Real Analysis
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Lecture - 13
Conditional Convergence

Now, there are a few things that are remaining in the discussion of the series and those we shall complete today. We discussed for a long time the series of non negative terms. Only in the last class we discussed the series of the terms which may have positive and negative signs and we discussed what is known as a bells test.

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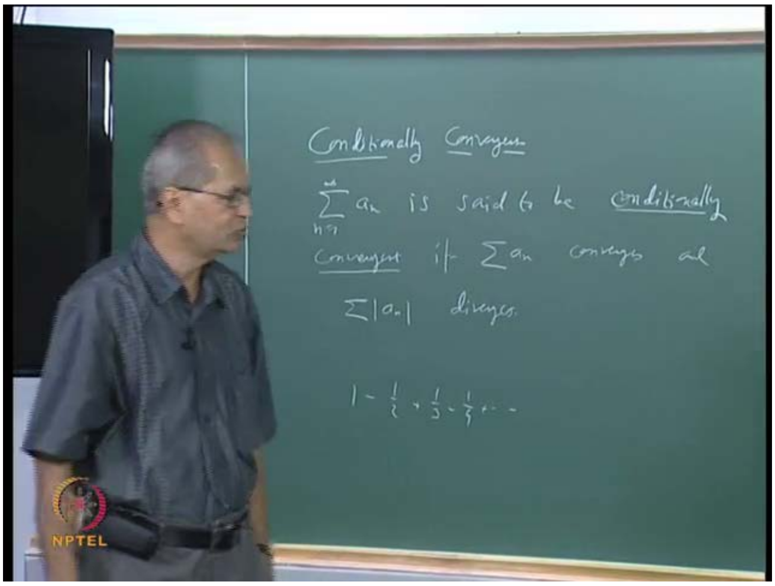

A series which is convergent, but which is not absolutely convergent it has a special name it is called conditionally convergent. That is to give of more formal definition, suppose if this is a series sigma a n, n going from 1 to infinity. This is said to be conditionally convergent or we say it converges conditionally. If the series converges that is if sigma a n converges, but it is not absolutely convergent. That means and a sigma mod a n diverges. We have seen an example of a series of this kind.

For example, we saw this series yesterday 1 minus half plus 1 by 3 minus 1 by 4 etcetera. We have seen that this series converges, but it is not an absolutely convergent series. So, this is an example of a conditionally convergent series. Now, in case of this conditional and absolute convergence, there appears one very important question and that is what we shall also discuss today.

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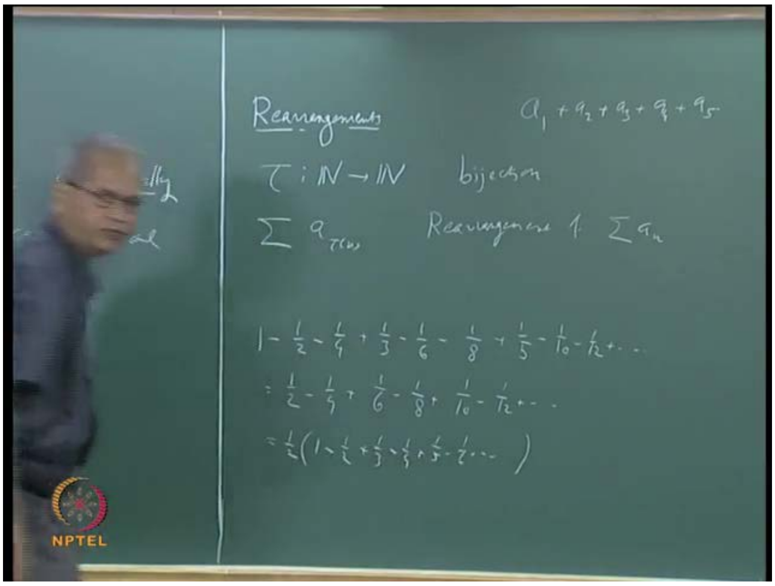

That is what is called rearrangement. Now, to just motivate this after all series is something like an infinite sum. Now, if we take a let us say a finite sum, suppose we take say something like a 1 plus, a 2 plus, a 3 plus, a 4 plus, a 5 etcetera. Since the addition is commutative as well as associative it does not matter.

In what way I take the sum, whether I add a 1 to a 2, then add to a 3 or I take a 1, a 5 here or a 3. There it does not matter whichever way you write this the sum is going to be the same right, but that may or may not happen in case of the infinite series. So, now here actually what is meant by rearrangement, it is that you write these terms of the series in some different order that is called a rearrangement. To make it more precise, suppose we take a map tau from n to n, suppose it is a bisection.

It is 1 1 on to as you know such maps are also called permutation. If it is a finite set, you call 1 1 on to map is permutation. So, if you consider a series sigma a suffix tau n that is, suppose your given series is sigma a n instead of that you consider sigma a suffix tau n. Then this is called rearrangement of sigma a n. That means basically what the term says that rearrangement, that you just rearrange the terms of the series write the terms in some different order. Now, what is the obvious question here, if the original series converges does this rearrangement also converge and does it converge to the same sum if it converges does it converge to the same sum

Now, it so turns out that that is true if the series is absolutely convergent and the things are very bad if the series is conditionally convergent. We can just see an example here, let us take this same example we have seen that this series is 1 minus etcetera. We have seen that this is a convergent series suppose its sum is s suppose its sum is s. Now, let us just rewrite the series in some different order what I will do now is that, I will write the series as follows. So, 1 minus half minus 1 by 4, that instead of taking 1 by 3 I take the next term as 1 by 4 then I will take plus 1 by 3, then I will take the next 2 immediate term 1 minus 6 minus what is a.

So, 1 minus 4 the next negative term will be minus 1 by 8 right, then take the next positive term that is 1 by 5 and then minus 1 by 10, minus 1 by 12 etcetera. I suppose you are not the original series was there was 1 positive term 1 negative term etcetera, followed by that. Now, what I am doing is that I am taking that same series I am taking the first positive term, then the next tqo negative terms then the next positive term.

Again followed by next two negative terms and do like that all right. So, it is a, this is a rearrangement of this same series. So, all right now let us see a few things here for example, this 1 minus half is half 1 minus half is half. So, this is 1 by 2 minus, 1 by 4 plus, again this 1 by 3 minus 1 by 6. You can say that is same as 1 by 6 that is same as. So, 1 minus. So, the next will be 1 by 6 minus 1 by 8.

Then similarly, you can say that 1 by 5 minus 1 by 10 again 1 by ten. So, that is. So, the next 2 terms will be 1 by 10 minus 1 by 12 etcetera this will be the series after simplification right. You can see that I can this half is a common factor from all of them ok. So, suppose we take that common factor out what remains half into 1 minus this will be 1 by 2, that will be plus 1 by 3 then minus 1 by 4 etcetera plus 1 by 5, minus 1 by 6 etcetera all right. Do you see that it is the same series here ok.

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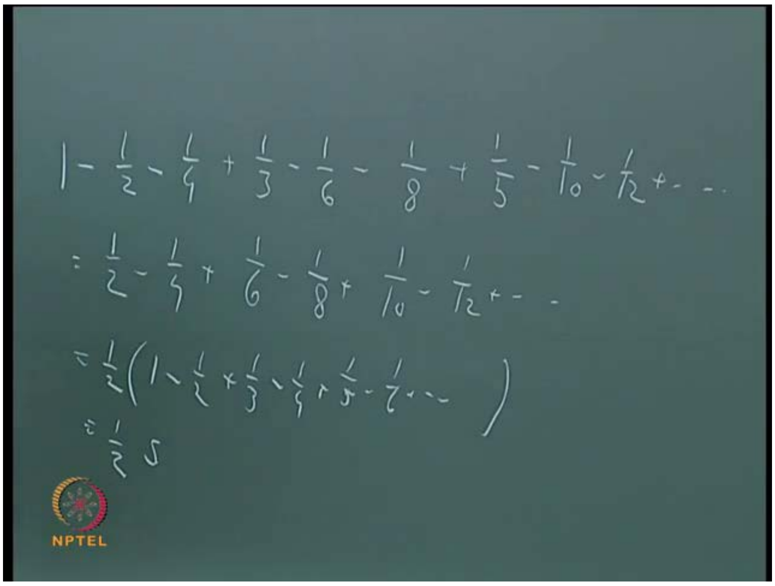

So, now what is its sum it is half s it 1 by 2 s right. So, it is the rearrangement of the same series, but it converges to a different number right, it converges to different number. So, this idea of this example is to show that, if the conditional convergence behaves very badly with respect to rearrangements.

In fact what is known is something much worse not only that rearrangements will converge different number, in fact given any real number we can find some rearrangement of the series, such that that rearrangement converges to that given real number. Not only that we can also find a rearrangement. So, that that real, the new series with that that new rearranges that new rearrangement diverges. On the other hand, if the series converges absolutely then all its rearrangements converge ok.

All those rearrangements converge to the same sum. So, let us just write this as a theorem. So, if sigma a n converges absolutely, then all it rearrangements converge and converge through the same sum. On the other hand, if sigma a n converges conditionally then given any real number, we can find a rearrangement such that that rearrangement converge to that re real number. So, what I was saying that for every s in R there exist a rearrangement sigma, I will call it rearrangement sigma a suffix tau n converging to s.

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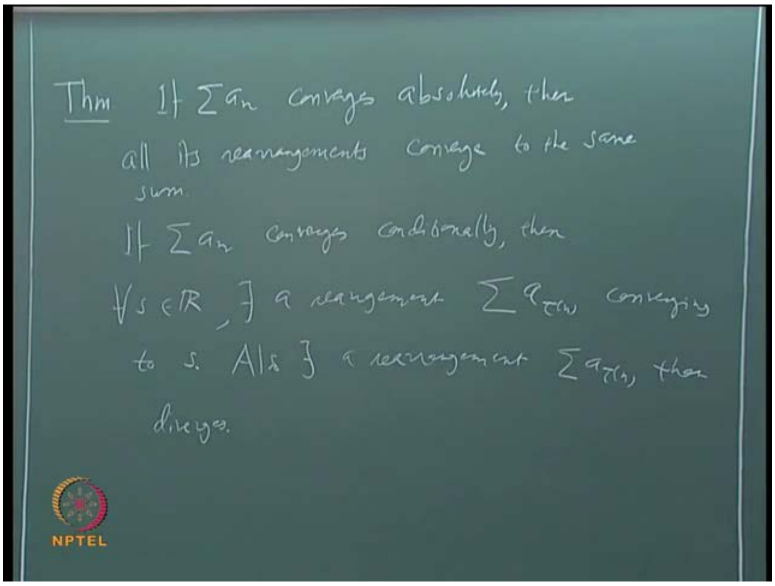

Also we can find a rearrangement. So, that that rearrangement diverges. So, we can say that also there exist rearrangement sigma a tau n that diverges. So, that is the importance of absolute convergence. If you know that a series is absolutely convergent or the series is of non- negative terms, remember this if the series is of non- negative terms, there is no difference between convergence and absolute convergence.

So, in that case you can rearrange the terms of the series in any manner you like and that will not change the convergence or divergence or it will not also change the sum. Whereas in case of conditional convergence things are quite bad all right. Now, there are a few elementary properties of the series which perhaps we should not discuss immediately after discussing the series and those are as follows.

Suppose we take two series sigma a n, n going from 1 to infinity and let us say sigma b n, n going from 1 to infinity. Suppose both of them converge then we can say that if you take the series sigma a n plus b n that should also converge. It is sum should be same as the sum of sigma a n and sigma b n. So, let us let us just see that, if sigma a n and sigma b n converge then, sigma a n plus b n also converges and sigma a n plus b n n going from 1 to infinity, this is same as sigma a n plus sigma b n equal to. By the way, you may wonder we are not going to discuss the proof of this theorem here because it is proof is somewhat lengthy.

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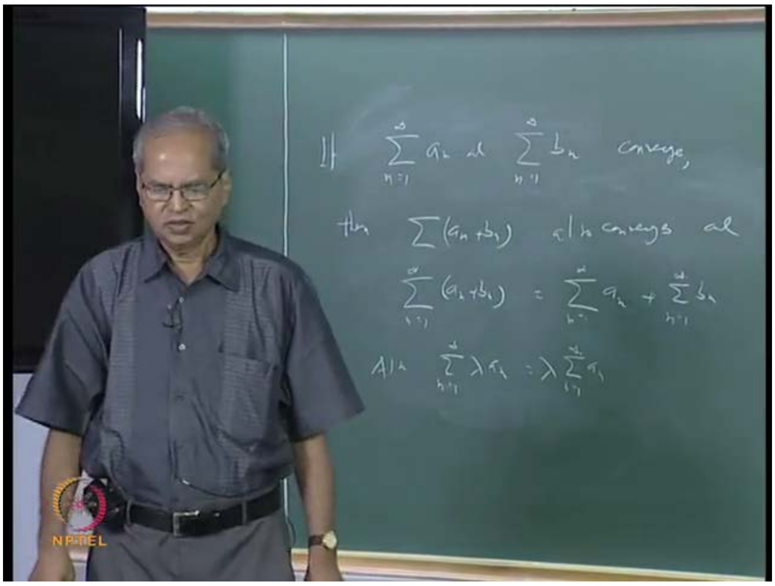

Those a few who are interested in proof you can see the proof in Roudin's book. This theorem is given in Ruded coming back to this. So, what it says is that if the two series converge their sum also converge. Similarly, if you multiply the series by some real number lambda then the new series will also converge. That is also we can say that also sigma lambda a n this is also convergent series and it sum will be same as lambda times sigma a n n going from 1 to infinity all right.

This will follow simply by taking the partial sums. Suppose s n is a partials of the series, suppose s n is a 1 plus a 2 plus a n and t n is say b 1 plus b 2 plus b n, then saying that sigma a n converges is same as saying that s n converges, s n converges to s and t n converges let us say t. Then use the corresponding theorem about the sequences then s n plus t n converges to s plus t.

Similarly, lambda s n converges to lambda times s that is all in there is a proof. I said already that, whatever we want to prove say or prove about the series, everything can be done using the sequence of partial sums. Using the corresponding theorem about a sequences, but what you will also notice further is that, we cannot give a similar characterization if we take the product. For example, if we take this series.

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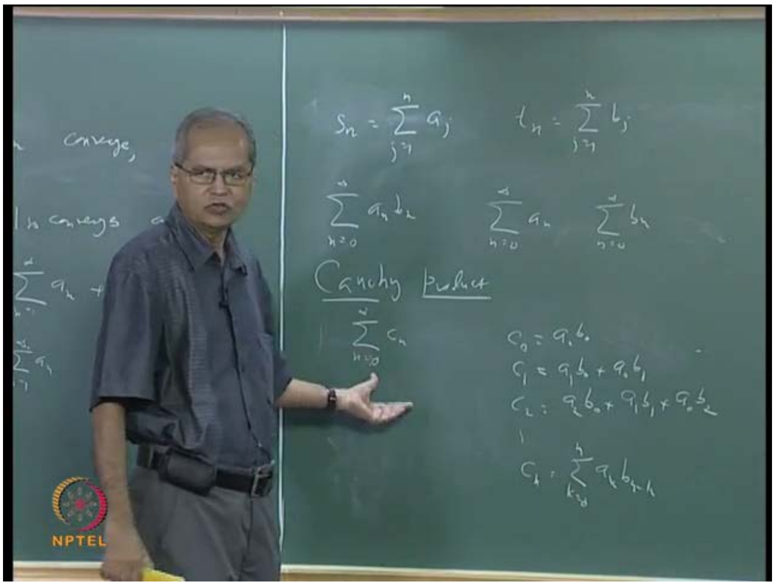

Let me again recall, that is suppose we take this s n as the sum sigma. Let us say sigma a j, j going from 1 to n and say t n as sigma b j, j going from 1 to n. Then partial sum of the series sigma a j a n plus b n that is same as s n plus t n that is fine, but suppose I take these series, sigma a n b n, n going from 1 to infinity. Then the partial sum of this will be a 1 b 1 plus a 2 b 2 plus a n b n and that is not the product of s n and t n right that is not the product of s n and t n. So, we cannot say that if s n converges and t n converges, this series also converges.

So, in general we cannot say that if the two series converges that is, if the sigma a n and sigma b n, if both of them converge we cannot infer from that that sigma a n b n is also a convergent series. So, to consider the products there is totally different notion, what is called cauchy products. Cauchy product is something like this for example, the first term suppose I denote the term, say product as sigma c n I denote that terms that are wrote as sigma c n. I think going from 1 to infinity, then this first term let, I think for this for considering this, it is convenient to start with 0 to infinity, take both the series, starting from 0 to infinity.

That is sigma a n also going from 0 to infinity and sigma b n also going from 3 to infinity just a minor convenience here. So, sigma c n will also I will take from o to infinity. So, the first number here is c naught, the first number here is c naught. So, that is taken as a naught b naught, that is just the product of the corresponding terms. Then the next number c 1, that is taken as a 1 b naught plus a naught b 1, a 1 b naught plus a naught b 1 right. Now, you can understand how we will proceed for example, next number c 2 that will be taken as a 2 b naught, plus a 1 b 1, plus a naught b 2 that is what we are doing here.

We are taking all those indices such that the sum becomes 2, here. Now, we can understand that how the general term will be. So, in general the term c n that will be sigma a k b n minus k, k going from o to n ok. So, suppose you form a series like this, then that series is called the cauchy product of these 2 series, sigma a n and sigma b n. We can say something about the cauchy product, if sigma a n and sigma b n converges then whether cauchy product also converges are not, there are some conditions for that, but since that is not very important right.

Now, for us we shall not go into theorems of those kind for the time being. All that you should remember is that convergence of these two series sigma a n and sigma b n certainly does not imply that sigma a n b n converges right. Of course, directly it also does not imply that sigma c n converges you need some additional conditions for that ok all right. Now I think for the time being we shall close the discussion of the series and let us move on to next topic, but to move out to the next topic, let us start something with which depends on the series. I shall define a new set.

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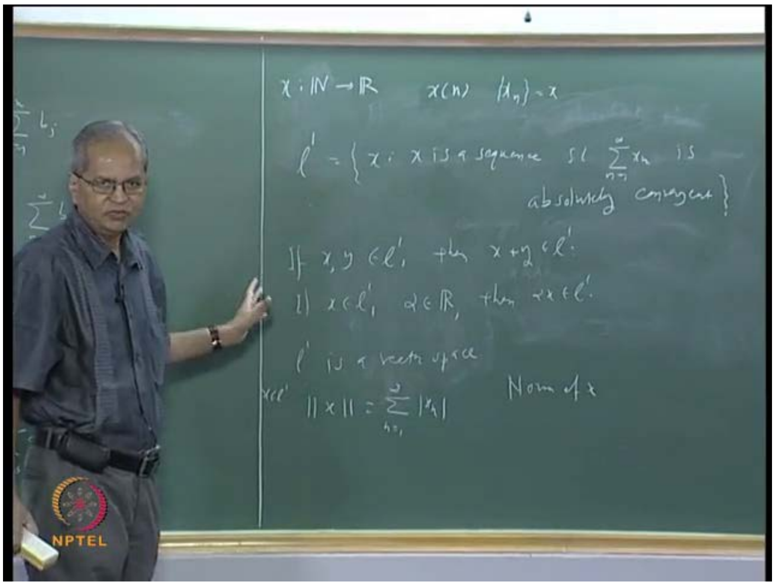

Let us say I will call that set l 1, l super script 1 this is also let us say this is a set of real sequences and a real sequence. As you know a sequence is nothing, but a real sequence is nothing but a function from n to r. So, suppose I denote any such function as x, x from n to r. Then image of any number n here by using this notation I should denote by x of n right, but in a sequence it is customary to denote it is as x suffix n all right. We can continue to use this notation. So, I will just say that x means this sequence x n, x means the function from n going to R and that is nothing but same as the sequence x n.

Now, this notation is some times more convenient, when you also want to talk about sequence of sequences. Then for example, suppose I want to talk of sequence of this sequences, then you will need some more either super script or sub script. So, instead of that for that this notation is more convenient all right. So, what I want to do is that I shall take the set of all sequences x. So, x is a sequence, such that sigma x n is absolutely convergent, that is take the series sigma x n, either write x n or this way whichever way you want like sigma x n is absolutely convergent all right.

That means what sigma mod x n is convergent, that is what sigma mod x n is convergent for example, if you look at these theorem. Here can I say if sig a sigma a n and sigma b n if both are absolutely convergent will it follow that sigma a n plus b n is also absolutely convergent right because you take mod a n plus b n, that is less nor equal to mod a n plus mod b n. Since sigma mod a n and sigma mod b n those are convergence. So, what you can say that sigma mod a n plus mod b n that is a convergent series.

Then use comparison test right. So, that is trivial. So, if there are two series which are absolutely convergent, then their sum is also absolutely convergent. In other words if I use this notation I can say that if x and y x and y belong to l 1, then x plus y also belongs to l 1 all right all right. Next is I will say that if x belongs to l 1 and let us say some alpha belongs to r, then alpha x also belongs to l 1 right because the series alpha x n will be nothing, but mod alpha times mod x n, that is also convergent series right.

So, what it means is that if you take these set l 1, then if you take any two elements in this set their sum is also in this set. Product of a real number and any element in this is also in this set, in other words this set these two operations, addition of two elements and multiplication of a scalar. A element in this right scalar real scalar is real number. So, what is the obvious thing to do next right, you are also learning linear algebra simultaneously right. So, you have all heard of vector spaces right.

So, vector spaces is structure with has which has these two operations. So, what is the next obvious question that we should ask that, whether these is a vector space and what is the answer right because see after all what is the operation here. If you take two any way it is a sum of two functions operation. So, we can say operation is co-ordinate wise x plus y nth entry of x plus y is x n plus y n right. So, what is the zero element, constant sequence 0 right. So, it is this operation of x plus y, it is associative commutative and simply also you can also verify all other axioms that alpha times x plus y is alpha x plus alpha etcetera all right. So, I will simply say that l 1 is a vector space.

Well that is the thing right, but there is something more about this. Now, because this I could have said even if I did not use this for absolutely convergent. Suppose I taken the sequences only of convergent sequences still I could have done this. I could have converted that into vector space, that is also done. Now I shall make use of this fact because of this fact what I can now is that the series sigma mod x n is a convergent series.

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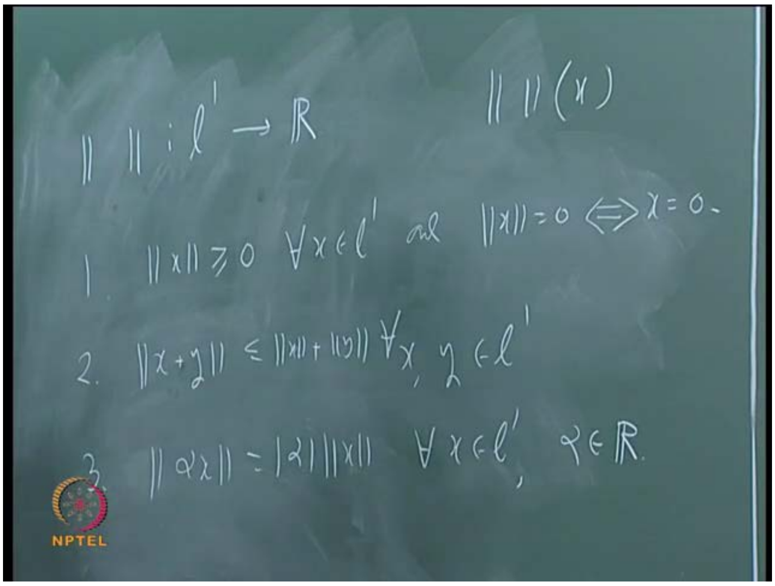

So, what I will do is that i shall call that, I shall give some notation for that sum I shall call it norm of x this is used it is called norm of x it is nothing but sigma n going from 1 to infinity, mod x n. We know that once x is in l 1, once x is in l 1 this is a real number that a it is absolutely convergent. So, this is defined. So, norm of x is defined. So, norm now it means it is a function from l 1 to R it is a function from l 1 to R right. Now, let ask some very obvious questions, what properties does this function have.

So, we have function norm, which goes from l 1 to R right. Of course, we are using the notation in a slightly different manner since if norm is a function from l 1 to R I should have denoted norm of x by this right. Something like f of x right, but any way this is customary to denote bring this x here all right. What are the properties let me just say first property. For example, can you say that this norm of x is bigger naught equal to 0 for all x in l 1, norm of x is bigger naught equals to 0 for all x in l 1 all right.

What is the norm of the 0 element 0 all right. Suppose norm of some element is 0, then what can you say if sigma mod x n is 0. Then all of this x n 's must be 0. So, can we say this that norm of x is equal to 0 if and only if x is equal to 0 norm of x is equal to 0 if and only if x is equal to 0 all right. Second property I want to say something about norm of x plus y norm of x, suppose you take 2 elements x and y in l 1. I want to know how is norm of x plus y related to norm x and norm y right. That is something we saw just now right. We can say that for example, what we are asking is this, what is the relationship between norm of x y is nothing but sigma mod x n plus y n right.

That is norm of x plus y and what is no R m x it is sigma mod x n and what is mod y it is sigma mod y n. Now, are these 3 numbers related, it is clear because mod of x n plus y n is less not equals to mod x n. So, this is less not equal to this right. So, we can say that this is less not equal to norm of x plus norm of y, for all x and y in l 1 ok. Lastly I want to say that norm of alpha times x norm of alpha that is alpha. Suppose, alpha is an real number, now alpha times x means sigma mod alpha x n. Is it clear that this should be same as mod alpha times norm x mod alpha times norm x.

This should be true for every x in l 1 and alpha in R right. Now, what can you say about these three properties, have you seen something similar earlier. It is these are properties very similar to the absolute value function on the real line, on the real line we have defined the function modulus of real number. When we listed a properties of that function those properties were very similar right. All these things for example, say suppose x were a real number all these things are true, if x and y are real number.

So, this is a function which basically has a properties which very similar to the properties what is called absolute value of real number. Now, such functions can be defined on several vector spaces. When it can be done that function is called a norm. The corresponding vector space is called a normed vector space or norm linear space ok.

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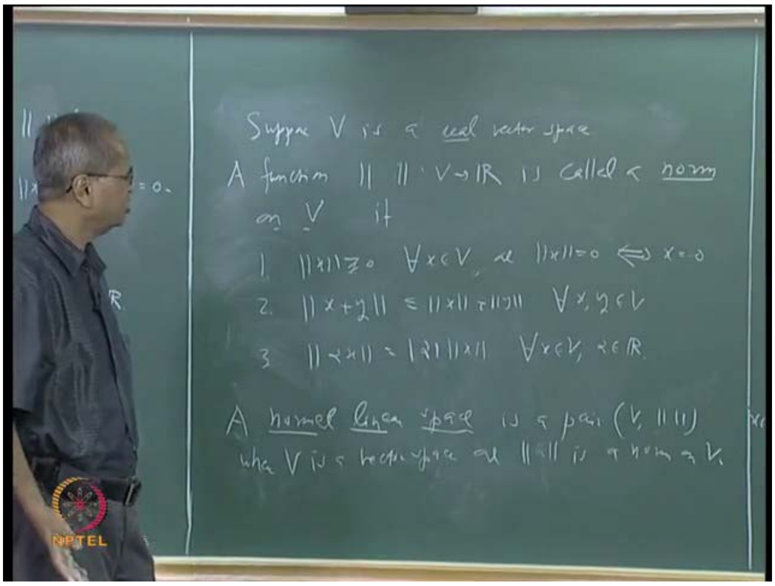

So, let us just make a formal definition as follows that is suppose v is a real vector space and a function, let us say norm going from v to R is called a norm is called a norm on v, if it satisfies these three properties all right, if you want we will write once again. If first is norm of x is bigger not equals to 0, for every x in v and norm of x is equal to 0 if and only if x is equal to 0 all right. Second property is norm of x plus y is less nor equal to norm of x plus norm of y, for every x y in v. Third property is norm of alpha times x is same as mod alpha times norm of x for every x in v and alpha in R right.

What I said last is that a norm linear space, a norm linear space is a pair v norm is a pair v norm. So, this is a term that where called norm linear space or we can also call norm vector space, where v is a vector space and this is a norm on v. So, is a pair where v is vector space and norm is a norm on v all right. Why we say that why we talk in terms of this pairs. So, it is possible that on the same vector space there may exist several functions satisfying this right. On the same vector space you may be able to define different we will see examples of these kind of things little later.

So, as a vector space those two objects will be same, but as norm linear spaces those two objects will be different. So, we with sum, let us say norm 1 and we with norm 2 as vector space is no other those are the same, but as norm linear spaces is those are different. So, that is why we usually talk about a pair of course, again as is the practice where it is clear from the discussion what is the norm that we are talking about. Then we will simply say that v is a norm linear space also there is one.

Obvious question here why we are taking real vector space. What can we not take vector space on some other field of course, we can also take vector space on complex numbers, you can take complex vector space. Then see as far as these first two are concerned there is it has no reference to the scalar. Only this last axiom, that refers to the scalars and that will change this will become for every x in v and alpha in c every x in v and alpha in c these two. That will be call complex vectors space and.

So, corresponding it will be a complex norm linear space and this is what we can call real norm linear space right. Since there is not much difference as far as the definition is concerned we shall not bother too much about this. Now, this is an example of a real vector space. Now, instead of taking the sequences from n to R, suppose I have taken sequences from n to c, then also you can define absolute convergence. All that in the usual way that would become an example of a complex vector space all right. Now, the next question why exactly we are discussing all these things. What is the idea of discussing this norm linear spaces. To understand that let us again look at our definition of the convergence of a sequence.

How did we define the convergence of a sequence. Suppose x n is a sequence in R, suppose we take a sequence in R, then when did we say x n converges to x we say that x n converges to x, x n converges to x. If you remember what will this we said that this bend for every epsilon bigger than 0 there exists n 0 in n, such that n bigger nor equal that is, if n is bigger than or equal to n 0 n bigger n 0 implies mod x n minus x less than epsilon. In other words the concept of convergence of a sequence depends on this function mod, x n minus x.

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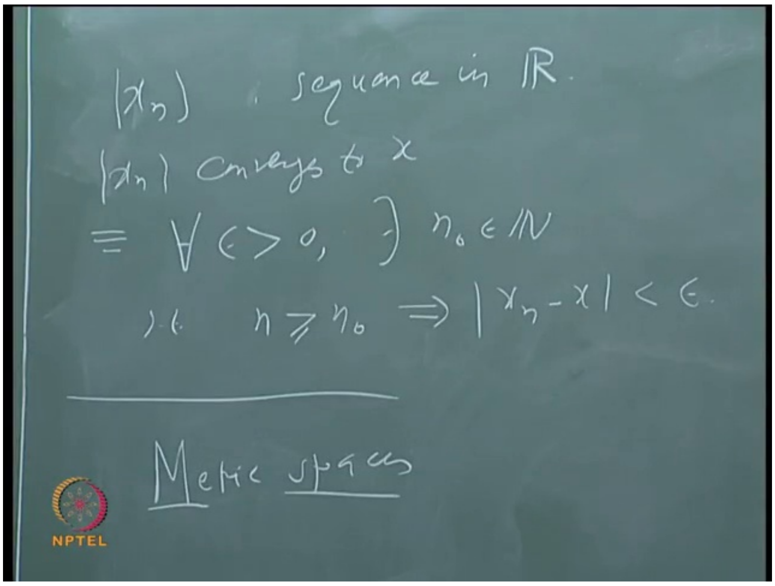

We proved several theorems about the convergent sequences about the real numbers, using this properties of this absolute value function and of course, some theorems using the all the completeness of the real numbers etcetera, but you can say that sequence can be defined on any set ok. After all what is a sequence, sequence is a function whose domain is the set of all natural numbers, core domain can be anything. So, instead of take considering sequence of real numbers I can consider the sequence of any objects right. Sequence of say elements in R 2 or R n, sequence of vectors, sequence of matrices sequence of functions.

Suppose I want to ask the question, how do we define what is meant by such a sequence converges. Suppose you are given sequence of matrices ok. Suppose that sequence is a n each a n is a matrix of some fix order, let us say 3 by 3 and I want to say that this sequence an converges to a what is the meaning of that or whole this one define. We can say that, if we had some notation is like this norm hundred. Then I put on simply imitate this will simply change to norm of x n minus x.

So, instead of sequences in a real line sequences, in a real line I can take sequence in any norm linear space and define what is meant by the sequence converges in that norm radius space right or more generally. This is one idea, that is the reason for discussing norm radius spaces. More generally see by mod x n minus x is the thing, but a distance between these two numbers x and x n x and x n and. So, if you remember what we had said all the time saying that sequence converges means distance between x n and x becomes small as n becomes large that was the idea.

So, one can say similarly, that if we have a concept of distance in any set. Suppose we have the concept of distance, then we can talk about the convergence of a sequence in that set. All that we need is the concept of distance right. Similarly, for example, other concept of limits continuity etcetera all those concepts depend in some sense. either on this like absolute value function or on the concept of distance. So, we can also develop all those concepts in more general sets like that. What is the advantage, see now we have proved let us say some theorems about convergent sequences.

For example, we have proved that every convergent sequences cauchy or every convergent sequences bounded we proved all these things for the sequences of real numbers. Let us say some time letter we talk of sequences in R 2 or R 3 or R n or sequences of matrices or sequences of functions. Then again we can define what is meant by convergence and again we may have to again separately prove, that every convergent sequences is cauchy or every convergent sequences is bounded and things like that. That means essentially you will be repeating the same proof again in various different context.

What is the way to avoid that instead of avoid this repetition, that is the method is what is called abstraction and very commonly used in mathematics. You may heard this word at mathematics is a very extract subject. People use it in a some sort of a negative way that mathematics is an extract subject, but extract subject is a very powerful tool. It is used in all sciences and as I said because of this abstraction, we can avoid these repetitions. It saves lot of time and energy and it is more efficient way of doing this. So, what we do is that we see for example, what you have done, we have this long linear space is an abstraction ok. Abstraction of what a real line and that l 1.

So, many spaces whatever common to all those spaces those properties we have taken and defined that as a norm linear space. So, similarly we will do about a distance and then follow the idea. Then after that we shall just develop all the theory in those particular either in norm linear spaces or those new x objects likewise let me just tell what this new objects are called those are called metric spaces. Once we develop in metric spaces it can applied to any different any of this other specific examples, R r 2 R n l 1 and all those things ok.

Now, let us come to this what is what is a metric space or what is a metric. This is something more general than norm linear spaces. Here what you have seen we shall subsequently show that every norm linear space is also metric space, but before that yeah which is basically same as saying that metric space is a more general concept because norm is defined only on a metric space. Starting point has to be a vector space, we the starting point has to be a vector space whereas, metric 10 can be defined on any set. So, we take x as a x as any non empty set.

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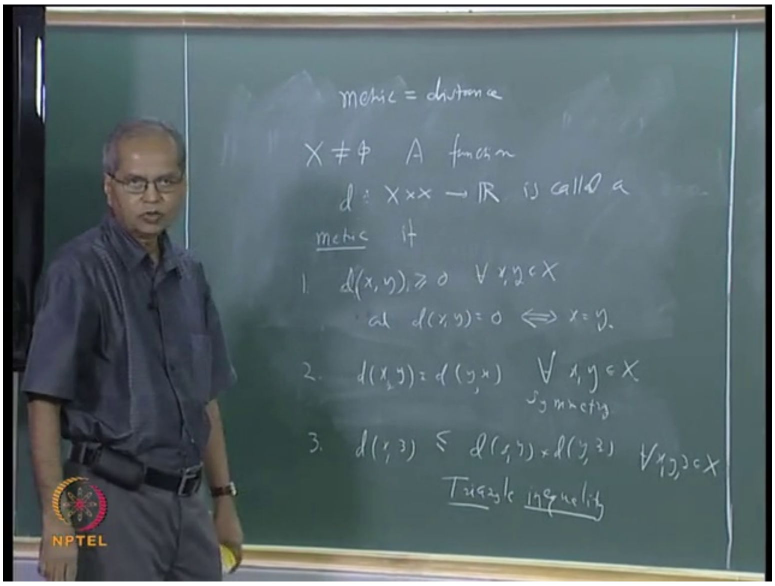

X as any non empty set then what is a metric it is nothing but a function which says something. Suppose we take two points x and y, in that it says what is the distance between those two points. So, that function which satisfies the property which we normally associate with the distance between the two functions. Whatever we commonly associate some of those properties are taken and those are taken as a definition of a distance or definition of metric space.

So, obviously we talk about the distance between the two points. So, it means it is a function from the pair of points, it will associate some real numbers to appear of points. So, we will say that a function, it is a function d from x cross x 2 R, function d from x cross x 2 R is called a metric. If it satisfies some properties.

What are those properties, those properties are again very similar to this. First property is that suppose you take two points. If you take distance between it is called a metric and metric is a must distance this is just a different word met. So, d x y distance that is a distance between in fact strictly speaking, I should write one more bracket here because it is d of this some element in x cross x, that element is x comma y. So, strictly speaking I should use this definition, but I will just what we will understand we mean is this.

So, d this is let me just remove this, just for the convenience. So, d x y this is bigger nor equal to 0 for every x y in x. That is what we normally expected the distance between any two function in non negative number and it should be 0 only when or it should if I that x and x distance between the points should be 0 and the distance between two points are 0, those two points must coincide. So, which is same as saying this and d x y is equal to 0, if and only if x is equal to y ok. Then second property is that distance between x and y this should be same as distance between y and x. It should not matter whether I call distance from x to y or from y to x, that should be the same. This should be true for every x y in x.

Lastly whether this property has a name in fact it is an obvious name this is called symmetry. So, we explain this by saying that distance is a symmetric function. This property is called symmetry. Then last property suppose we take 3 points x y and z, then we want to compare the distance between x and z and the distance between x and y and distance between y and z, suppose we take 3 points. You imagine that those 3 points form a triangle, then the distance between x and z is the length of one side and distance between x y is are the other two sides. So, what we should expect is that, this should be less nor equal to that this should be less nor equal to that.

So, this is true for every x y z in x and because of the comma is which might. Just now this last property is called triangle inequality. This last property is called triangle inequality. By the way similarly, in this definition of a norm this property 2 is also called triangle inequality, this is also called triangle inequality. We will there is a reason for this a little later.

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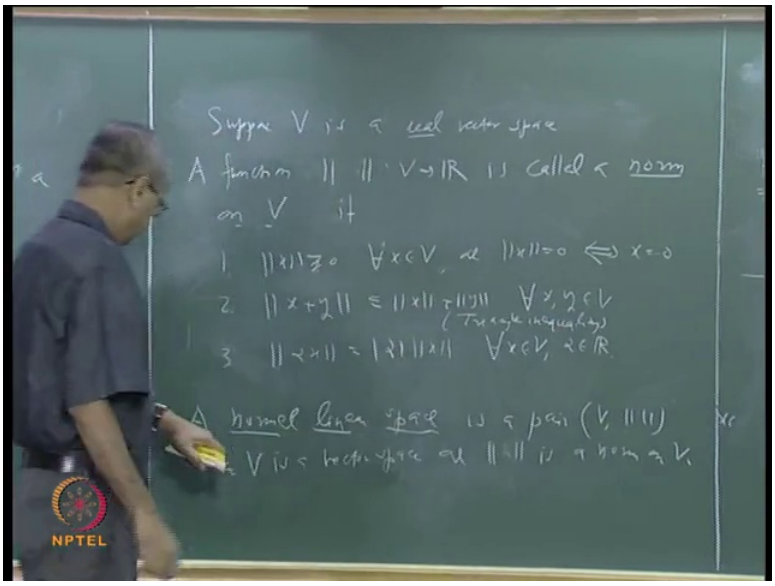

So, that is about a metric and. So, what is a metric space, again in a similar way metric space, is a pair x d where x is a non empty set and d is a metric defined on it. So, let us just recall it once again.

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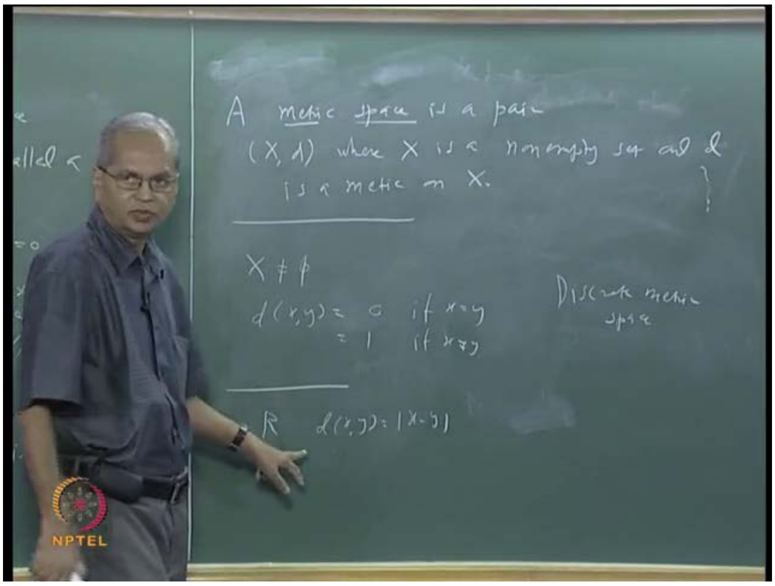

So, metric space is a pair x d where x is a non empty set. Here also I should have said is called a metric on x, this is called metric on x. So, coming back to this a metric space is a pair x d, where x is a non empty set and d is a metric on x. Again why pair again because when one can define several metric on the same set x. So, for example, I can define say d

1 as 1 metric d 2 as the arbitrary d 3 as. So, the set underline set with the same, but a metrics may be different.

So, in this case those become different metric spaces. Now, you can see that all these actions which we have written here or the properties which you have. So, said with the usual concept of distance and those are the ones which are taken for defining the distance. Now, you may ask there are some many other things also which we associate with a distance. For example, we also know that given two points we can talk of something like a midpoint of the two. Then that has not come here in the actions, but again which of the property is to be chosen for making definition.

That is a matter of convenience and also matter of history because this definitions like this arise after several years of efforts from various mathematicians by trying various axioms. Which were better etcetera and ultimately it will decided which exactly the things that go into the definition. So, let us not into that end of history right. Now, let us see some examples of the metric spaces. So, in example is what it should be some non empty set and function defined like this. Usually ah with given function like this to check whether it found a metric or not. Usually these two properties are very easy to check.

In fact by trivial and if at all anything takes some time it is this last property triangle inequality. There is one very famous example where this a metric which you can define on any set. Suppose x is any non empty set and suppose you define set d x y is equal to 0 if x equal to y and one if x not equal to one ok. It is easy to see that it is only this last property will take some time to check as I said other two properties are trivial. So, this is also a well known metric, it is called a discrete metric. The space is called as a discrete metric space it is called discrete metric space.

The main use of this discrete metric space is basically for understanding. It is not much of practical importance you do not come across discrete metric in any applications, but in order to understand the various concepts in metric spaces and to check whether you have understood or not, this example is very useful all right. Then the next obvious example is that of a real line, you can take the real line and define d x y as mod x minus y distance between x. That is the usual distance between the two real numbers. Again it is easy to see that that satisfies all this three properties ok.

Now, let us come back to this, will just finish this. So, we have seen that this function norm is nothing but the generalization of the function of the absolute value. So, we can use this idea in any norm linear space. Suppose I take this instead of taking x and y as two real numbers. Suppose I take x and y as 2 elements in a vector space and define distance between x and y as norm of x minus y. Then that should also satisfy all this properties because those are basically followed from the properties of the absolute value. Let us just quickly see how this happens and then we will stop with this.

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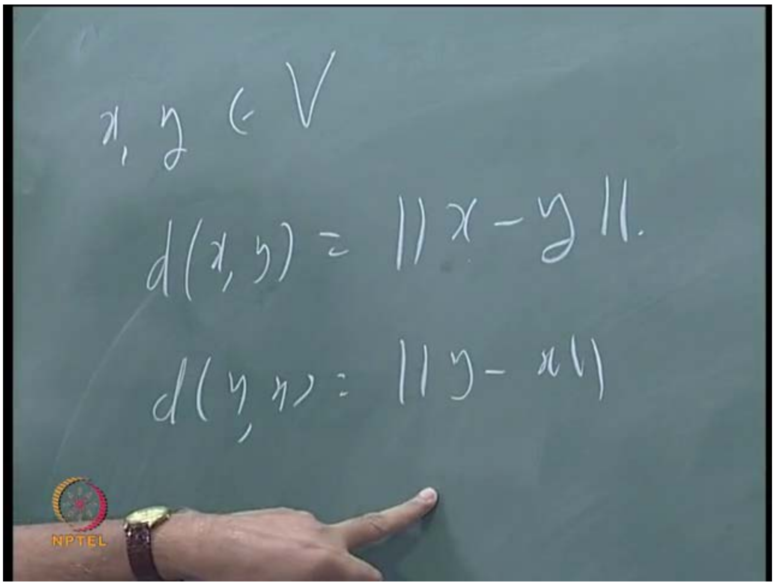

Now, suppose let us say v is a norm linear space and take any 2 x and y in the. Define d x y s norm of x minus y right. Then we will just quickly verify these properties one by one, but in the first thing that we require. The distance between x and y should be bigger nor equal to 0 right. Is it true norm of and this should be 0, if and only x equal to is does the norm true that follows from this pattern norm of x minus y will be 0, if and only x minus which is same as x equal to y all right. What about this distance between x y is equal to distance between y x.

So, distance between y x will be norm y minus x right. By this definition are these two things same y, what it follows from what you, it is nothing but minus 1 time is this. That follows from this last property, if you take alpha is equal to minus that is the basically form norm of minus x is same as norm of x for every x. So, this symmetry follows from this property 3 right. What about the triangle inequality, this is norm of x minus z this is norm of x minus y and that is norm of y minus z. Is that true that norm of x minus z is less nor equal to norm of x minus y plus norm of y minus z. Again you see you can first for example, suppose you take a as x minus z, b as x minus y and c as y minus z, then you can say that norm of a plus b is less nor equal to norm a plus norm b. It will give this.

So, this property 2 implies this triangle inequality here and that is why that is also called triangle inequality right. So, what it what follows from it is that, every norm linear space can be made into a metric space. Every norm can will lead to a metric on that factor space and. So, this is big class of examples of metric spaces and that is what. Most important in applications most metric spaces which are important from the point of few applications are basically norm linear spaces. I think let us stop with that, we shall see more examples of this in the next class.