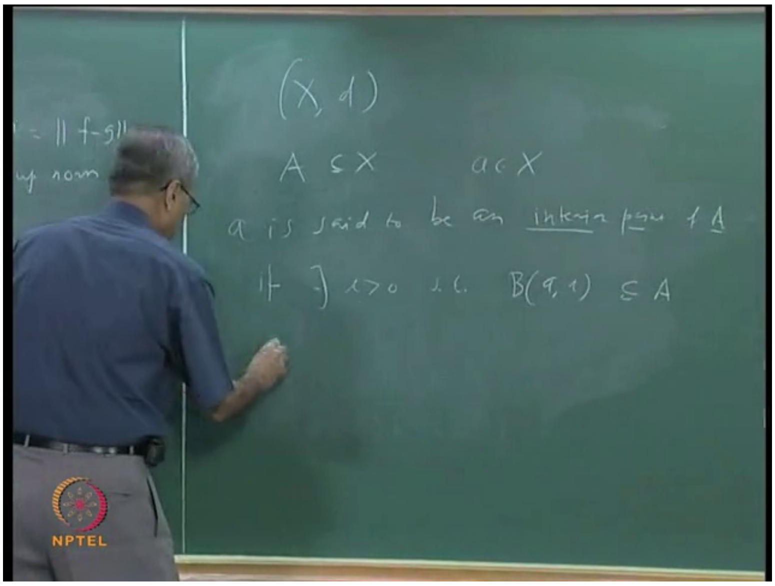

So, let us say we have again have metric space X and d. Now, what we want to consider is the position. Let us say given a subset. Suppose A is a subset of x and let us say we take some point. Suppose I call that point a, a in x, small a in x. This point may or may not belong to A. We will consider a general case. We shall say that this point a is an interior point of A. That is what we want to define.

So, we will say a is said to be an interior point of A, interior point of A. If you can, if let us say there exists some real number r bigger than the 0 such that open ball with centre at a and radius r is completely inside A, is completely inside A such that ball with centre at a and radius r is inside A. This obviously means that the point a also is inside A. So, every interior point is point of A. Suppose we collect all such points that will form a set. That will form a set. That set is called interior of A. That set is called interior of A.

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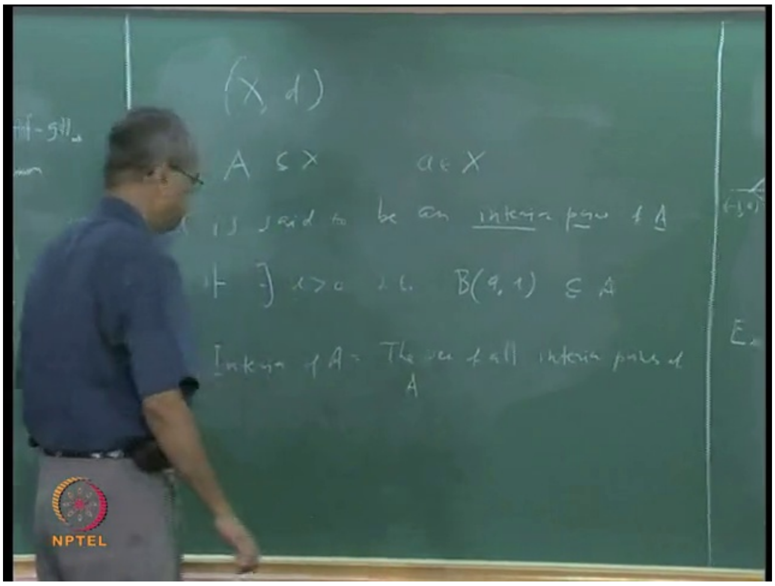

It is the set of all interior points of A.

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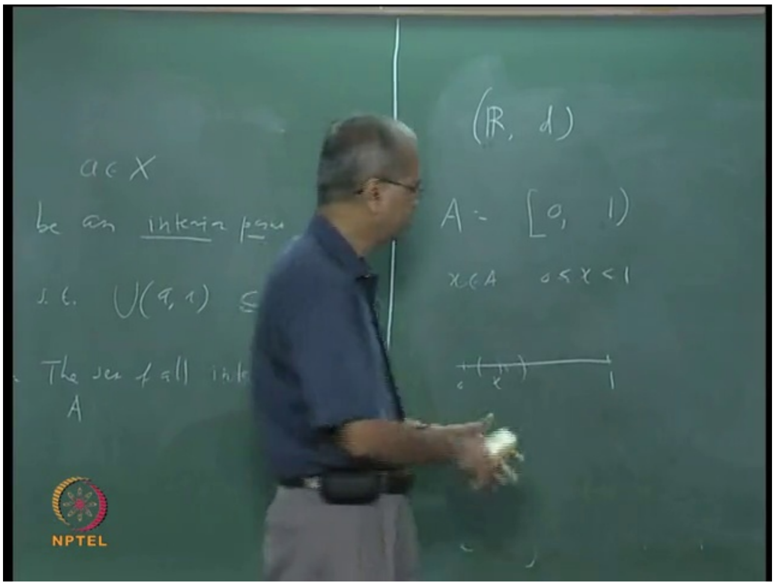

For example, let us take the set R with the usual metric. Let us take R with the usual metric. Yes, this should be open ball. That is right, open ball with separate a and radius r. So, that should be U. So, coming to back to this example, so suppose I take this set A as something like this. Suppose I take say 0 to 1. Now, suppose we ask that what are the interior points? Of course, remember since every interior point must be a point of, every interior point must lie inside A. We can forget about the points, which are outside A. For example, this point 1 not in A. So, there is no question.

So, coming back to the points set 0 to 1, so suppose you take some, let us say we take some point let us say x belongs to A, then 0 less nor equals to x less than 1. Let us forget about 0 for the time being. Suppose I take this one, 0 less than x less than 1, 0 less than x less than 1. Then, can we say that every such point must be an interior point and how does that follow because suppose this is 0 and that is 1. If I take any point x here, I can always find because we know that the ball with centre at x and radius r, we have already seen it ball with centre x radius r. That is nothing but the interval x minus r to x plus r.

So, the whole question is can we choose r such that the interval x minus r to x plus r lies completely inside A? The answer is obvious. You take x and 1 minus x take minimum of that and take any positive number, which is less than minimum of these two. Then, for that r, this interval x minus r to x plus r will lie completely inside A. So, every such x for which 0 is less than x less than 1 that is an interior point.

Now, which point is left out? It is the point 0. Now, is that an interior point? Why again x is not an interior point? This is because whatever r you take, whatever r you take, open ball centre at 0 and radius r that will be the ball of the form minus r to r. That will be the ball of the form minus r to r. Every such ball will contain some point which is outside A. so, this is not an interior point, 0 is not an interior point and rest all are interior points.

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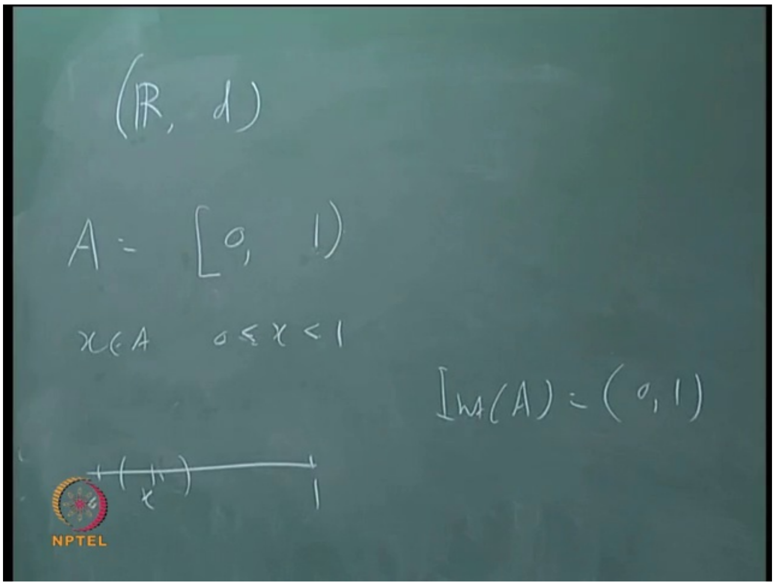

In other words, interior of A is this open interval 0 to 1. In this case interior of A is nothing but the open interval 0 to 1. Is it clear?

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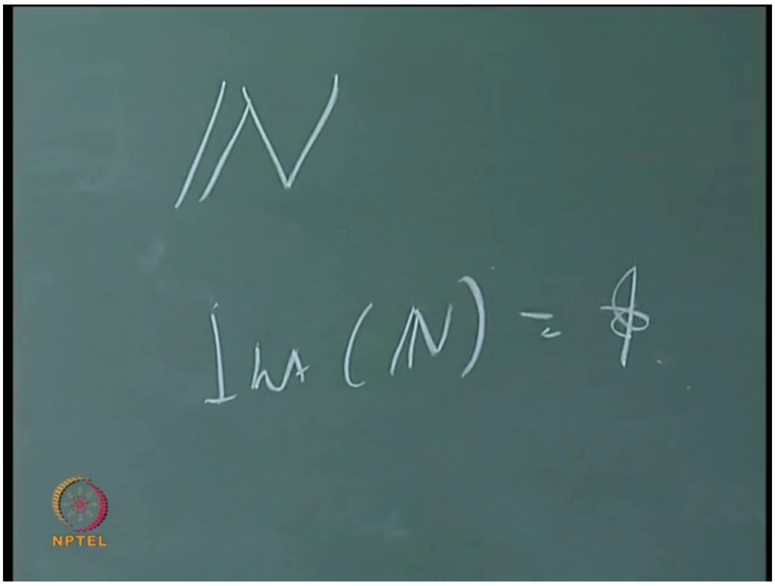

What about suppose I take this set N, set of all natural numbers? What is the interior of that? The interior of that is empty. The interior of that is empty because there is no interior point for that. So, interior of this N is empty. What about interior of Q? Yes, again it is empty set. Interior of the set of our national numbers again is an empty set. Suppose you take the set of all irrational numbers. Again, it is an empty set because again because the same thing which we have seen that between any two real numbers, there exists a rational numbers.

So, if you take any interval, if you take any interval that interval has to contain at least one rational number and at least one irrational number So, if you take simply the set of all rational points, its interior is empty. Similarly, if you take the set of all rational points, its interior is also empty. I think we will stop with that. We will continue with this in the next class.