Real Analysis

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Lecture - 14
Metric Spaces: Definition and Examples

So, we were discussing metrics spaces in the last class. So, let us recall a few things once again.

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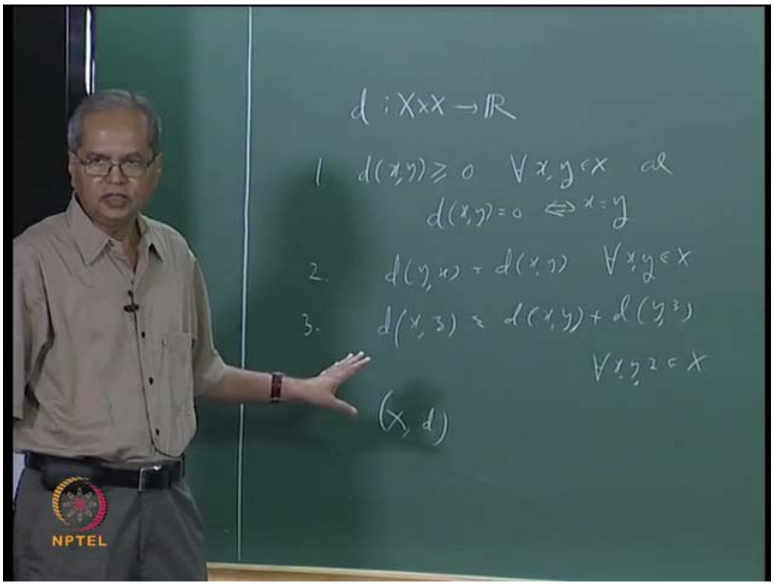

Recall that metrics metric is a function d from X cross X to R satisfying some properties, let us again recall those properties first property was this that this distance is always bigger than or equal to 0. This is for all x y in X and it is equal to 0 if and only if x equal to y equal to 0, if and only if x is equal to y and second thing is what is called symmetry that is distance between y and x is same as distance between x and y for all x y in X.

The last property which we had called triangle inequality that is if you take three points there is distance between x and z is less not equal to distance between x and y plus distance between y and z for every x y z in X. We had said that a metric space is a pair x, d where x is on empty set and d is a metric defined on this then we saw some examples for metric space one example is what we saw discrete metric discrete metric can be defined on any set.

The definition is fairly simple that a distance between any two points is 0 if the two points coincide and if 1 if the two, those two points are different that is a trivial example, but that is an example which helps in understanding several things and that we will see as we go ahead. Then coming to the other class of examples we had seen that one of the major class of examples of metrics spaces that are very important from the practical point of view.

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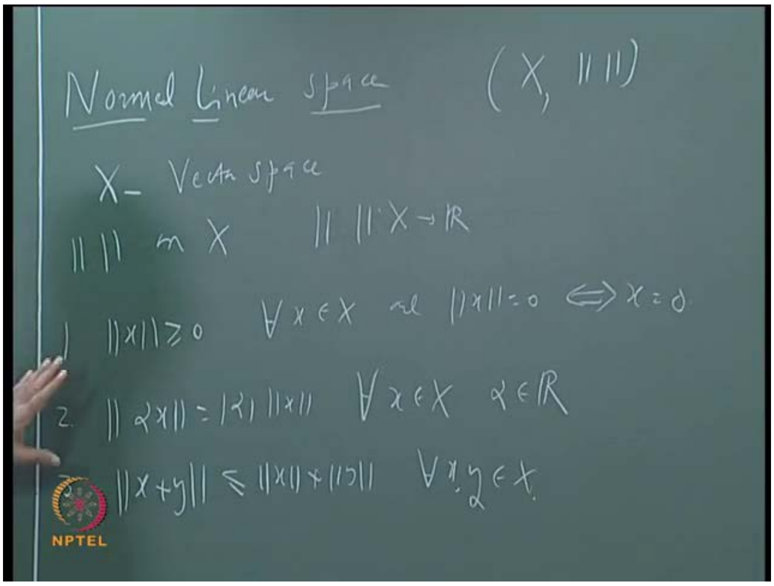

What are called normed linear spaces, now here in case of normed linear space the starting point is not an arbitrary set, but it is a vector space x should be a vector space. So, it means there is a meaning in saying the sum of two vectors product of a scalar and vector and all those things and we have we have also seen that a norm on X that is a function non set is going from X to R. That satisfies again a three similar properties like this the first property is norm of X is bigger not equal to 0 for all x in X and norm of X is equal to 0, if and only x is equal to 0 and second properties norm of I may or may not have written those properties in the same order.

But, the point is there are three properties that is alpha times x is mod alpha times norm X this is true for every x in X and alpha in R. And finally the property what we had called is triangle property that takes this form norm of x plus y is less not equal to norm x less norm y. This is true for every x y in X and norm linear space is again a pair X norm where x is a vector space and norm is a norm on it if it is a vector space over R, we call it is a real norm linear space if it is a vector space over C space, we call it is a complex norm linear space.

As far as these properties are concerned 1 and 3 this one depend on what are the scalar will R only this property two depends on what are the scalars. Here, only appropriate change will be instead of taking here alpha in R you have to take alpha in C. And what we have seen is that every norm linear space is a metric space, and how does it become a metric space.

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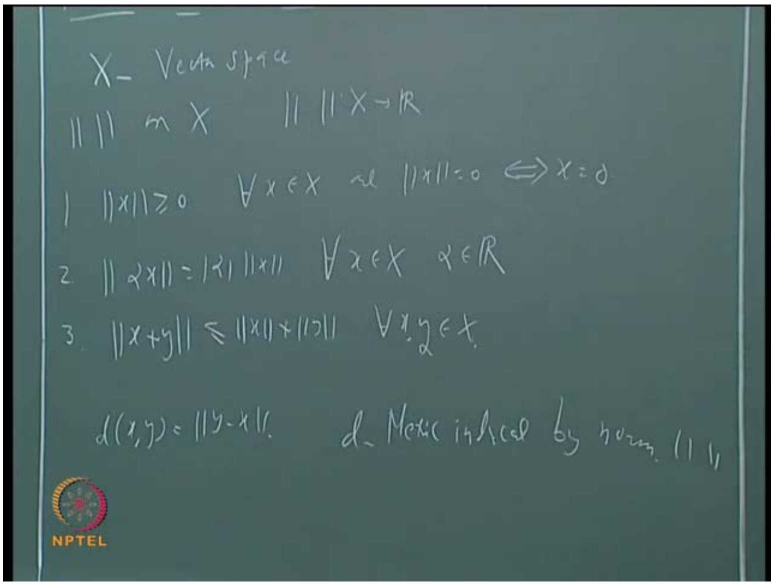

Taking any two vectors x and y we define distance between x and y adds norm y minus x, x norm y minus x and this d is called a metrics induce by this norm. So, we can say that d is called a metric induced by norm by the norm this norm if this is a relationship between the norm d see the metric d and this norm then we say that d is a metric induced by a norm. So, every norm induces a metric every norm on a vector space induces a metric and, so that gives us a big class of examples of course, here there is it is very natural to ask question whether every metric is induced by some norm.

But, of course that is obviously not true because metric can be there even on the arbitrary set metric can be define even on arbitrary set that need not be a vectors space even on a vectors space. Also there can be a metric we does not come from any norm right because if a metric is induce by a norm then it has to satisfies some extra properties, what are the extra properties.

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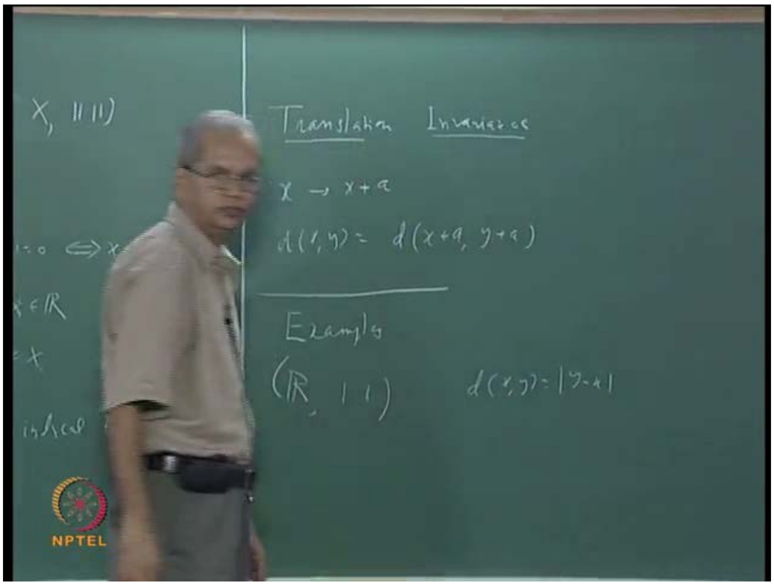

Let me mention here that extra properties what is called translation invariance translation invariance that is if metric is induced by a norm then that metric we say that metric is in variant under translation what is the meaning of translation. Translation means adding a fix vector to each that is the max which takes x to x plus a that is called a translation that means you add just a fix vector to all. So, suppose x goes to x plus a, y will go to y plus a then the distance between x and y should be same as distance between x plus a and y plus a.

That is in this case distance between x and y that will be same as distance between x plus a and y plus a why because this is norm of y minus x that is norm of y plus a minus x plus a because again same as y minus x. So, if you metric is induced by a norm that, so this is a property which is called translation in variance that is for any a distance between x and y is same, a distance between x plus a and y plus a this property may or may not be true of an arbitrary metric.

For example, in a vector space you can also give a discrete metric that will not satisfy this property or I would say that where I, when I that may or may not satisfy this property. So, we cannot imitates this so give it is possible that metric may not be induced by any norm, so one way of checking that is this similarly something will follow from this property also. But, I will not go into that discussion towards that is not very important right now let us know go to discussing various example that is concrete example of metric spaces and in particular I would say would since the measure class of example out of this state.

We shall discuss those example of norm linear space let us start with very well known examples obviously this R, R itself is metric is a norm linear space I would say that you take this absolute value function, here then this absolute value function satisfies all these three properties that something we already seen. So, that is a norm linear space hence an metric space because and this metric that is solve the metric will be d x y given by mode by minus x this metric is usually called usual metric on.

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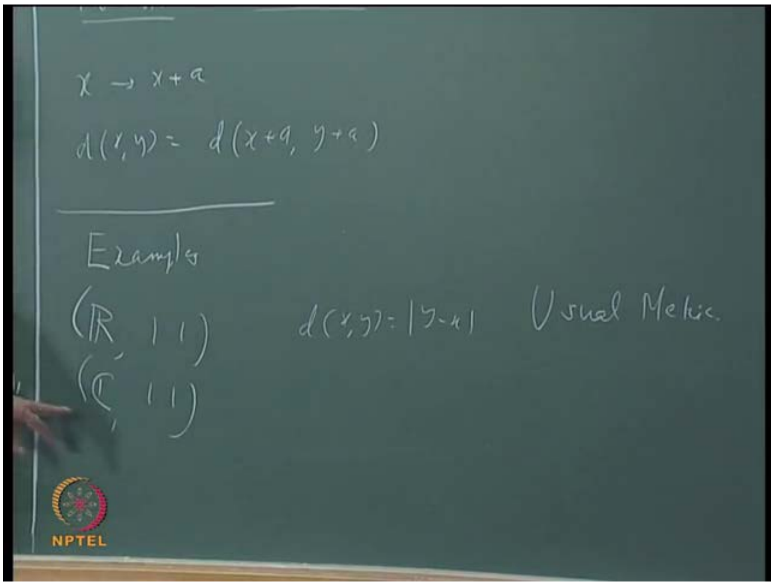

This is called usual metric or usual distance, now you can see that whatever we can do about R the same thing you can be done on C also, on C also you have absolute value function. So, you can also talk of absolute value of a complex number, so we can take this also and that is also called usual metric on C, that is also called usual metric on C why it is called usual metric because it is possible to define several other metric metrics it is.

So, is it for example clear for you that suppose I have such a function d suppose I have and suppose I define say d 1 of x y as two times d x y, let us say then that will also satisfy all this properties that is also satisfy all. So, that will be a new metric, so there is nothing unique about metrics several metrics can be defined on the same set, but as a metric space we regard those as different metrics spaces. Now, let us continue with the examples, now let us go to let somewhat had I mention instead of talking about R let us take R 2, R 3 excreta and similarly C 2, C 3 excreta.

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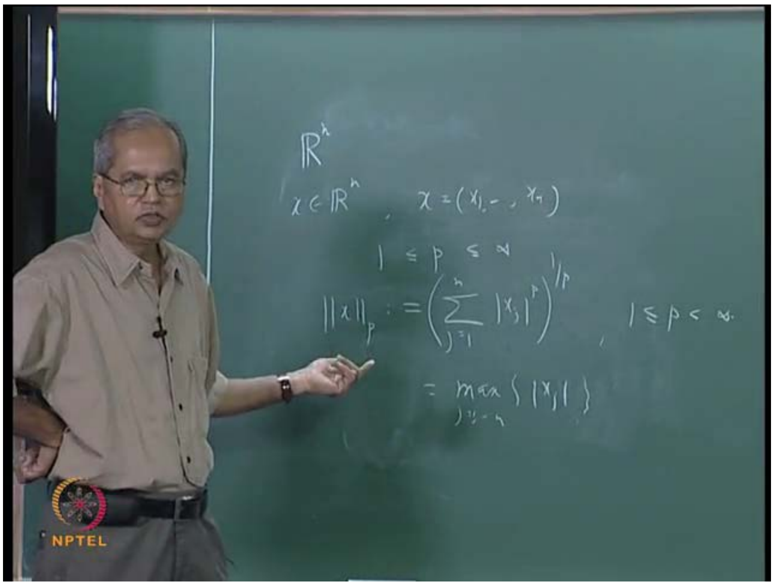

So, in general let me talk about R n R n, now what are the elements of R n, R n is nothing but the partition product of R with itself take an end time, so every element is ordered n triple. So, x belongs to R n this means x is of the form x 1, x 2, x n as you know these numbers x 1, x 2, x n they all are called co ordinates x j is the j th coordinate, so I will define a norm on this in fact what I would say is not just one norm. But, I will define what is called a family of norms, so to do that let me just take this number p let us say 1 less not equal to p less not equal to infinite and I will define an norm depending on this number p depending on this number p norm of x.

So, I will call it norm suffix p, norm suffix p and this defined as follows it is sigma j going from 1 to n mod x j raise to the power p whole to the rest to the whole thing rest to the power 1 by p wait, wait I will come to that. So, this is for this particular definite for 1 less not equal to p less than infinity and for p equal to infinity we defined that as maximum of mod x j equal to 1 to, so this last norm is called norm suffix infinity. Now, and all these others are non suffix p for 1 less nor equal p less then strictly less than infinity, so these are called p norms or l p norms p norms are sometime also called l p norms of course we have to prove that these are the norms that is something we are not proved yet.

But, as I told you proving that these are the norms we have to verify these three properties and out of these three properties, these two are very trivial those follows more or less immediately from the round only. Only thing that we will require some work is this property 3 which is called triangle inequality which is called triangle inequality in this case also for example some of well take this case suppose this is each of this bigger nor equal 0. Whatever definition you take suppose it is equal to 0 then this summation is equal to 0, but it is a sum of non negative numbers, so if that is 0 each of them must be 0 so that gives it each x j must be 0.

Similarly, if the maximum of mod x j is 0 then each mod x j is must be 0, so each x j must be 0, so that is easy to verify similarly suppose you take alpha time 6 that then the coordinates are alpha x will alpha x n if you put it, here that mod 5 will come outside. So, second property is also very easy to verify, so what remains to verifies the third property namely the triangle equality and it is that property which require some work and there also you will see that it is very easy to check that also for these two case.

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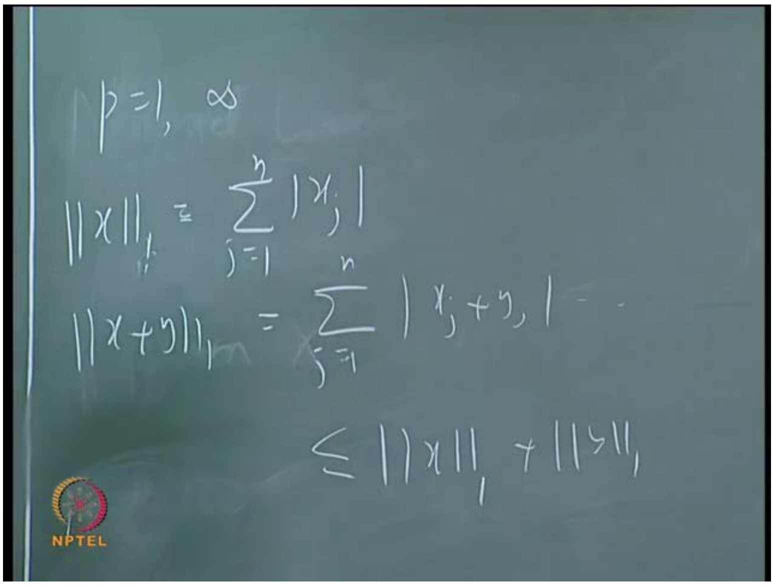

The case p equal to 1 and p equal to infinity these two case this triangle inequality is also very easy because for p equal to 1 what happens let us call this norm suffix 1 then it nothing but sigma mode x j sigma mod x z. So, what is norm of x plus y suffix 1, so this is sigma j equal 1 to n mod x j plus 5 j and what is to be done after this mod x j plus y j is less not equal to mod x j mod y j.

So, separate the two sums and you get you get that is the right hand side, so continue this will be less not equal to norm suffix one plus norm y suffix 1 in a similarly further case p equal to infinity this norm suffix infinity will be maximum of mod x j plus y j and then you can easily show that. That is since each for each j mod x j plus y j is less not equal to mod x j plus mode y j by taking maximum of all this quantities you will be able to show that this is also true that norm of x plus y suffix infinity is less not equal to norm x plus norm for p equal to infinity, so this two cases are easy.

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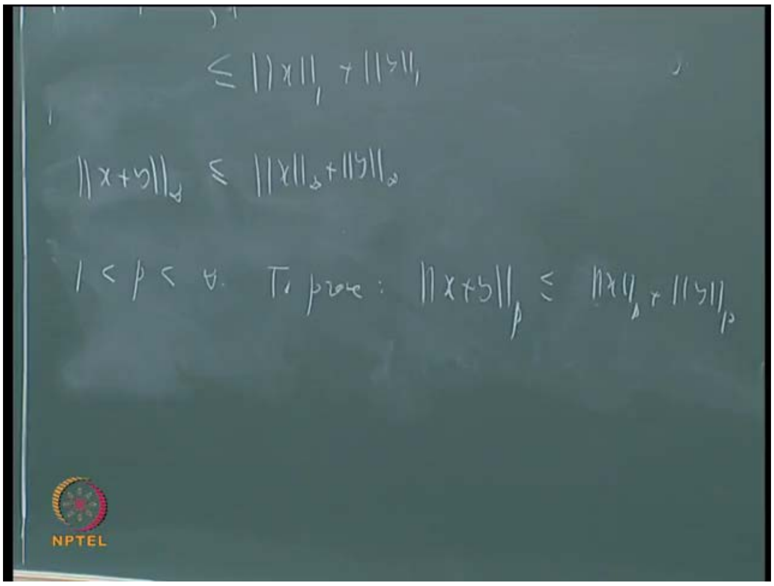

What requires some not revival is the case when one strictly less than p strictly less than infinity and in this case to prove that to prove that norm of x plus y. That means what is to prove in this case, we need to prove this norm of x plus y suffix p is less not equal to norm suffix p plus norm y suffix p and let me write this in the full form instead of this.

So, what is the, it is mod x j plus y j is the raise to p j going combined to a and then this whole thing raised to one by p this should be less not equal to sigma mod x j to the power p this raise to 1 by p plus again. Here, same thing for y j sigma mod, but j to the power p j going from 1 to a whole to the power 1 by p and this what I have written here is a very famous inequality about the real numbers it is called Minkowskys inequality. Suppose we prove this inequality then it will follow that this is true this is true and then from there it will follow that this is a norm it will follow that this is a norm, now will try to prove that in proving this.

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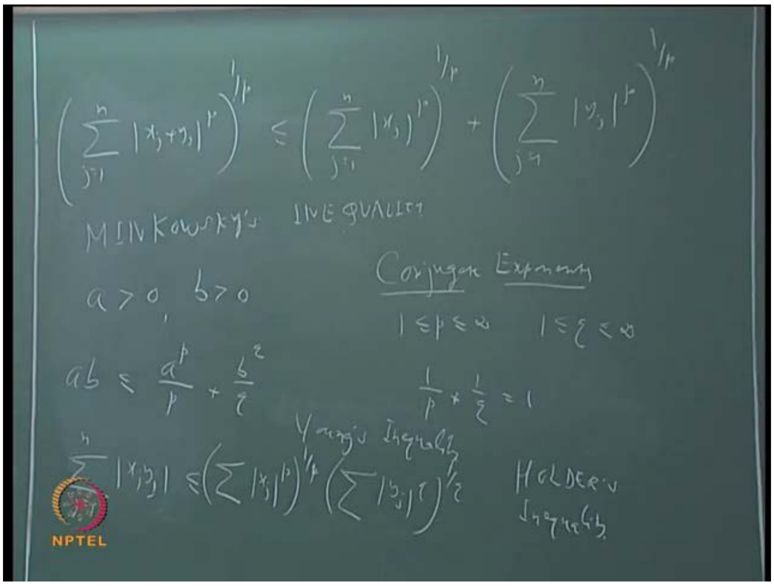

We have to go through sums steps there are some inequalities which we require to prove this inequality and those we have to prove in between. So, for that suppose we take to begin with let us say we take two numbers a bigger than 0 and b bigger than 0 and for that we also need a notion of what is called conjugant exponents. Conjugant exponents, conjugant exponents to this p is called an exponent in number 1 less not p less, so this p because we are taking the power p-th power excreta.

So, conjugant exponent to p means what it is again a number like between in 1 infinity this again a like number lying between 1 and infinity and satisfying this that is we have these two numbers 1 less not equal to p less not equal to infinity and 1 less not equal to q less not equal to infinity. If these two number satisfy this property 1 by p plus 1 by q is equal to 1, 1 by p plus 1 by q is equal to 1 then p and q are called conjugant exponents of each other. Most obvious case is when p and q both are 2, if p is 2 and q is also 2 then this is one that is the symmetric case, so 2 is a conjugant exponent of itself, so from this that if p is less than 2 q will be bigger than two and vice versa.

By convention we take if p is 1 q is infinity because if p is 1 this number become 1, so this 2 is satisfied q has to b infinity and vice versa. So, we take 1 an infinity as the conjugant exponent of each other and for 1 less than p less than infinity it is defined by this, so where is that coming to picture here. So, it is this inverse if you take positive numbers then this a b less not equal to a to the power p divided by p plus b to the power q is divided by q this is sometimes called Young’s inequality. Let me also write one more equality it is, it is like this if you take sigma nor x j y j, j going from 1 to n sigma x j by then is less not equal to sigma mod x j to the power p, I will forget about j going from and bounded this j goes from 1 to n every wherever this mentions.

Here, this raise to 1 by p plus, sorry not plus multiplied by sigma norm by j to the power q this whole thing raise to 1 by q where p and q are conjugant exponents of each other alright this last inequality that is called Holders inequality. Of course it is if you if you remember that norms suffix p defined like this then Minkowskys inequality is nothing but norm of x plus y is less not equal to norm x into norm y, one can give a similar way of remembering this Holders inequality, see if you take the vector x as x 1, x 2, x n.

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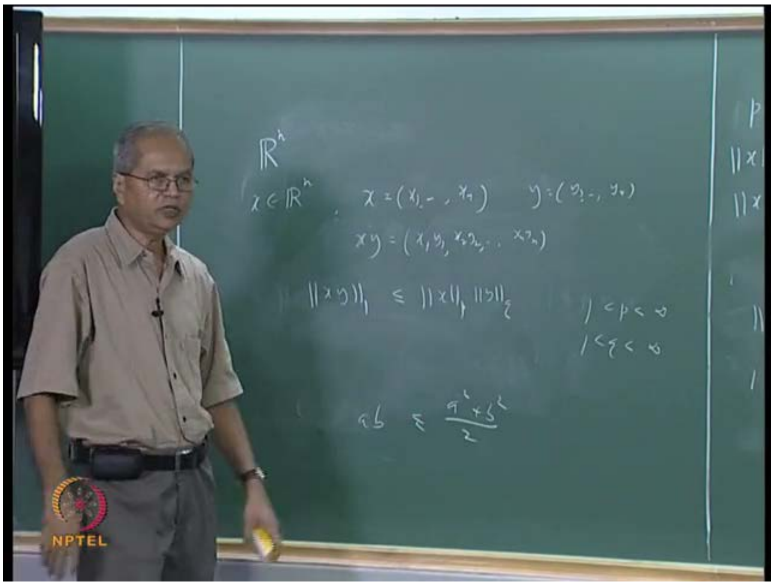

Suppose, we take y as y 1, y 2, y n and suppose it write the vector x y as the vector that you obtained by just co ordinate wise multiplication right just multiply co ordinate wise and write that is an vector x y. So, it will be x 1 y 1, x 2 y 2 excreta x n y n then what we have, here left hand side is nothing but the one norm, one norm of X 1 right this is p norm of x and that is q norm of y. So, another way of writing holders inequality which is easy to remember this that is norm of x y suffix one this is less not equal to normic suffix p plus norm y suffix q and this is true for all possible p and q where p and q are conjugant exponents of each other.

All possible p and q where p and q are conjugant exponents of each other and from what we have seen earlier to prove these inequalities for p equal to 1 and infinity is easy. So, we will you I will give that case to you for proving that for p equal to 1 and p equal to infinity that is easy, so we shall only discuss this particular case 1 less than p less than infinity.

So, that will also imply 1 less than q less than infinity if p lies strictly between 1 and infinity q also lies strictly between 1 and infinity let us start from this first. So, called Young’s inequality is it also creative that if p and q both are two there conjugant exponent then this becomes a well known inequality suppose p and q both are 2 then that simply becomes a b is less not equal to a square plus b square by 2. That is very easy to prove s square this is same as seen that 0 is less not equal to a square plus b square minus 2 a b that is a square minus plus b square minus two b is nothing but a minus b whole square.

So, that is always bigger not equal to 0, so that case it is very easy to prove and in fact you will see that always inequalities will also fairly easy to prove when p and q are equal to 2, general case will involve some work. But, again not very difficult basically we use the methods of elementary calculus to prove this required that is why we take some function and show that that function increases or decreases in some intervals and take an appropriate values of t.

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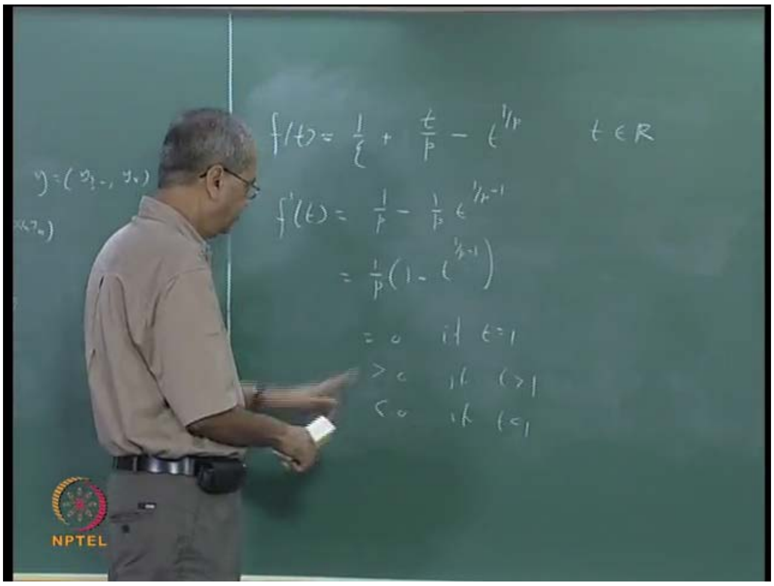

For example suppose of this function f t is equal to 1 by q plus t by p minus t to the power 1 by 2 for t in R then, so look at, look at that the derivative f prime t derivative this will be 0. So, it will 1 by p minus derivative remember 1 by p is bigger than 1, sorry not bigger than 1 less than 1, so it 1 by, 1 by p or derivate this 1 by, minus 1 by p into t raise to 1 by p minus 1. So, that is 1 by into 1 minus t raise to 1 by p minus 1, now it is obviously 0 if t equal to 1 at other places where it is 0 or not depends on what is this exponent 1 by p minus 1.

Now, 1 by p minus 1 is a negative exponent if p is bigger than 1, 1 by p less than 1, but of course it is basically same as saying that it is 1 by t, so if, so what it means is that for one of the cases it will be bigger not equal to 0 and for the other case it will be less not, so less not equal to 0. So, what is the what is the case tell me when will it be bigger not equal to 0 it will bigger not equal to 0 if and less nor 0 if t is greater than 1 yes and less not equal to 0 t is less than 1 in fact, here you have strict inequality actually right just check. So, what follows from this that the function is 0 at t equal to 1 and it is increasing for t bigger than 1 and decreasing for t less than 1, so suppose you were to draw a graph of the function.

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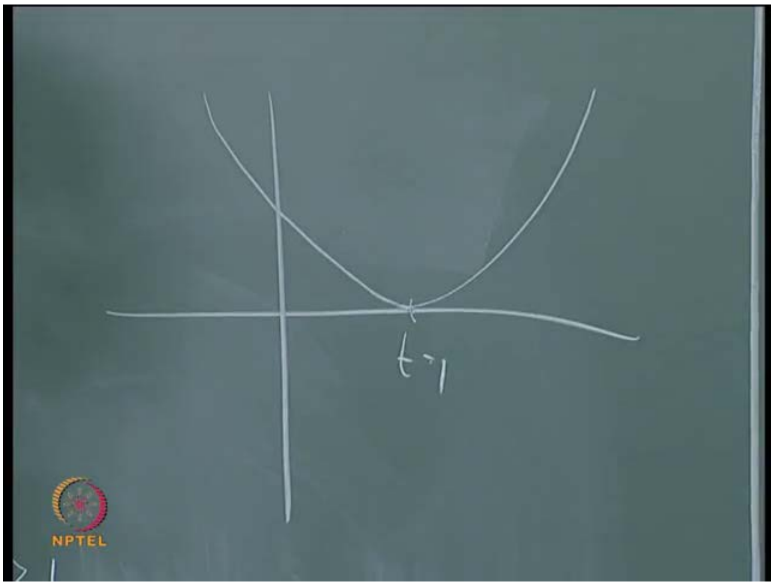

Suppose this is, suppose this is t equal to 1 and this point the function with 0 and t bigger than 1 it increases and for t less than 1 it decreases. So, does it follow from where as it must be bigger not equal to 0 everywhere, so what it means is that if t is bigger not equal to 0 for all t.

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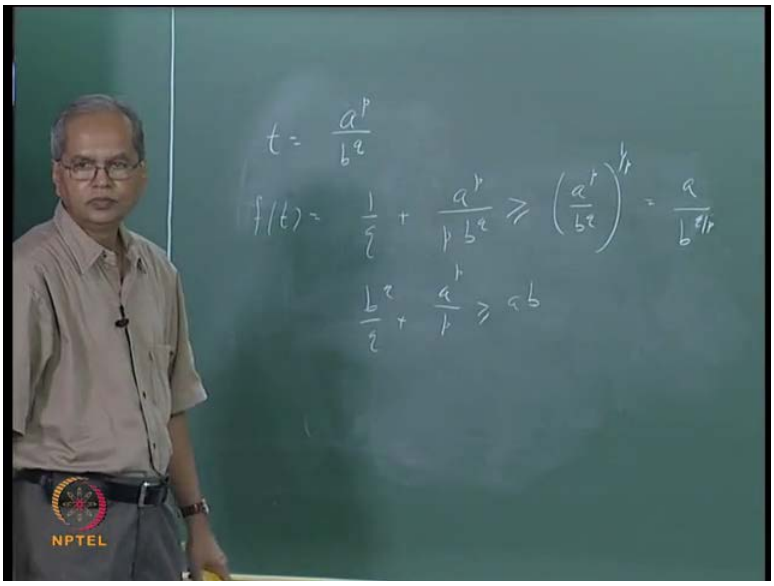

Then, now what I will do is I will take t is equal to a by p plus b a raise to p plus divide by b raise to q t is equal to a raise to p divided by b raise to q and I will use the for this t also f of that should be bigger not equal to 0. So, what is f of this, so f of t what will be that it will be 1 by q plus t by p that is a raise to p plus p into b raise to q minus t is again a raise to p divided by b raise to q. Then this whole thing raise to 1 by 2 this must be bigger not equal to 0, now simplify this and you should get that inequality, so simplifying this what will get suppose I just keep at take this term on the right hand side.

So, I can, I can write this as, so this will be a to the power p whole to the power 1 by p, so this is nothing but a, numerator is nothing but a and this is divided by b to the power q by p. Then what to be done after that you multiply whole thing by this b to the power q by p, so this will become when you bring it, here it will become b to the power q divided by q plus a to the power p divided by p and that is bigger than equal to a b. So, that is that is what we wanted to prove that is that is Young's inequality, of course you can you can check this calculations this nothing much there.

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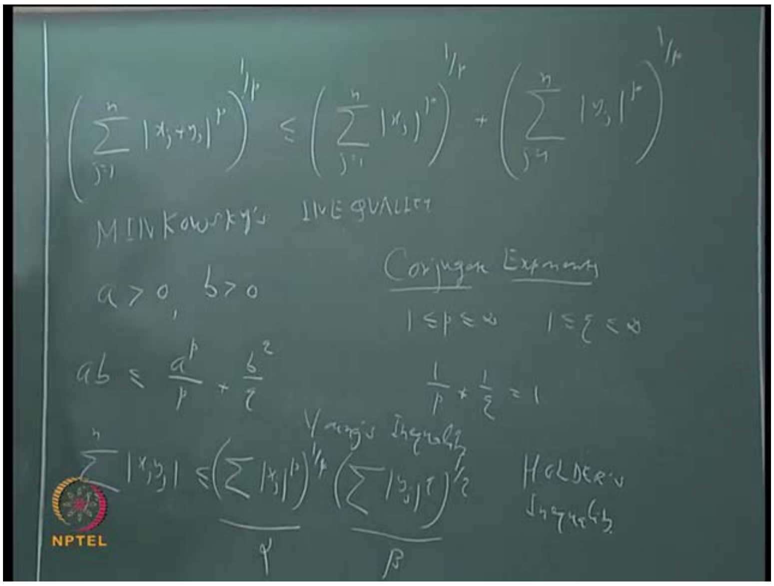

Now, let us see how we want to prove this let us go to Holders inequality, now for Holders inequality let me call this right hand sides this and this as alpha and beta, so suppose this is alpha that is beta, so what is it meaning of this it means.

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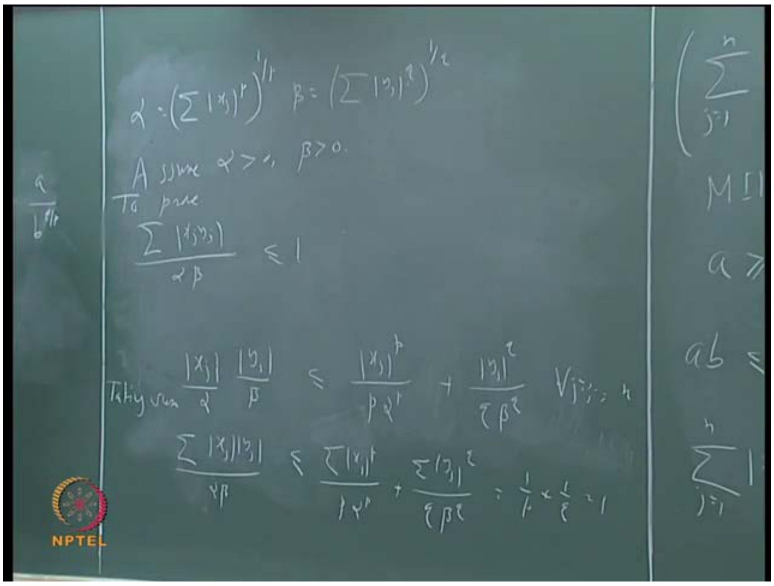

Let me write, here it means alpha is sigma mod x j to the power p whole to the power 1 by p and sigma mod by j, so beta is this sigma mod by j to the power not p that is q to the power q whole to the power q whole to the power 1 by q. Now, let us first dispose of trivial case suppose one of them is, here that is say suppose alphaic 0, if alphaic 0 then all x is are 0 then the left hand side is 0 then nothing is to 0 right hand side is any way non negative number.

So, alpha is 0 nothing is it follows trivially similar case with beta is equal to 0 similar case with beta is equal to 0, so if alpha is equal to 0 or beta is equal to 0 this follows trivially. So, wish only need to consider the case been alpha and beta are both strictly positive, so let us say assume alpha bigger than 0 beta bigger than 0, so what I can say is that basically what we need to prove is this that this sigma j going from 1 to n mod x j by j is less not equal to alpha into beta.

Which is same as saying that because alpha and beta both are bigger not equal to, I can say that sigma mod x j by j this divided by alpha beta less not equal to 1 this is what we want to prove. Now, what we will do is that I will take I want to apply this Young's inequality to these numbers mod x j live divided by alpha and mod y j divided by beta take this as a and that is b take this as a and that is as b of course I should say one more thing here though. Here, I have said a greater than 0 and b greater than 0 remember that 1 if that is equal to 0 this inequality is again trivia, if a is equal to 0 then left hand side is 0.

So, whatever is done in the right hand side this will be still true, so I will just make a small change, here a greater not equal to 0 and b greater not equal to 0 then also this inequality is true. Suppose I apply this Young's inequality to this then what do I get, I will get this sigma see this raise to p divided by p plus this raise to q divided by q. So, that is mod x j to the power p divided by alpha to the power p into p that is the first step, a to the power p by p what is second term it will be mode y j to the power q divided by q beta to the power q. This I can do for each j, this I can do for each j this true for all j going from 1 to n all j going from 1 to n and then take the sum.

So, taking the sum what will get is this will be sigma mod x j into mod y j this whole thing divided by alpha beta that will be left hand side that is taking sum and what will be the right hand side. It will be sigma mode x j to the power p divided by this will remain as this p alpha to the power p plus sigma mode y j to the power q divided by q beta to the power q. But, this is nothing but, alpha to the power p because alpha is this raise to 1 by p, so this right j hand side is nothing but 1 by p plus 1 by q and that is equal to 1. So, that proves holders inequality of course you may say that, here you have mod x j into mod of y j, but here we had mode x j into y j, but you see this no difference this two thing are the same.

So, we proved Holders inequality, so only a thing remains that is to prove this Minkowskys case inequality again will start with the obvious point if any one of this there you have a sum there you have a sum if both of them are 0. If both of them are 0 then it means each x j is 0 and each y j is 0 and in which case the left hand side is also 0, so the inequality follows trivially inequality follows trivially. So, we can only with need to consider only the case when both of them are strictly positive why we have to go to such types because we may need to divide by those numbers somewhere in their proof. So, to justify that we must say right in the beginning that those numbers are different from 0 at since they are already non negative it means that they are strictly bigger than 0.

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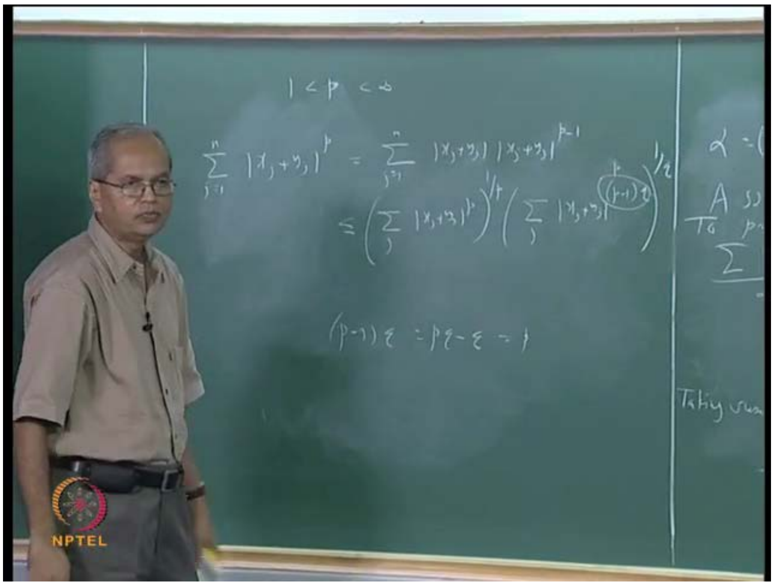

So, what we can start for our is this look at sigma j going from 1 to n let me say mod x j plus y j raise to the power p, let us remember p is strictly bigger than 1 this is the case we are considering 1 less than p less than infinity. So, I can write this as let me write do it in next step write this is a sigma j point of going to n small x j plus y j into mod x j plus y j into p minus 1 I can do that fine. Now, to this I want to apply Holders inequality I will take these numbers as x j and these numbers as y j right because we have something like this we have sigma mod x j by j.

So, similarly we have sigma over j something depending on j multiplied by something depending on j, so for this product I will, now apply holders inequality. So, what follows from here, so what I should do is that for this I should raise to p take the sum and then raise whole thing to one by p, so what is this is sigma over j I will forget about this 1 to n.

So, mod x j plus y j raise to the power p whole 3 power 1 by p what about this again sigma over j mod x j plus y j to the power p minus 1 and we have to raise this to the power q, we have to raise this to the power. So, it is p minus 1 into q p minus 1 into q and then this whole thing raise to q that is why applying Holders inequality to this do you agree with this. Now, let us just look at this exponent what is the exponent, here it is it is p minus 1 into q, so what is that it is p q minus q p q minus q see remember till now we have not use the fact that p and q are conjugant exponents we shall use it.

Now, what is p q minus q what is relationship between p and q it is this 1 by p plus 1 by q is 1, so what follows from here it is that suppose you multiply by p q throughout it is p plus q is equal to p q. So, this p q minus q is nothing but p, p q minus p is nothing but p, so this exponent here is nothing but p, I should have done one more thing before this I will go back to this step will come to this step, see mod x j plus y j is always less not equal to mod x j plus mode y j.

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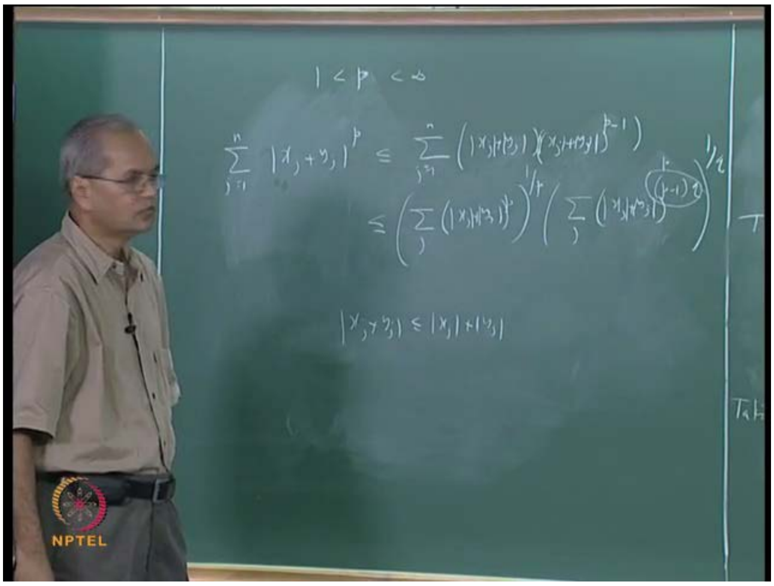

So, I can instead of this inequality sign, here itself I will take this less not equal to then I make this first term as mod x j plus mod y j and similarly the second term also as mod x j plus mod y j then this whole thing is to p minus 1. In other words wherever there is x j plus y j will replace it by mod x j plus mod y j, so this mod x j plus mod y j raise to p and here also mod x j plus mod y j this raise to p its this no problem with that right because see earlier everywhere we had mod of x j plus y j. But, we all know that mod of x j plus y is always less not equal to mod x j plus mod y j no I think we have to have some more steps in let us let us not do it in this fashion.

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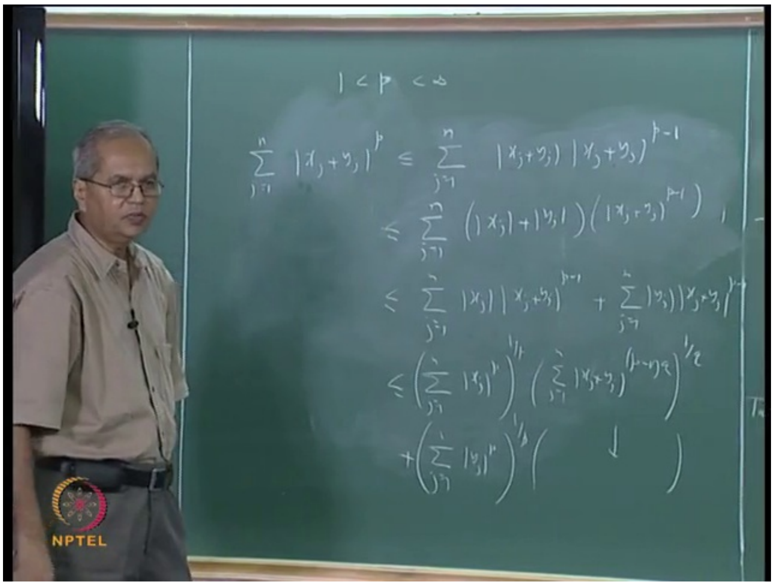

Let us say this is sigma j going from 1 to n mod x j plus mod y j into p minus 1 and what I will do is that there is I forgot about one there is one more step in between and then we will do what we talk. So, this is say by sigma j going from 1 to n mod x j into, sorry I think again not this way it is mod x j plus y j into mod x j plus y j to the power p minus 1.

Now, I will try this is sigma j is going from 1 to n is this follow I will do it for only this first term this mod x j plus mod y j, so this is less not equal to mod x j plus mod y j and then this I will keep as it is mod of x j plus y j to the power p minus 1. Then I will split this sum into two, I will split this sum into two, so this is less not equal to sigma j is going 1 to n mod x j it to the second term is this mod x j plus y j to the power p minus 1.

Then plus sigma j going from 1 to n mod by j into mod x j plus y j the power p minus one do you agree with this, now after this we apply Holders inequality to each of this.

Whatever we have done earlier these in between a steps were missing in the earlier calculus this are important, now we apply holders inequality, so what is we do is that, so suppose I apply the holders inequality for this first sum. Here, then this will be less not equal to sigma j is going from 1 to n mod x j to the power p whole thing raise to 1 by p into this second term will be sigma j is going from 1 to one mod x j plus y j raise to p minus one into q. Then this whole thing raise to 1 by q that is for the first sum that is for the first sum for the second sum what will happen this will be instead of mod x j to the power of p 1 by it will be we saw the plus sigma j going from 1 to n mod y j to the power p then whole thing to the power 1 by p.

The second term will be the same second term will be mod x j plus y j to the power of p minus 1 into q and then the whole to the power 1 by q. Now, you see that this second term there are two theme triple notice, here we are one theme which we have already notice on this that this is nothing but p minus 1 into q that is nothing but p and that is a term which is common to both this terms, that is term which is common to both this terms.

So, what I want to do is that I want to take that term to the left hand side I want to take that term to left hand side, but what you can see is that left hand side that term is actual the same which is there in the bracket whatever you have in the bracket it is exactly the same thing that is here. So, suppose I divide by this term that is where this earlier argument to assuming that everything is different from 0 consent essentially, what it means is I want to divide by this term I want to divide by this term, so suppose I do that.

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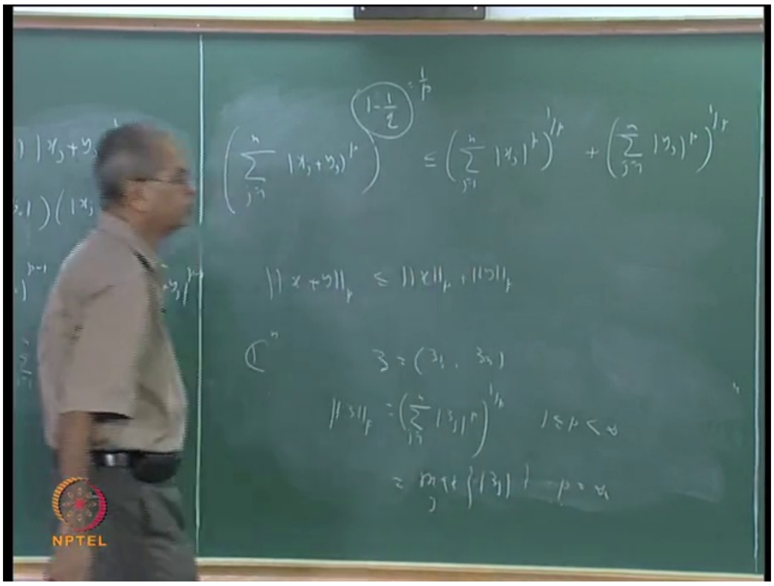

Then this will be left hand side will be sigma j going from 1 to n mod x j plus y j to the power p this whole thing raise to 1 minus 1 by q that will be the left hand side after dividing and what will remain on the right hand side. It will be sigma mod x j to the power p that is from this first term and then from the second term sigma j is going from 1 to n mod by j to the power p put the power when meant that follows simply by dividing by this step, dividing by this step.

But, again use the Frank that this 1 minus 1 by q that is nothing but 1 by p that is how the we started with 1 by p plus 1 by q is equal to 1, so this 1 minus 1 by q that is nothing but 1 by p and using that you will get. But, this is nothing but what we have written, here if you put 1 by p here then that is nothing but the Minkowskys inequality, so is it clear. So, we have proved Minkowskys inequality and what it means is that we have proved this inequality that is norm of x plus y suffix p is less not equal to norm of suffix p plus norm y suffix p.

In other words this p norm is a norm on r n p norm is a norm on r n is it clear what ever done so far, now there are just one or two extra points that we can see, here and they are the following. Instead of taking R n I could have also taken C n only difference there will be that the components will be order in triples of complex numbers. So, for example typical point will be suppose I call that point z it will be other from z 1, z 2, z n z 1, z 2, z n and I can simply define norm of this as follows norm of z suffix p see it will be sigma mod z j the power p j going from 1 to n whole thing to the power 1 by p for 1 less not equal to p less than infinity. If p is equal to infinity, I will take maximum of mode z j maximum what is say mod z j or p equal to infinity that will define a norm on C n.

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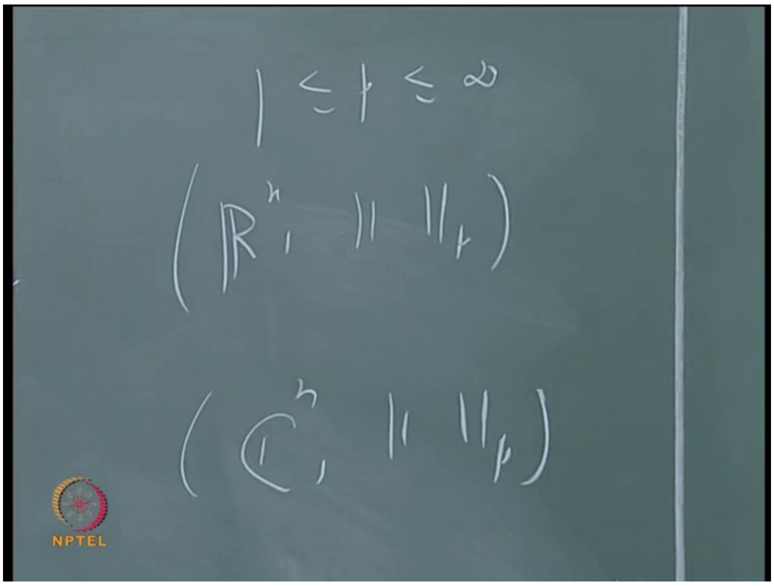

So, for each p what we have seen for each p 1 less not equal to p less not equal to infinity this R n with this norm suffix p and similarly this C n with norm suffix p these are non linear spaces. These are all non linear spaces because this suffix p is a norm on each of this and since we are already seen that every norm linear space is a metric space all of these are metric spaces.

So, we have got a big class in fact infinitely many metrics spaces by this either next class what we will see is that not only for this R n and C n we can also conquer the spaces of sequences they are called sequence spaces at those can also be made into norm linear spaces by giving similar norms. So, for that again we shall require inequalities similar to these Minkowskys inequalities, but there the sum will go from 1 to infinity instead of going instant taking finite sums we will need the sum going from 1 to infinity and then the questions of convergence extra will arise.

But, anyways you will see that whenever we want to talk about the convergence of a series, basically we have to say something about the partial sums and the partial sums are always the finite sums. So, basically using that idea we still use this inequalities to prove a similar inequality for infinite sequences and then will use those things to prove something about the norms of the spaces of sequences that we shall do in the tomorrows class, we stop with this.