Real Analysis

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Lecture - 15
Metric Spaces: Examples and Elementary Concepts

Well we had defined this norm, norm suffix p yesterday on these spaces that is Rn and Cn.

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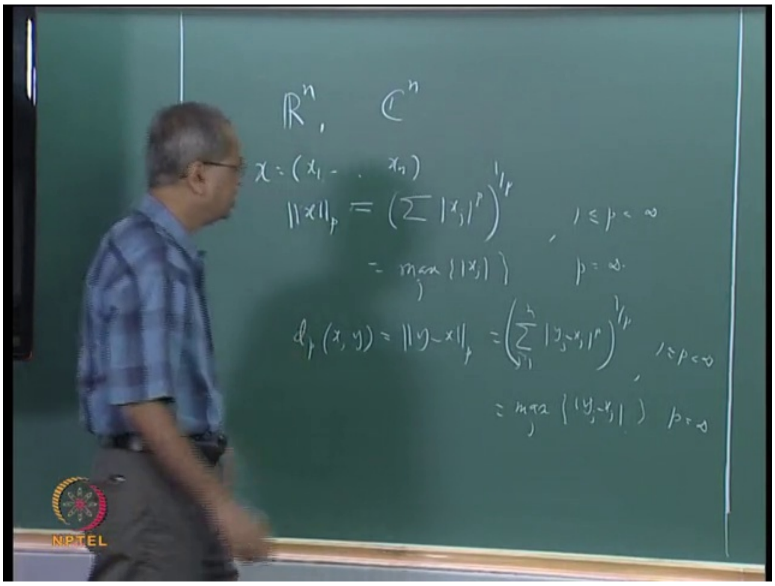

Let us again recall suppose take x as x 1, x 2, x n, then we have defined norm suffix p of this by this formula, sigma mod x j to the power p, whole thing rest to 1 by p if 1 less than or equal to p less than p less than infinity. This is maximum over j mod x j if p is equal to infinity. In all cases, we have proved this is a norm on Rn or if you take a similar if x is in Cn, it is a norm on Cn. Then, each of these norms will induce a metric on Rn.

So, we can denote that metric also by d suffix p d suffix of p of xy. We take that norm of xy suffix p that will be by our definition sigma j going from 1 to n mod yj minus xj, whole power p and this whole thing or 1 by p, if 1 less than or equal to p less than infinity. In a similar way, it will be maximum of mod yj minus xj, maximum taken over j for p equal to infinity. Each of this will be on Rn and similarly on Cn.

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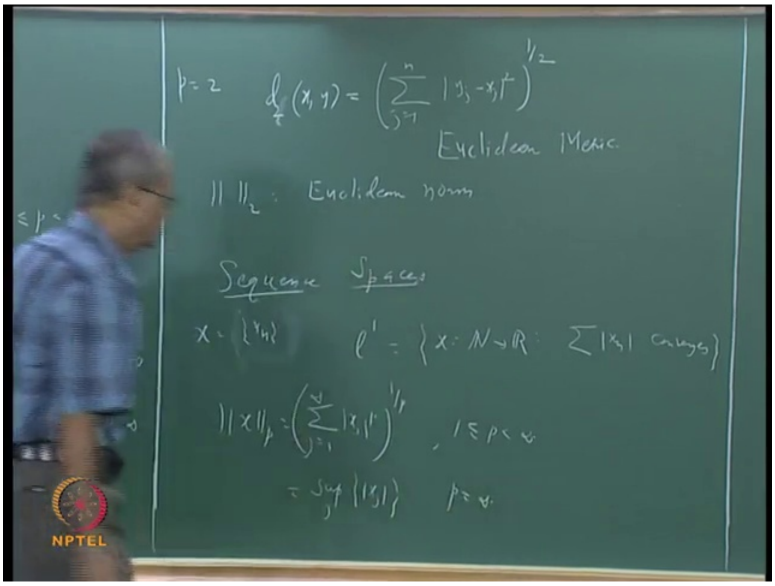

The case p equal to 2, let us take the case p equal to 2. Then, that is called let us say d suffix 2 xy that is not, but sigma j going from 1 to n mod y j minus x j square and then the whole thing that is square root of whole thing, this particular distance is most popular distance and that is that is called Euclidean distance or Euclidean metric. As you realize that if n is equal to 2 or 3, x and y are the points in R2 and R3 and the distance between those two points is given by this formula. That is usual distance that we have. That is why this is called usual metric or usual distance. It should be also clear to you that if n is equal to y and the whole thing is happening in R1, all of these will coincide. For various values of p, all this formulae will simply come down to just one single formula. So, this is about these spaces Rn and Cn with various metric. Similarly, for p equal to 2, the corresponding norm is also called Euclidean that is norm suffix 2 is also called Euclidean norm.

Now, let us consider slight extension of these things. Here, we are considering the spaces Rn and Cn. So, these are spaces where the points have n coordinates. Suppose we let the coordinate to be infinity or which is same as saying that we consider the sequences just n because these are of sequences. So, those are called sequence spaces or spaces of sequences. So, here the objects are each x is a sequence. So, we will denote x as this sequence x whose n th, n th element, n th element is x suffix n. The sequence can be of real numbers or complex numbers depending on the spaces as we are considering the real vector spaces or complex vector spaces.

Now, one of the spaces, we have already discussed that is the space we show as l 1. This was one of the first examples. What was the space? It was the space of all such sequences x that is x from N to R such that sigma mod xn converges that xn absolutely sigma xn is absolutely convergent. Now, in a similar way, as we have done here, we can define norm suffix p. We can define norm suffix p. The only thing is that now the sound instead of going j from 1 to n, they will go from 1 to infinity. They will go from 1 to infinity.

So, let us not define norm suffix p of x in the same way sigma j i going from 1 to infinity mod x to the power p, whole thing 1 by p, this is for 1 less than or equal to p less than infinity. Similarly, for p equal to infinity, here we had taken of maximum of mod x j. We could take maximum here because there are only n numbers here. So, it will have maximum. So, here there are infinitely many. We cannot talk of maximum. We cannot talk of maximum. So, we take supremum or least upper bound. So, this will be supremum over j mod x j. This is if p equal to infinity.

That is one more difference, this norm suffix p as defined here or l that is defined for every x. It is defined for every x, but this norm suffix p as defined p, here you can see that not defined for every sequence x. It is defined only if this is a convergent series. If sigma mod x j to the power p is the convergent series, then only we can talk of the norm x suffix p like that only.

Similarly, here also, you can talk of supremum of mod x j only if x j is a bounded sequence. If it is an unbounded sequence, again we can anything about norm suffix infinity. So, that is restriction. So, we cannot consider all possible sequences. We have to consider sequences for which the series sigma xj to the power p converges and in this case bounded spaces.

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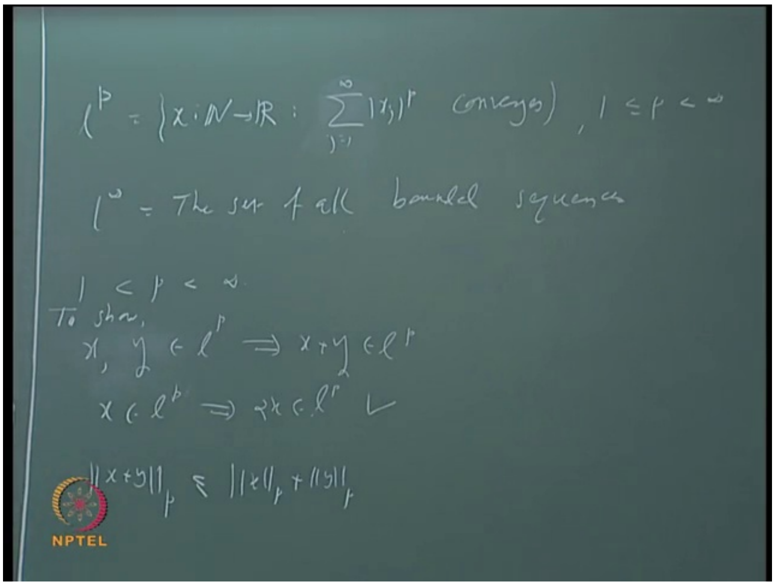

Those spaces are called l super script p that is this space. That is all those sequences x from let us say N to R for which this norm suffix p converges. Let us say I will write in this sigma mod xj to the power p, they are going from 1 to p converges by the way. Let me also say that when the series is of non negative terms, we have seen that the series is convergent. It is same as saying that sequence of partial sums are bounded. That is why, for example, what I have written here, many books will also find this notation sigma, mod xn strictly less than infinity. That means the same thing sigma mod xn converges.

So, similarly, mod x j to the power p strictly less than infinity that is the meeting of that it is a convergent series. So, for such x, we can now define norm x suffix p. Of course, this is for 1 less than or equal to p less than infinity if is that thing, but this set of all bounded sequences. Now, we have to verify each of these is a vector space. Each of these is a vector space and this defines the norm on that there only it becomes the metric. Now, that is easy. For example, 1, we have already verified p is equal to 1, we already verified. Similarly, p equal to infinity that is easy.

Suppose if xn and yn these are two bounded sequences, then xn plus yn is also bounded sequence. Similarly, alpha times xn is also bounded sequence, so in infinity, its vector space is trivial. Similarly, l 1 vector space follows from the properties of convergent series. Also that this is a norm as you have seen, as I have mentioned earlier also that where ever you want to change that something is norm, only thing that needs real checking is the triangle inequality, triangle inequality. Also, the case p equal to 1 and infinity is trivial here. p equal to 1, we have already seen.

p equal to infinity, what will that involve? You will have to take the supremum of xj plus yj. We show that it is less than or equal to supremum of mod x j plus supremum of mod j. That is also fairly straight forward thing to see. Let us just look at this particular case, 1 less than p less than infinity and p less than anything. Now, to show that this is a norm, what is that required to be shown? We will let us show this that is norm of x plus y; not just that we will have to show this that is first. First thing is this x and y belongs to lp. We must show from here x plus y also belongs to l that is what we need to show and also x belongs to lp employs alpha x belongs to l.

We also need to show that this is out of these two things, this is trivial. If x belongs to sigma mod x to the power of p is equal to convergent series, if I replace it by xj by alpha times xj mod alpha is nothing but mod alpha into mod xj that mod alpha will come outside. So, this part is trivial. There is nothing to do. We need to show this. We need to show this. This is also shown that norm of x plus y suffix p, this is less than or equal to norm x suffix p plus norm y suffix p. What we can do is if we show this last thing, this will also follow, this will also follow. So, let me just write this in the complete expanded form. What is the meaning of this?

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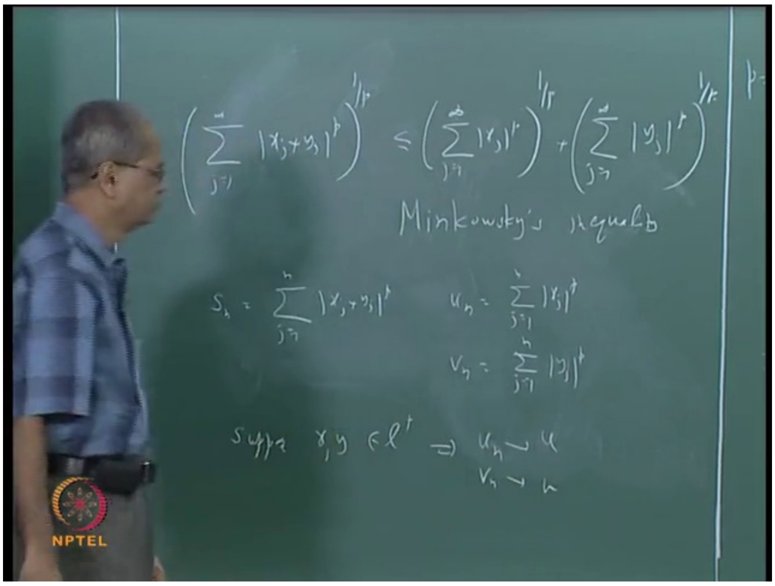

It means that norm of x plus y suffix is p is nothing but xj plus yj, this to the power p, this whole thing to the 1 by p, this is less than or equal to sigma mod xj to the power p j going from 1 to infinity, whole to the power 1 by p. Similarly, for sigma mod yj, whole to the power 1 by p. You remember we have proved for this to this. The only difference for it was going from 1 to n instead of 1 to infinity. If we replace j going to 1 to n, everywhere we have already proved this. That is what we have to call Minkowsky's inequality. Even this is also called Minkowsky's inequality. Since, we have observed earlier whatever we need to prove over the series, the only way to prove is going to the partial sums, sequence partial sums.

So, suppose we consider sequence of partial sums for the time. We let us say forget about this over 1. I will just consider partial sum of the series. So, what is that? It will be suppose I call Sn. Sn is sigma mod xj plus yj to the power p j going from 1 to n. Similarly, let us talk about use of the notation for the partial subsidies. Suppose I call as un sigma j going from 1 to n mod xj to the power p. Let vn is sigma j going from 1 to n mod j to the power p.

Let us now look at this suppose x and y are in lp. Then, suppose x and y belong to l. What does it mean? It means that sigma mod xj to the power p and sigma mod, they are convergent series. They are convergent series that means tends to something and vn tends to something, it has limit. So, this means let us say un tends to say v. Let us say, sorry, un tends to u and let vn tends to v. Now, do we know all the relations between Sn, un and vn? What is that that is Minkowsky? What we have proved is that this rests to 1 by p is less than or equal to 1 by p. That was the Minkowsky only finite sum. Let us use that.

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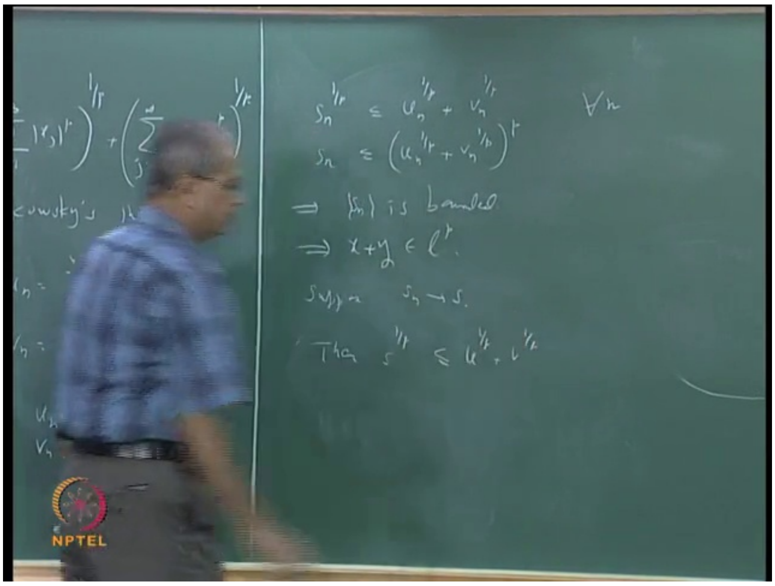

So, what we can say is that Sn rests to 1 by p is less than or equal to un rests to 1 by p plus vn rest to 1 by p. We can say suppose I take p forward for everywhere, I can say that Sn is less than or equal to un to the power 1 by p plus vn to the power 1 by p and whole thing is to p. Now, is the argument clear of this? Does this what you have on right hand side, what you have on right hand side, is it a convergent sequence etcetera? Does it mean it is a bounded sequence? Every convergent sequence is bounded. Does it mean Sn is a bounded sequence? Does it mean Sn converges? This is because I said is is not increasing because it is a partial series of non negative terms.

So, that proves that Sn is convergent. Once we say Sn converges, and then it is same as the x plus 1 belong to lp. So, we can, I will simply write this. This Sn is bounded. Sn is bounded and we have seen that in case of series of non negative terms, sequence of partial bounded in the series converges. That means this series converges, which is the same as saying x plus y belongs to lp.

So, this implies that x plus y belongs to l and we have already observed if x belongs to lp and alpha follows to lp; that shows that lp is a vector space. Now, the only thing that remains to be shown is this last thing, 1 x plus in etcetera. We need to prove this in Minkowsky inequality.

Now, you can see that once we know that Sn is a convergent sequence, then you go back to this. Now, this is true for every n. This is true for every n. so, the limit of n should be less than or equal to limit of whatever you have on the right hand side and we know all the limits. We know all the limits. Suppose the limit of Sn is S. So, Sn is bounded. So, suppose that it converges to S. Then, what follows from this equation here is Sn to the power n by S to the power 1 by p is less than or equal to the u to the power 1 by p plus v to the power 1 by p. But, that is same because this, whatever is here in the bracket here is S because Sn converges to this number that is the some other series. Similarly, this is u that is v, so each of these lp is normally near stress and hence a metric space. This will induce the metric. Now, just for the sake of understanding, let us see couple of examples of sequences, which belongs to some other spaces, do not belong these spaces etcetera.

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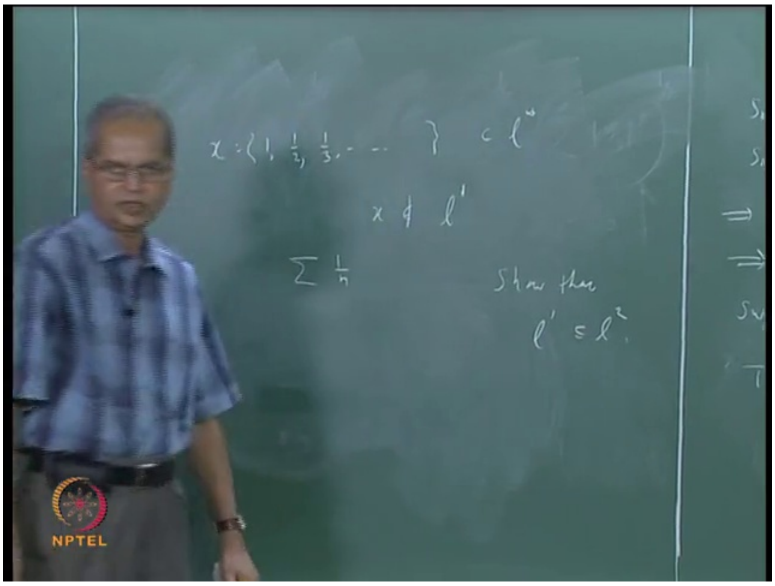

Let us take the sequence 1 by n, let us say x is this sequence 1, 1 by 2, 1 by 3 etcetera. Let me ask a question that x belongs to element in infinity. What is infinity? Is it a bounded sequence or not? It is bounded in the sequence; every of non degree terms is bigger than or equal to 0 less than or equal to 1. So, this belongs to l infinity. That is no problem. Does it belong to l 1? Yes. No. Why was sigma more than xj bounded by n sigma 1 by j. That is coinciding. So, it does not belong to x. It does not belong to l 1. What about l 2? Then, we will consider sigma n by 1 square sigma n by 1 square. So, it will belong to l 2.

So, remember when we decide Rn and Cn in the spaces are all same, but when we are talking about p, each p space, this is different for each, this space is different. They are all different metric spaces. Can you all also see that each of these lp square contains an infinity? If we noted, if sigma mod x j is to the power p is a convergent series, n term converges and every convergent sequence is bounded. So, each of these lp is contained in infinity. So, in infinity, biggest space among all this is, let me also give this as an exercise.

Show that l 1 is containing l 2. Show that l 1 is contained to l 2. Take this as an exercise. Those of who can do this, generalize this, take any two numbers p 1 and p 2, if p 1 is less than or equal to p 2, then l p 1 is contained in l p 2. The same idea will work of course. Do this after your quiz. Now, since an infinity is the biggest possible space of all these, it has some of well known sub spaces.

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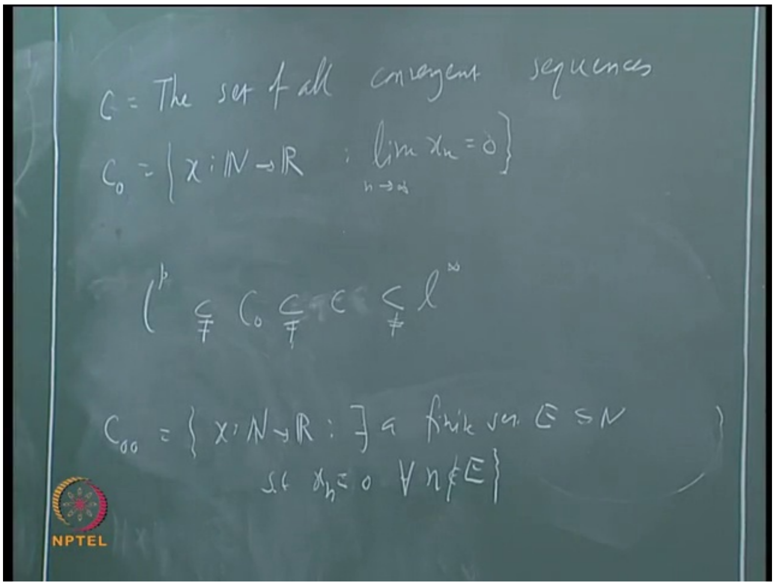

Let us also talk about these spaces. Those also have standard notations, the space we call small c, and this is the set of all convergent sequences. It is also a vector space. If xn and yn are two convergent sequences, xn plus yn alpha times xn that is also convergent sequence. Is it clear that that is contained of infinity? Every convergent sequence is bounded. Then, let me take one more called c naught, it is called c naught. See when we take c in the set of all convergent sequence, we do not bother about what limit of the sequence is.

What I have to do is take those sequences and converge to 0 only. So, let us have this sequence x from N into R, find the property of limit xn as n tends to infinity is 0 is that also vector space. That is clear if x tends to 0 alpha times x times to 0 and if xn and yn are two sequences, limit 0, xn plus yn also tend to 0 that is a vector space. What is the relationship between these two? c naught is containing c and c is containing infinity. This is the inclusion strict every time. We can say in fact, we can easily find the sequence, but bounded convergent. This is small c. Now, what about the relationship between all those lps and c naught.

If the series sigma x converges, then the sequence xn converges to 0. So, once we say that mod xj, it is a convergent series, it means that xj goes to 0. As it goes to infinity that means each of these lp is contained in c naught. Each of these l ps contained in c naught. This is inclusion strict. Again, you can say that this is yes. What is an example of this? Just take this sequence. This belongs to c naught. This belongs to l 1. So, this belongs to l 1. This is also equation is strict. Then, let me take one more space, which is also very important; by the way, this is just fairly standard followed in many books. The space which is called c naught will take two substrates.

This is the space of all those sequences where what we say it, but finite numbers of terms are 0. That means in a sequence, at the most, finite numbers of xn are non zero. So, we can say that so set of all x in x from N to R such that xn is equal to 0 for all n belongs to n minus E, where E is a finite set. That is where E is we can say that such that let us there exist a finite set E with a finite subset of n such that xn is not 0 for all n belonging to E. In fact, I should say other way actual this is a wrong description what I should say is that it is 0 except finite limit in n.

So, what should I say is this xn is 0 for all n outside E, xn is 0 for all n that is xn is 0 for all n not in E. That is the correct description. That is that exists, a finite subset of n such that outside the finite set, all xn are 0. Think over it. I will not explain it not because it will take some time. So, since E is finite set, E is a finite set, I can take some number which is bigger than some of all numbers in E, some number which is bigger than all of those numbers in E. Suppose I call that number n 0 that is n 0 is a number which maximum, bigger than all numbers in E. Then, what can I say about xn for bigger than or equal to 0? They are all 0. They are all 0.

So, that means what we can say about relationship in naught, naught and c naught, c naught, naught is contained in c naught. So, since that means from what I have said just now, remember we had seen in equality, eventually something happens. This is also described by all those sequences, which become 0 eventually all these sequences, which become 0, eventually 0 that is that is c naught, naught. What is the relationship between c naught, naught and lp. Is it clear, each c naught, naught is containing lp because finitely many of these terms are going to be non zero.

Everything else is 0 because if you take series like this, only finite numbers of terms are non zero. Of course, the finite number on terms may depend on a sequence. For a different sequence, may be different. So, c naught, naught is the smallest among all these spaces. As I said this is sometimes described as a space of all those sequences, which are 0, which are eventually 0 and this is something described as space of all sequences, space of all null sequences. In some books, they use these words, if a sequence goes to 0, it is called a null sequence and c naught is called space of all null sequences. So, I think this is enough for all the examples for these spaces of sequences.

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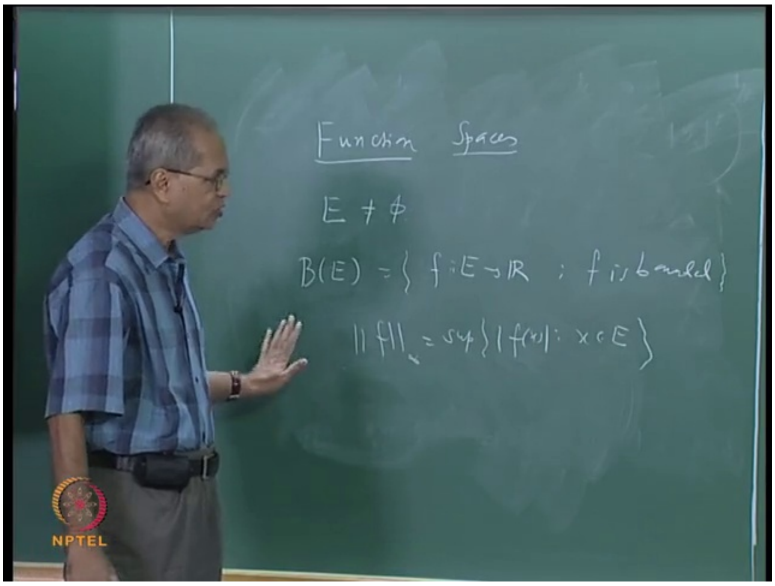

Let us go to the next type. Those are called functions spaces or function of spaces. Here it is also possible to go through various spaces like this E less than 0. This is also properly contained. Can you give me an example, which is c naught, naught, which is in same l 1, I said l 1, I have located as this and l 1, 1 by n is not this. So, give an example, which is c naught, naught, naught in which is in c naught, naught and l 1. That is right 1 by n is not in, it is not in c naught, naught also. So, what we want is an example, which belongs to n 1, but not in c naught, naught.

So, what is a sequence? What about 1 by n square 1 by square is in l 1 it is not in c naught, naught. If the infinitely terms are not wrong in c naught, naught, let me go to the spaces of functions. Suppose you say E is a non empty set. E is an not empty set and we can consider the set of all bounded functions from E to R. That is suppose I call easily, this notation is used B of E. B of E, this is set of all functions f from E to R and bounded, f is bounded. Then, if f is bounded, we shall define this for f in B. Define this norm f suffix infinity as supremum of mod fx, x belongs to E.

Suppose we have taken the x for which x belongs to E. Is it clear this supremum should be finite because we taking all bounded this should be exist? Some alpha sense like mod x is less than or equal to mod x in E in supremum exist as a finite real number. So, this is a well defined thing. It is easy to put this is a norm. It will set as all the property of norm, it is obvious to see that this is bigger than or equal to 0. If it is 0, each of these effects, it is also vector space set of all bounded function on a.

E is a vector space. E is an option, f by j is fx plus j etcetera, usually all are called plotline operations on functions. This becomes the metric space with respect to metric with induced by this. Is it also clear to you that this is called plot? Here is the special case of this B of E. In fact, function is a sequence, if you take E is equal to n that what you have called this B of E. Let me also consider one more space here.

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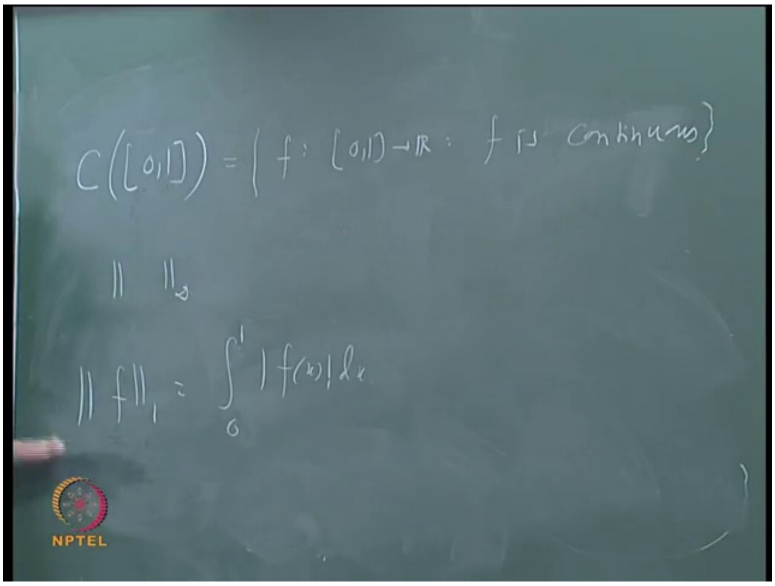

Let us assume that what is meant by continuous function on an interval. Let us say take this space, which is divided by c 0 1. We shall go to into detailed discussion of continuous functions of metric spaces later. But, you all know from your undergraduate courses what is meant by continuous function on 0, so we take option n from 0, 1 to R f is continuous. Is it obvious that this is vector space? It is a vector space. Do you all know that if a function is continuous on this, those involved interval, those are bounded again a well known property of this.

So, every function here is bounded. We do not have separate. It is ordered. So, you can define this non suffix infinity here. Also, we can define this non suffix infinity here also, but here we can define 1 and that is as follows. Let us assume that again we shall discuss in more detailed file in integration time. We assume that you all know that every continuous function is integral that is function is continuous. We can talk of its integral. So, we define a norm of f x follows, this is also derived as norm suffix 1 it is integral a to b mod fx dx 0 to 1. Now, is it clear to you that this is also a norm, this is also a norm? Then, there is only one problem here, which needs proof here that it is bigger than or equal to 0 is here. If it is 0, if the norm is 0, it means the integral mod fx is 0, but that will not immediately imply that mod fx is 0.

This is because it can happen integral is 0, but a function is non zero. But, that is avoided by continuity. If the function is non negative and continuous and if the integral is 0, then we can show that the function must be 0 everywhere. That is where this continuous property is used and that is the continuity is used. Then, all other properties are easy. For example, to show that norm of f plus g is less than or equal to norm of E will follow from the similar property of these integrals. So, this is also a norm linear space and that will also induce a metric. Just as we have done here, one can similarly define these spaces with various values for a. For example, I can take this to the power p and this whole integral to the power 1 by p. We can define various metric on this also.

Again, to show the triangular inequality, it will require proving some inequalities on the integrals. Those are also well known equalities, those are also minkowsky etcetera, but we shall not go to those things. So, I think now you know what sufficiently many metric spaces and large number of them are spaces, so that any concept of metric spaces here, you can look at all these various examples and try to understand what exactly is happening.

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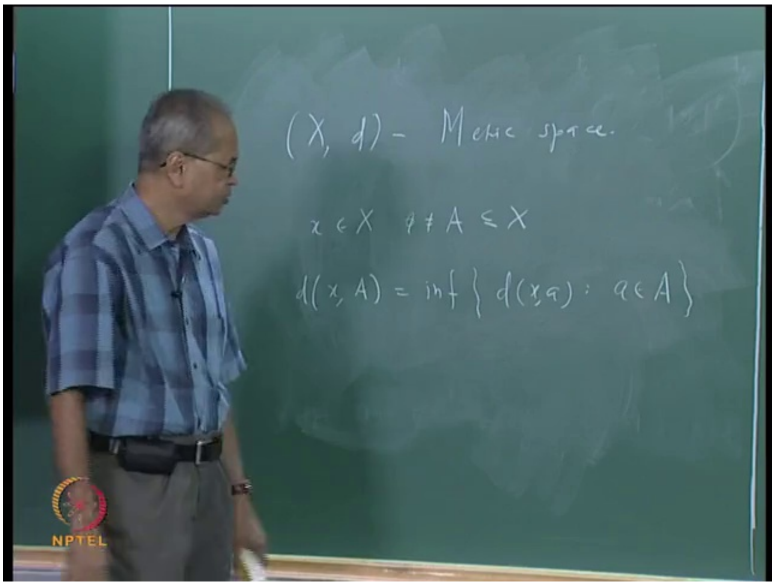

Let us go to some general theory of take an X, d; say X, d is a metric space. Once something is a metric space, this d is a function of on x plus, let me say given two points x and y, x y in X. I can talk over distance of x and y. I can talk about the distance between x and y. Now, let me take slightly different issue here. Suppose I take let us say x some in X and A is some subset of X. Let us to avoid a trinity, let us take a non empty subset. Suppose I want to talk about the distance between x and A. I want to talk about distance x and A.

What is the most natural way of defining because A will have so many points? A will have so many different points. So, what should be taken as differential distance between x and A. Infimum of the property of course one can take, do not exist, minimum may not exist. So, what we do is we take this infimum of all these points, distance between x and a taken over all small a belonging to and why does the infimum exist? Certainly, it is non empty set because we have taken non empty and distance is always bigger than 0. So, it is bounded below and every non empty set bounded below infinity follows from the axiom. So, this is a well defined thing. What if x belong to a? The distance will be this. What about the converse?

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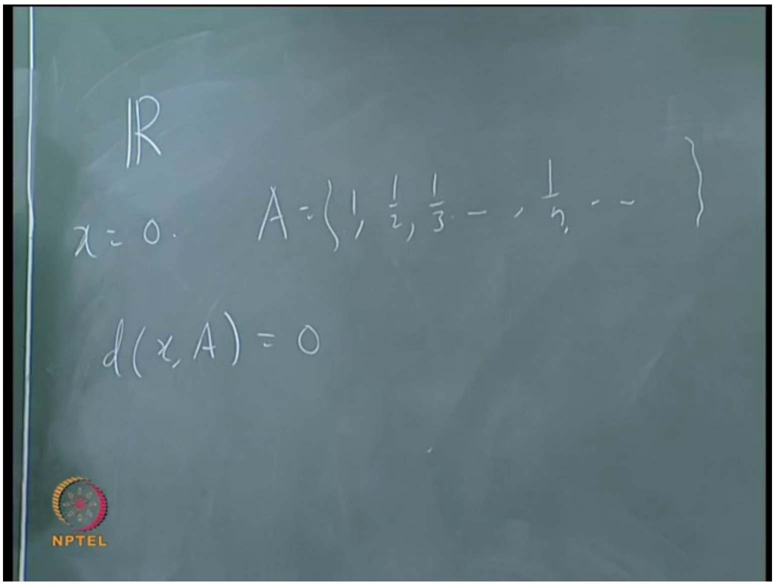

Let us take it R with the usual, this metric, let us take R with the usual metric. So, suppose I take x as 0 x as 0 and A as 1, 1 by 2, 1 by 3 etcetera 1 by n. So, what is distance between x and A? It is the infimum of mod 0 minus 1. So, it is 1 by 1. So, what is the infimum that is 0? Now, that x belong to A. So, it is possible that the distance between x and A is 0, but that x does not belong to A. So, the converse is false if x belongs to a distance is 0, but converges falls, it is possible that x does not belong to A, but distance is 0. Now, here I have taken a point and the set. Let us go to the next thing. I can also take two sets. Suppose I take two sets, A and B, let us say both are non empty sets. Anyway, A I have already taken.

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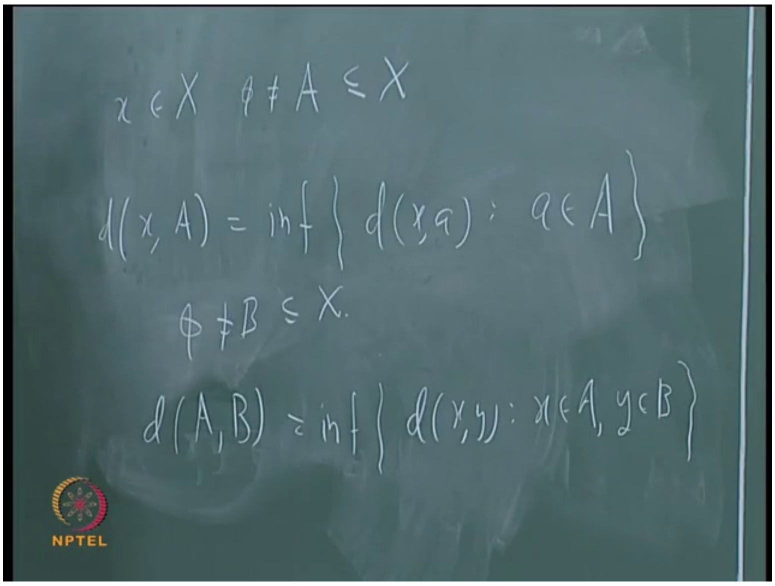

Let me take one more set non empty set B continuous to x. I want to talk about distance between A and B. Again, the most natural way of defining that is you take all possible pairs of points 1 in A, 1 in B, take infimum of all those numbers. So, it is, let us say infimum of distance between where x and y where x belongs to A, y belongs to b. Can you see that this is nothing but a special case of this? You can take here B as a singleton set x. So, that is the same thing.

Now, one more thing suppose A and B are non empty intersection that means suppose some point is common to both A and B. Then, what can we say about distance? Distance will be 0. Distance will be 0. Can you say converse also holds? Suppose distance is 0. Can we say that they must have at least one point? You can again take a similar example, there two sets can be disjoint still the distance may be 0; in fact, one can also consider better example there.

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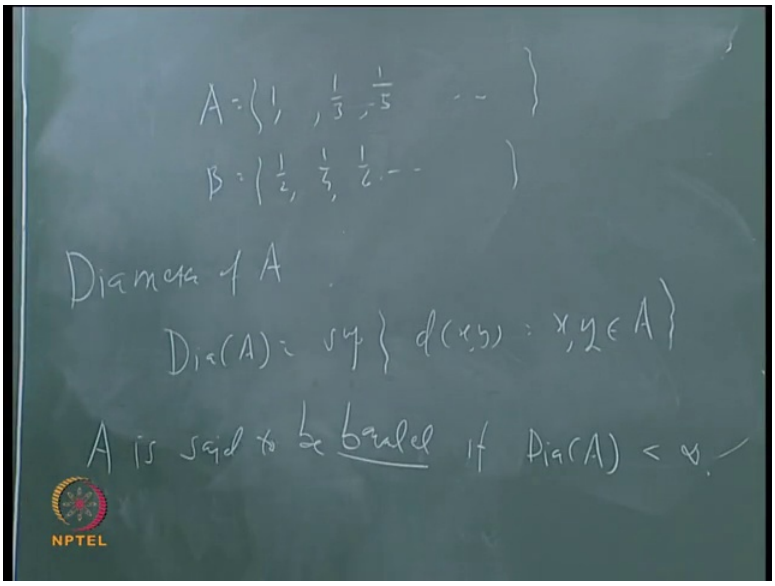

For example, suppose I take A as 1, 1 by 3, and 1 by 5 etcetera. Suppose I take B as 1 by 2, 1 by 4, and 1 by 6 etcetera. Then, these two are disjoint sets, but what about distance between the two sets? The distance is 0. Then, we also need one more concept what is called diameter of A. Again, if you look at the diameter of A of a disc, then it is nothing but the distance, the biggest possible between the two points. We basically use the same idea.

So, take any two points, look at the distance and take the supremum of all those. The diameter of A, denote by this diameter of A, that is supremum of distance between x and y. x and y goes for infinity. Wherever x and y go to infinity that take all possible pairs and take the corresponding distances and then that will be taken, the supremum that will be the diameter. The only thing is that we do not know whether this set is bounded above. We do not know whether this set is bounded above, bounded above. We can talk about this supremum. It is in spaces like this that extended real line comes into help.

So, what we can say is that if the set is unbounded, you take the supremum as infinity. You take the supremum as infinity. So, in that case, we set diameter that is A is infinity. If this set is unbounded, we take that diameter is infinity. Now, important definition, we will say that set A is bounded if diameter is finite. So, A is said to be bounded if diameter of A is finite real number set that A is strictly less than infinity, remember always bigger than or equal to 0. It will be always bigger than or equal to 0. If it is a finite real number, then this set is called bounded. Let us define what is meant by bounded set in real line. We will define say that set is bounded below, what is meant bounded above and we have said a set is bounded above as well as below that will be called the bounded set.

Now, we have extended that concept for any subset of any metric set. Given any metric space, you can say meant by bounded set. What is bounded set? Now, I shall give you an exercise when check that in the case of real line, this definition is bounded set coincides with the earlier definition of bounded set because we have definition. Suppose with the real line with a usual metric by this definition bounded set whose diameter is finite where as we have already defined by some other method. Now, check that these two definitions coincide. There should be also one more very important thing that you should realize.

I will just make and then I will just stop. This diameter these diameter depends on what is the metric. It is the distance between; it can happen the same set is bounded with the one metric and not bounded with the some other metric. Is it clear? Is it also clear that to you suppose we have defined what is meant by space metric? So, suppose A is subset of a discrete metric space. What can we say about this because this number has 0 and 1? So, the diameter is always less than or equal to 1. So, every set in the discrete metric space is bounded. Every non empty set in discrete metric space is bounded. So, we will stop.