Real Analysis
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Lecture - 12
Series of Non-Negative Real Numbers

So, we were discussing these series of a non negative terms in the last class and close that we saw.

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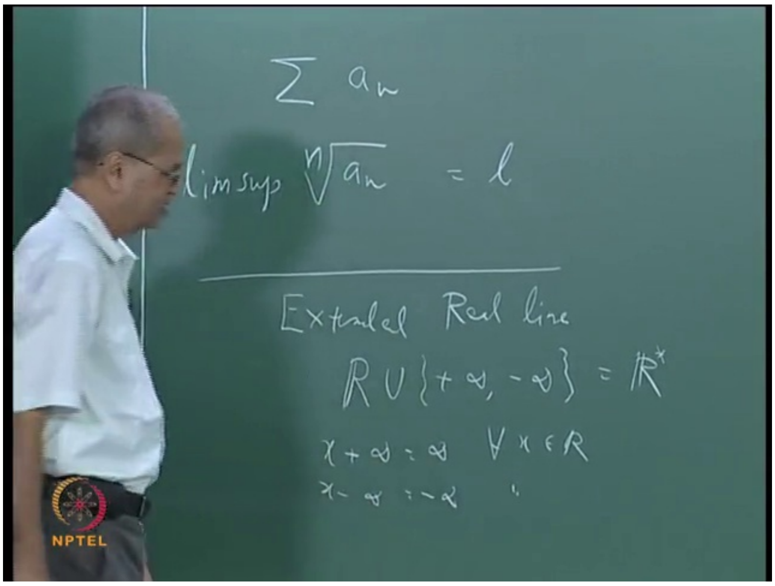

Let me just recall what we suppose this sigma a n is the series. Then we looked at n route of an and lim sup of this. What we said is that, if this is less than 1, the series converges. If this is bigger than 1 the series diverges and if this is equal to 1 this test gives no information. We also seen an example of both the types. Now in this connection there is one more thing that is a required when we discuss this series of non negative terms.

Till now we have been talking about the real numbers and real numbers systems, but for the discussion of this convergence or divergence of a series, it is also convenient to discuss what is called extended real line means nothing, but your usual real line along with these two symbols plus infinity and minus infinity. Some books denote it as R star this usual real line R star. This two additional symbols with the usual properties associated with them for example, properties like x plus infinity is infinity for every x in R etcetera.

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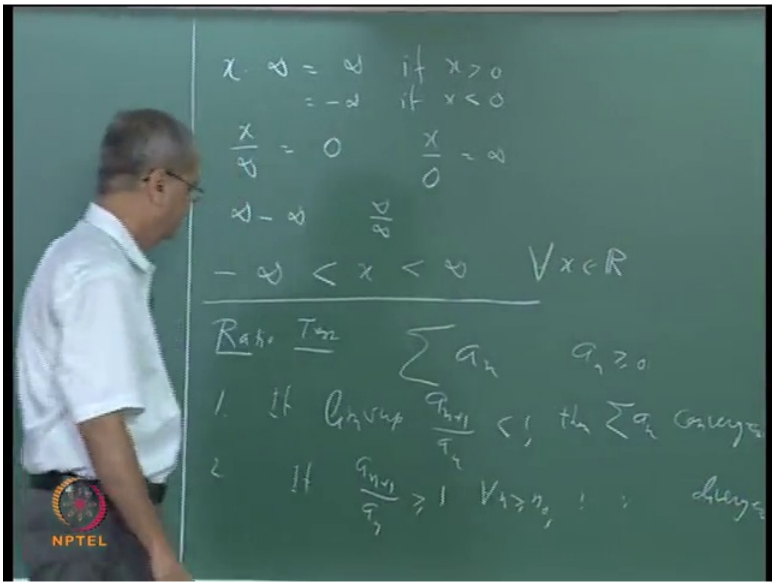

Similarly, x minus infinity is minus infinity for every x in R and x multiplied by infinity, that is infinity if x is bigger than 0, minus infinity if x is less than 0. If x equal to 0 that is usually left undefined and sometimes it is taken as 0, that you will see when you learn major theory. In most of the major theory equals 0 into infinities taken as zero, but in general it is left undefined, x divided by infinity that is taken to be 0, x divided by 0 that is taken as infinity. We avoid things like just as 0 into infinity is not defined.

Similarly, infinity thing like infinity minus infinity or infinity by infinity. These things are left undefined, that is about the algebraic operations. About the order, this is a thing that is for every real number x minus infinity is strictly less than x and strictly less than plus infinity, for all x in R, all right. Now, this has some convenience for example, we have seen that, if a monotonically increasing sequence is bounded above, then it converges, but if we are using extended real line we can say that, if a sequence is puritanically increasing, then it either converges or diverges to infinity.

If it is not bounded above then it diverges, then we say it diverges to infinity. Similarly, for monotonically decreasing sequence, if it is bounded below the it converges to some real number otherwise we say it diverges to minus infinity. In fact some books also use a rotation converges to plus infinity or minus infinity, but we shall not use that. Now, what does the relevance of this two this route test. It is that this l can be infinity. So, that comes under the case l bigger than 1. That comes under the case l bigger than 1. So, even if this l.

So, for example if you look at define a limit superior we have taken the sequence let us say alpha k were alpha k were suprimum of a n for n bigger nor equal to k. So, that limit of that sequence can be infinity. So, limit superior of the sequence says. So, early can be infinity limit inferior can be minus infinity. So, even those cases can be included with this convention all right. Then let us similarly, look at the another popular test namely ratio test . So, here all also we are getting that the series let us say sigma a n and each a n is bigger nor equal to 0. So, we look at limit superior of a n plus 1 y n, that is why it is called ratio test. Limit superior of a n plus 1 y n.

So, this is the let us say that if this is less than 1, then sigma a n converges. So, instead of saying that this limit superior is bigger than 1, what I will say is the following, if a n plus 1 by a n if this is bigger nor equal to 1. Let us say if this happens for all n bigger nor equal to some n 0, then sigma a n diverges. If it is we know that for any sequence limit inferior is less nor equal to limit superior right. So, you look at this sequence a n plus 1 by a n.

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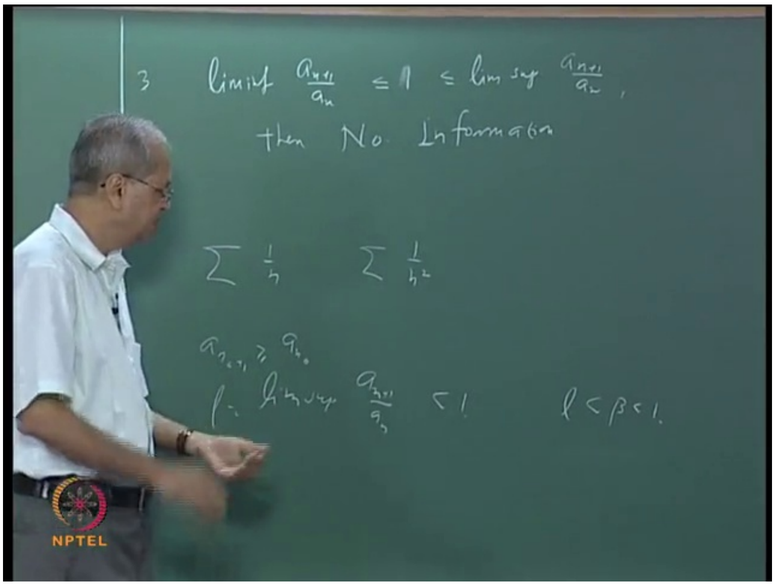

So, suppose the following things happens that limit inferior of a n plus 1 by a n. Suppose, this is less nor equal to 1 and limit superior is bigger nor equal to 1. Then the test gives no information then no information, that means the series make converge or series made diverge we can give the examples of both the kinds. We can say that as far as the proof's are concerned let us dispose of something certain things which are trivial. As far as the last thing is concerned we can say that we are already seen the examples of this.

For example, if you look at say sigma 1 by n and let us say sigma 1 by n square. Then a n plus 1 by a n will be a either something like n plus 1 by n or n plus 1 square by n square and the limit of that will be 1. Once the limit is 1, limit inferior limit superior everything is one. Once set will be this case and we know that this series converges and this diverges. So, there is nothing new here. So, in this case this test gives no information right. Similarly, if you look at this part two if a n plus 1 is bigger nor equal to a n for all n bigger nor equal to n 0, then it will mean that in particular.

For example, a n a n 0 plus 1 is bigger nor equal to a n 0 right. Then a n 0 plus 2 is bigger nor that that means all a n will be bigger nor equal to a n 0 for n bigger nor equal to n 0. So, in which case a n cannot tend to 0, since all of them are bigger nor equal to a n 0 and a n 0 is strictly bigger than 0. So, the sequence a n cannot tend to 0 and hence the series cannot converge all right. Now, let us at this first case if limit superior of a n plus 1 by a n is less than 1. We can suppose that limit superior is l. So, let us say l let us say l is limit superior a n plus 1 by a n.

If l is less than 1 can, as we have done in the previous case, we can always find some number beta which lies between l and 1. So, consider some number beta which lies between l and 1. Then we can say that ah there you exist some n 0, such that for all n bigger nor equal to l 0, a n plus 1 by a n is less than beta right, n because l is limit superior. In fact exactly the same argument we use even for the even for the proof of the similar case in the route test also.

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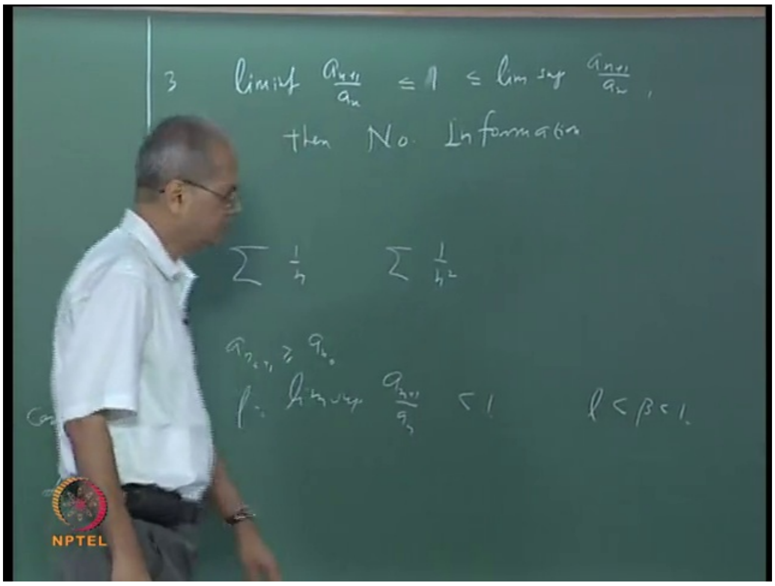

We can say that there exist n 0 in n.

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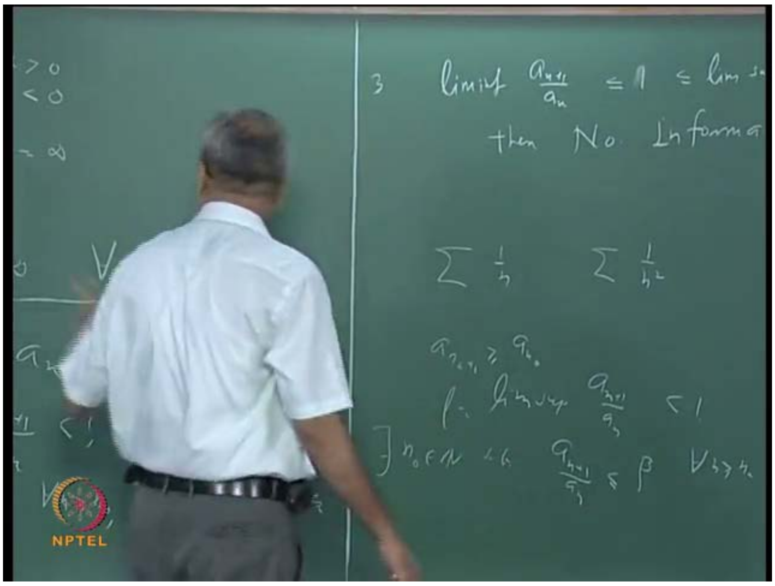

Such that a n plus 1 by a n is less nor equal to beta for all n bigger nor equal to n 0. Then what we can say after this is that if that is the case for example, apply this for l equal to n 0 and you will, I will continue there.

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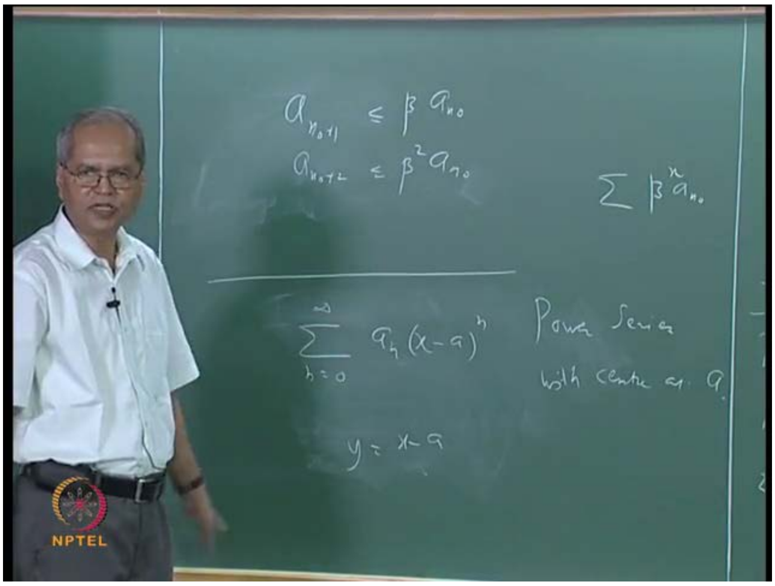

So, if you applied for n equal to n 0, you will get the following that is a n 0 plus 1 this will be less nor equal to beta times a n 0 ok. Then a n 0 plus 2 will be less nor equal to beta times a n 0 plus 1. So, it will be less nor equal to beta square times a n 0 all right. So, you can say for example, a n 0 plus 2 this will be less nor equal to beta square times a n 0 right. In general you can say that a n 0 plus k will be less nor equal to beta to the power k a n 0. In other words you can compare this given series to the series beta to the power n, sigma beta to the power n.

This is a convergent series, now beta to the power n let us say beta to the power n in to a n 0. So, for n bigger nor equal to n 0 is term of the given series sigma a n is less nor equal to the corresponding term of this series. This is a convergent series, now beta to the power n let us say beta to the power n in to a n 0. So, for n bigger nor equal to n 0 is term of the given series sigma a n is less nor equal to the corresponding term of this series. This is a convergent series and hence by the comparison is the given series converges right. So, that is the argument in this case all right.

Of course, given a series of positive terms, you can either apply route test or ratio test to decide the convergence or divergence of that series of course. It can happen that either the test is applicable right. In general the ratio test is easy to apply compare easier compare to the route test, but route test is more applicable there are instances.

Where route test will give some positive answer, but the ratio test will fail one can easily construct examples like that, but let us not go to that kind of thing. Since we are here at this convergence or divergence of the series let us also discuss one or two points which follow immediately from this. Let us also look at the series like this, sigma n going from 0 to infinity a n x to the power n.

Such a series is called power series. Of course, this is again when we subsequently when we discuss the sequences and series of functions, we shall come back to the series like this. So, this the special case of this the reason for discussing it now is that, certain things about the power series follow immediately from whatever we had discussed so far. So, I am discussing that here or more general we can think of a series like this, a n x minus a to the power n, all right?

Now, this is call power series with centre at a and whenever we are given a series like this the question to be asked is, for what value is of x does this series converge. Obviously for what value is of x does it diverge ,those are the questions. You can notice one thing very clearly here is that, you can make a change of variable. You can write let us say y equal to x minus a. We can write then this series will becomes a n y to the power n. So, whenever this series converges for a particular value of y, it will converge for the particular that value of x. You can put y equal to x minus a and draw the appropriate conclusion for x.

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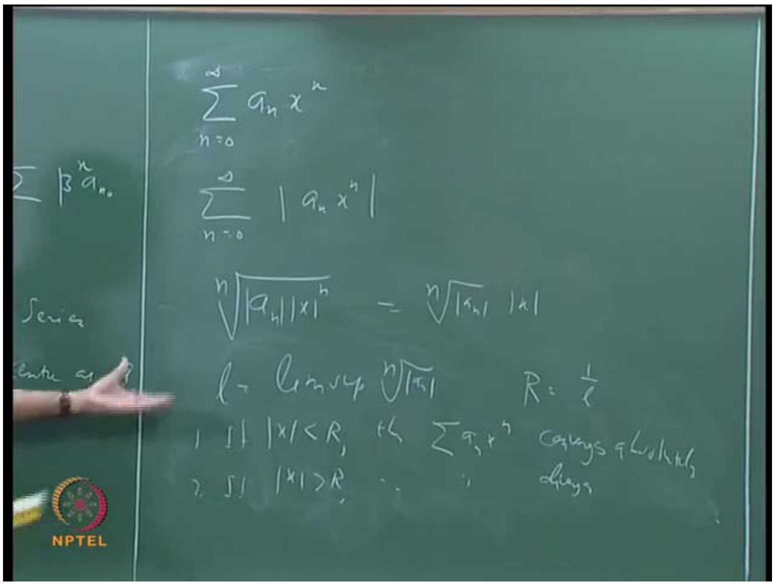

In another words what I want to say is that it is enough to discuss. It is enough to discuss the series like this, sigma a n x to the power n, n going from 0 to infinity. Which is same as saying that without loss of generality we can assume that the centre at 0. It is enough to discuss that case and if centre is something else, we can easily draw the conclusions about those series by using the knowledge about this series. Now, if we instead of this series. Now, suppose I look at this series sigma n going from 0 to n mod a n x to the power n, then we know that whenever this series converges. Then series also converges because we have shown that every absolutely convergence series is convergent.

So, we look at this and let us apply for example, the route test for this. So, if you look at the nth term here, it is a n x to the power n. So, if you look at the nth route of that nth route of mod a n in to mod x to the power n, then that is nothing but nth route of mod a n into mod x. So, if you look at the lim sup of the whole thing there is nothing but, see this is independent of n. So, it will be lim sup of this multiplied by mad x. So, that is why what is done usually is that, we take let us say l as lim sup of a nth route of mad a n right.

What it means is that, if l in to mod x is less than 1 if l in to mod x is less than 1, then this series converges. If l in to mod x is bigger than 1, the series does not converge. And if l into mod x is equal to 1 we cannot conclude anything from this information. Since everywhere type we are talking of things like l in to mod x is less than 1 etcetera. It is convenient to take this number R as 1 by l ok R as 1 by l. Then say that whenever mod x is less than R, that is same as saying that l in to mod x less than.

So, what it what we can say from here is that, if mod x is less than R, then this series converges then and which is same as a that. This series sigma a n x to the power n converges absolute all right. Similarly, one can by it if mod x bigger than R, then we can similarly, show that this it is diverges. We have to discuss, if mod x equal to R we do not know we have to takes that case independently right. Now, what we can see from here is that if mod x is less than R, that is same as saying that x lies between minus R to plus R that is same as in x lies between by.

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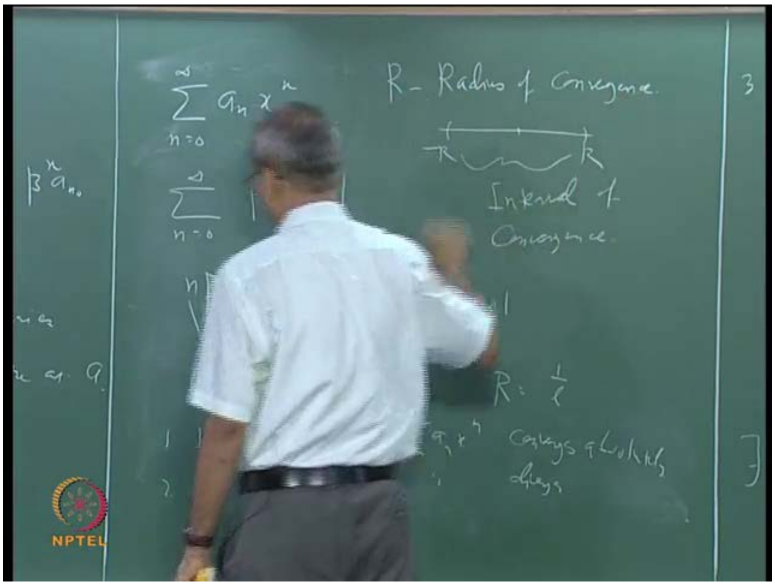

So, that is if you look at this interval minus R to plus R. So, what we are saying is that whenever x is inside this interval the series converges absolutely right. Whenever x is outside this interval the series diverges. At the n points of course, we do not know, we have test it independently and that is why this interval is called interval of convergence of the power series, we call this interval of convergence interval of convergence. This R is called the radius of convergence. Let us take one of the example.

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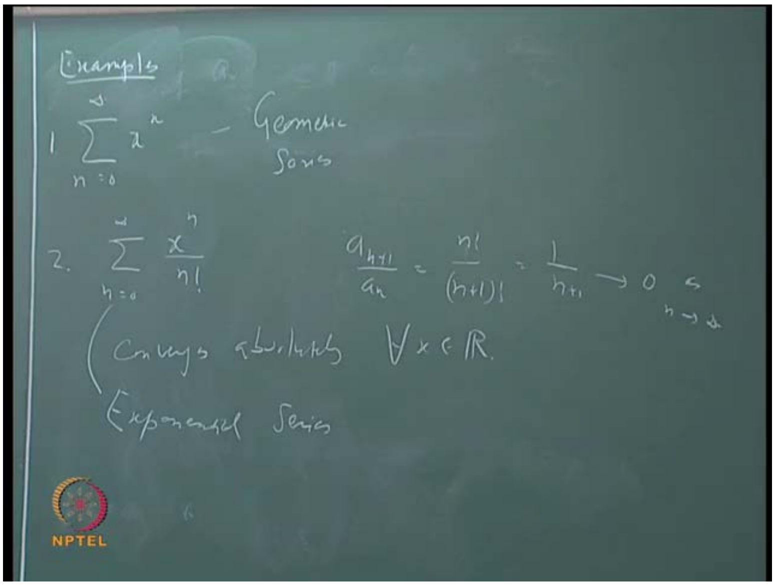

This one is example which we have already seen, sigma x to power n, n going from 0 to infinity. Examples this is the geometrics series and we have seen that this series converges if mod x is less than 1 and diverges in this particular case it will diverge for mod x bigger nor that is it diverges for mod x equal to 1. It also diverges for this n point also, when R is plus 1 and R is R is minus 1. Now, what is to be noted under this is at we are here we are using the extended real line, that is if this l is infinity then the radius is 0.

We make that convergence, if R is equal to 1 by l with the convention that, if this l is 0 R is infinity and if l is infinity R is R is 0 ok. So, to get an example like that let us look at this next example. Sigma x to the power n by factorial n, n going from 0 to infinity. In this case it is easy to look at the ratio test, instead of the route test because here a n is 1 by n factorial. So, if you look at a n plus 1 by a n it is 1 by n plus 1 factorial and divided by again 1 by. So, it is n factorial by n plus 1 factorial. So, it is 1 by n.

So, 1 by n plus 1 right 1 by n plus 1 and that tends to 0 as n tends to infinity. Using this we can show, that is same as saying that this of course, I should have taken n th route of mod n, but we can show that that is also 0 n th route of mod n, that is also 0. Then that will that will say that l is 0, that is saying that R is infinity. That means this series converges absolute for every value of x. So, this converges absolute converges absolute for all x in R. This is what you have already called this is what is called geometric series and you perhaps also know that this is called exponential series is this clear. Whatever we have done so far. Now, let me take a special case of this at x, x equal to 1 that x equal to 1.

Then this will sigma 1 by n factorial n going from 0 to infinity. Our idea is to show that this some of this series is e, this is what we want to show all right. What is that we already shown about this number e, it is the limit of this that is. Suppose I take x n as 1 plus 1 by n plus 2 n, then we have shown that this tends to e. Now, what we want to show is that limit of this is e, limit of this is e means we already know that this converges from whatever we have discussed, we already know that this series converges it converges means what it sequence of its partial some converges.

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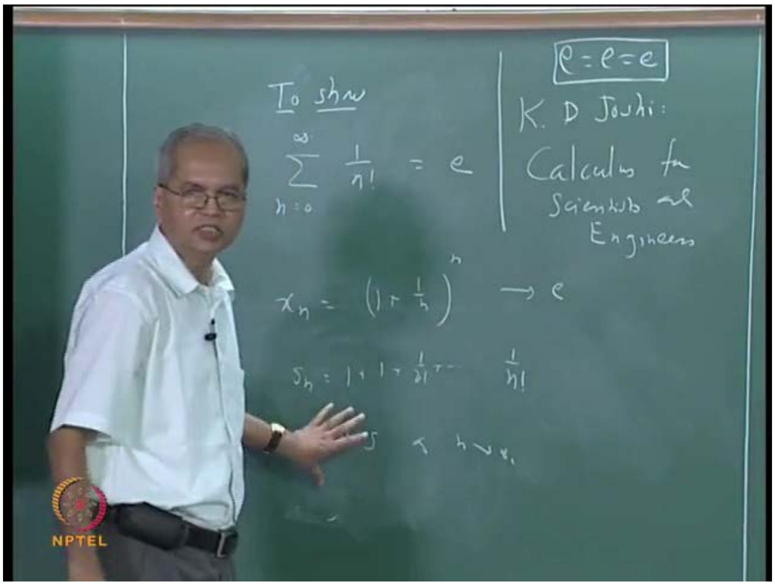

So, suppose we take this sequence it is S n is this 1 plus 1 by 1 factorial etcetera. It is starting from 0. So, it is etcetera up to 1 same 1 by 2 factorial a is up to 1 by n factorial 1 by n factorial. Suppose since we know that will be it is converges. Suppose we call that limit of this as let me just say suppose S n tends to S as n tends to infinity. Then that is then what we want to show is that S is equal to e, right?

What we want to show is that S is equal to e all right. Now, in this case let me also mention see essentially what we are saying that, the number e can be defined in more than 1 ways. This is one way you get the number e. This is another way it which you get number e. In fact it can be also defined in one or two more ways, only thing is it whenever you do it you have show that, all those definitions are the same. So, since we are not going to consider all those things, let me just give you one reference.

It is a book by K D Joshi is a professor in I I T Bombay, that is Joshi. The title is calculus for scientist and engineers, calculus for scientist and engineers. In this book there is a section and the title of that section is this, e equal to e equal to e ok. You can predict from this title, that the number is defined in three different ways and this section shows the equivalence of all those three definitions. So, two of those ways are this, what we had discuss there is one more way. So, if you are interested in that, you can just have look at this. For the time being we shall just show the equivalence of these two definitions.

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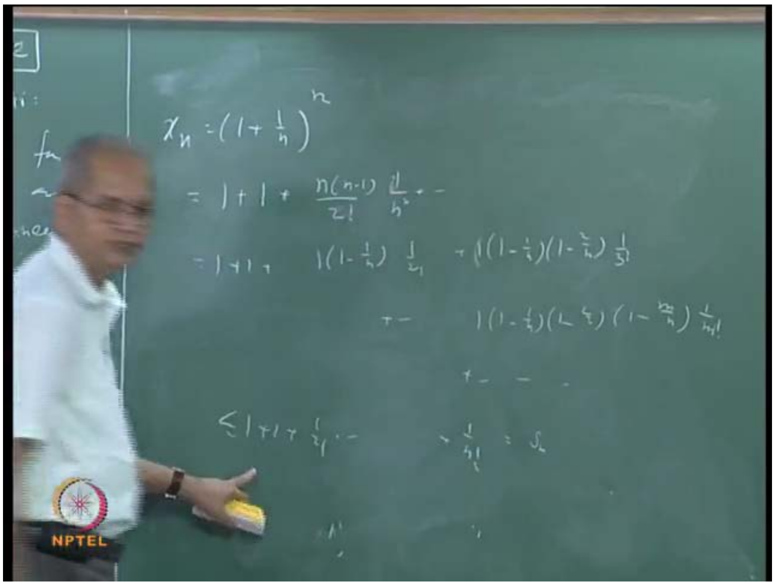

Let me recall that while, when we are proved that this sequence converges, we had got this equation x n is 1 plus 1 by n rest to n. Then we had expanded this by using binomial theorem. So, it is 1 plus 1 plus, we relate n in to n minus 1 by 2 into 1 by n into n minus 1 in into 1 by n square etcetera all right. Then we had simplified this two the following 1 plus 1 plus 1 into 1 minus 1 by n into 1 by 2 factorial. Similarly, there are next term will be 1 into 1 minus 1 by n into 1 minus 2 by n into 1 by factorial 3 etcetera. So, suppose we go on like this the nth term will be 1 into 1 minus 1 by n in to1 minus 2 by n etcetera. 1 1 minus m y n into 1 by m factorial this way.

At that time we had notice that each of this term, 1 minus 1 by and whatever factors are occurring here there are less nor equal to 1. You are separating some positive quantity from this, so, this will be less nor equal to 1 plus 1 plus 1 by factorial 2 etcetera up to 1 by n factorial, right? Which is nothing but S n.

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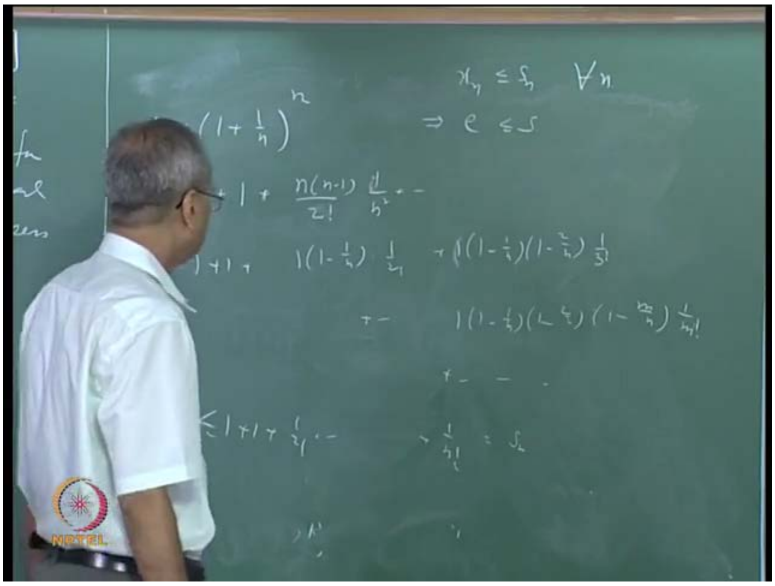

So, suppose you recall this. So, what does this do, that x n is less nor equal to S n for all n x n is less nor equal to S n for all n all right. So, taking in the limit what follows. So, limit of x n should be less nor equal to limit of S n. So, we have call this limit of this as S limit of this as e. So, this proves that e is. So, this imply the e is less nor equal to S ok. Our aim is to show that e equal to S remember this is what we want to show.

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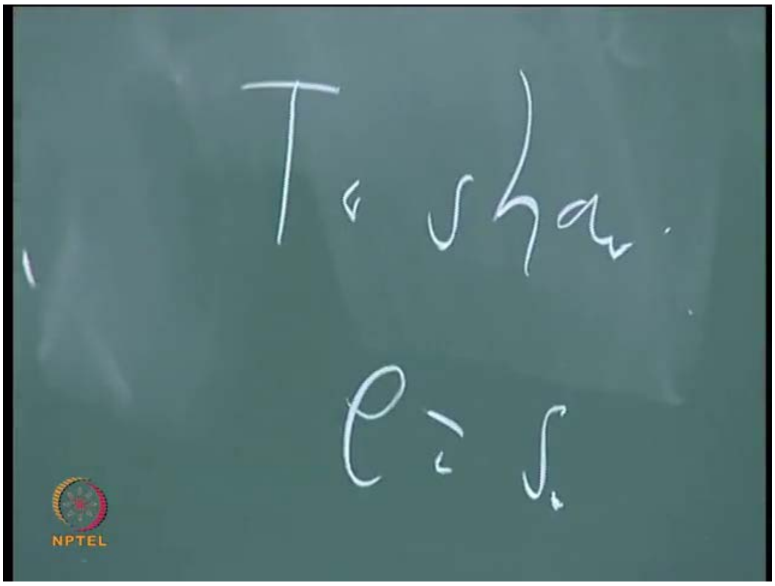

To show e is equal to S. Now, what we are shown is it e less nor equal to S right. So, what is left we shall also show that S is less nor equal to e. Now, from this calculation you will see, that suppose is take some m which is less than n suppose is take some m which is less than n. Let us say 1 less nor equal to m less than n. Then what we can see from here is that ah x n is this etcetera. going up to up to this nth term. So, suppose I ignore all this subsequent terms then I can say that x n is bigger nor equal to the sum up to this. So, what I can say is that x n is bigger than or equal to 1 plus 1 minus 1 by n in to 1 by factorial two etcetera.

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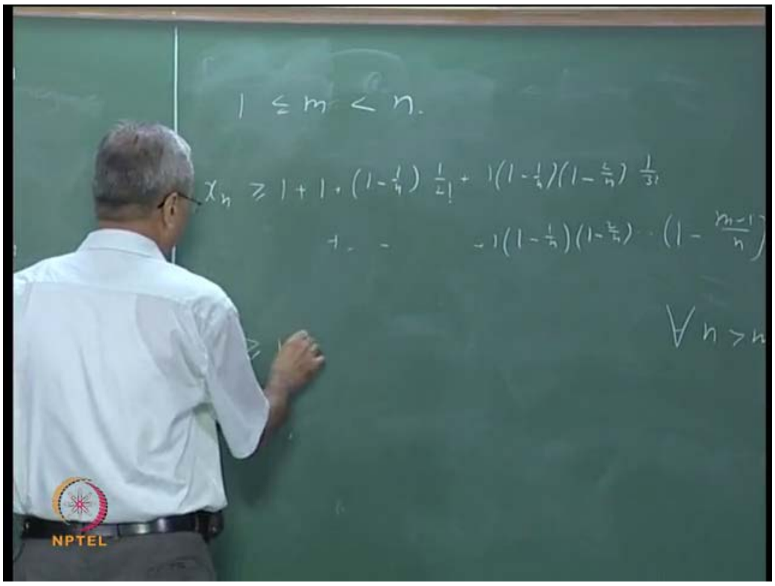

Let us say, let me write one more term plus 1 into 1 minus 1 by n into 1 minus 2 by n into 1 by factorial 3 etcetera. Suppose I go up to m and stop there. So, that will be 1 into 1 minus 1 by n into 1 minus 2 by n etcetera. It will be I think this should have been m minus 1 by n here because you compare with the previous term it is 2 by n m they dint followed by 1 by factorial 3. So, this goes up to 1 minus m minus 1 by n into 1 by factorial m.

Now, this is true for all n satisfying this, for you can say that this is true for all n bigger than m. So, what I say now is that keep some value of m fixed. So, keep m fixed and let n vary keep m fixed and let n vary, let n be bigger than m. So, keeping m fixed and getting n vary. Suppose, I take the limit of both sides. Suppose I take the limit of both sides, then what is a limit of this. This will go to e, what will happen to the left hand side remember m is fixed. So, this term 1 by n will go to 0. So, similarly, 1 by n, 2 by n etcetera m minus 1 by n, all of them will go to 0. So, let me this terms are going to become 1.

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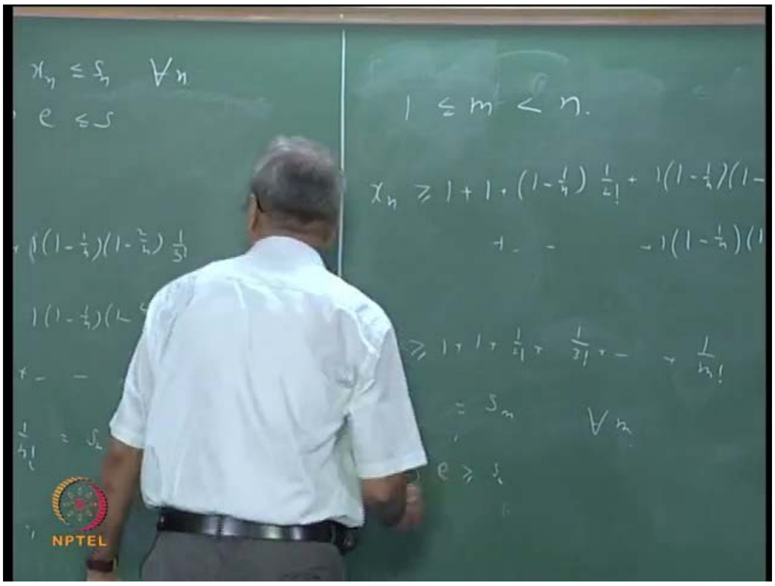

So, what we can say is that, that will imply that taking limit e will be bigger nor equal to 1 plus 1 plus 1 by factorial 2 plus 1 by factorial 3 etcetera up to 1 by m factorial ok. That is nothing but S m right that is nothing but S m ok. What did we prove, now we proved that e is bigger nor equal to S m, but after taking relevant with respect to n is bigger nor equal S m. We can say for all m e is bigger nor equal to S m for all m. Hence we can say that since e is bigger nor equal to S m, for all m, e is also bigger nor equal to S ok.

So, this implies e is bigger nor equal to S right. So, what it shows is that, these two limits e is equal to S are same. That means the number e can be defined in these two different fashion. It is a limit of 1 plus 1 by n, rest to n. It is also a limit of this series all right. Suppose, now we want to also say how this S n, this is a partial some all the series. So, that means S n goes to e means for large values of n, the difference between S n and e is small. That means you can take S n as an approximation of e S n as an approximation of e for large values of n. Now, the question is suppose we want to know how good or bad this approximation is.

For example like an have a problem like this, I want to find a value of e, let us say correct to two decimal places. Then I should know what is a difference between then I should know what n I should take. So, that difference between e and S n is less than 1 by

100. That will be correct true, then I can take S n to be the value of e correct 2 to decimal places. So, to do that, we shall have a some estimate for the difference between e and S n. Of course, one thing is there, all of this S n are less nor equal to e, right? So, this a series of positive terms and S n is a monotonically increasing sequence that is converging to e. So, this e is the least upper bound of S n.

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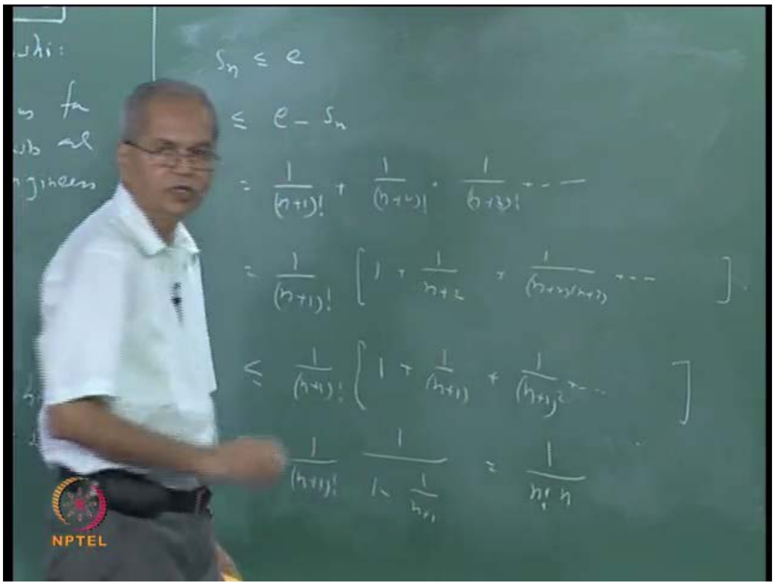

So, this is something that we know already that S n is less nor equal to e or which is same as saying that 0 is less nor equal to e minus etcetera ok all right. Now, we let us have some estimate for e minus S n. S n is e is this let us take this definition e is this and S n is 1 etcetera up to 1 by n factorial. So, what can we say about e minus S n, e minus S n should be the remaining terms of this series. We just sometimes call tail of the series. So, let us let us have some estimate for this term e minus S n will be this 1 by n plus 1 factorial plus 1 by n plus 2 factorial etcetera plus 1 by n plus 3 factorial etcetera, all right?

Now, suppose I want to have a some estimate for this term. Then what I can do is that I will take this 1 by n plus 1 factorial as a common fact of everything. Write the terms which had inside as 1 by this will be 1, 1 by n plus 2 this will be 1 by n plus 2 into n plus 3 alright. Now, there is one observation, here this term add about to anything about this is it clear that this is less than or equal to 1 by n plus 1 right. Is it clear this is less than or equal to 1 by n plus 1 square and subsequent will be less than or equal to 1 by n plus n q is that clear. So, we can say this is less than or equal to this 1 by n plus 1 factorial into 1 plus 1 by n plus 1 plus 1 by n plus 1 square etcetera.

Now, what can we say about this last thing, what is occurring in this bracket is a geometric series right. It is a geometric series that its common ratio is 1 by n plus 1. So, we can take exact surround this series and what is that sum it is 1 by n plus 1 factorial it remains as it is. This will be 1 minus 1 minus 1 by n plus 1 right. So, what you will get after simplification, it will be n plus 1 by n. So, here you have n plus 1 factorial. So, that 1 by that 1 n plus 1 will cancel.

So, what we remind is this 1 by n factorial into n 1 by n factorial into n. So, the difference between e minus S n e and S n will be always less than or equal to 1 by n factorial into n. This tells you how many terms you should take to get the good approximation of e. So, for example, suppose this number is less than say 1 by 100. If this number is less than 1 by 100. Then it means that S n is a correct approximation of e up to 2 decimal places right. What is that for example, this number is going to grow very going to become the n into n factor. It is going to become very large for even for a small values of n for example, if n is equal to 3, 3 factorial is 6.

So, this is three into three fact that is that will be already 18. If you take it is n as 4, it is 24 into 4, it is already bigger than 100. So, it will less than 1 by 0. So, you can easily calculate the values of e value of various, values of e to whatever approximation you want alright. Now, this also has another very interesting consequences. That is not very easy to prove otherwise and namely that let me write that as a theorem, e is irrational ok. By the way what we have proved here is, this let me write it here. So, what we have put is 0 less than or equal to e minus S n and this is less than or equal to 1 by n factorial into n right for all n right this is important. Till now you only know how to prove that root two is irrational right. No other number you know how to prove that it is irrational.

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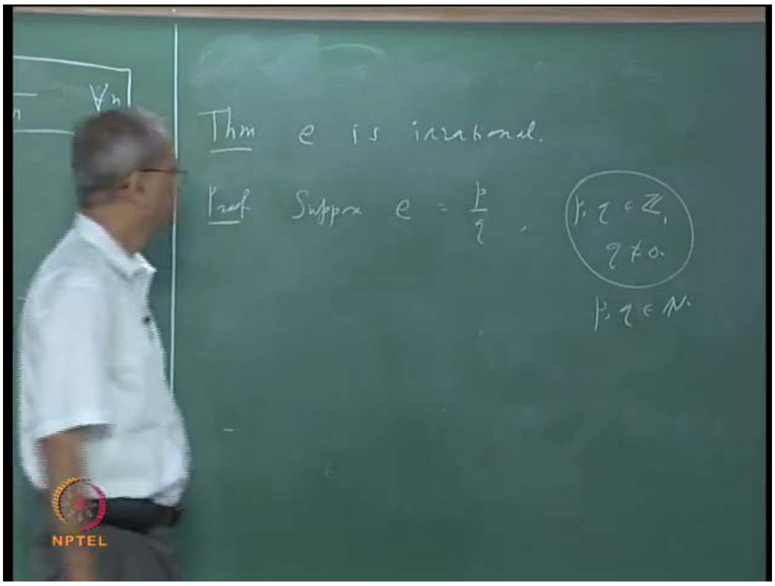

Now, let us take this prove, it is quite different the proof that the root two is irrational. For proving that we shall use this proof. Anyway how one does go on proving something is irrational. There is only one way, you assume that it is rational. Suppose, it is irrational get it get it into some contradiction ok. So, suppose e is rational, rational means what e whether must be form p by q e is p by q, where p and q are integers and q not equal to 0 right. Can we also assume that p and q are positive, we already know e lies between 2 and that is something we know. S0, p and q need not be, we need take negative numbers.

So, in fact I can say instead of this I can simply say p and q belong to m. That is clear alright. Now, look at this, this is true for every n this is true for every n. So, I can take n is equal to q, I can take n is equal to q. So, what I can say is that look at I will rewrite this in a slightly different manner.

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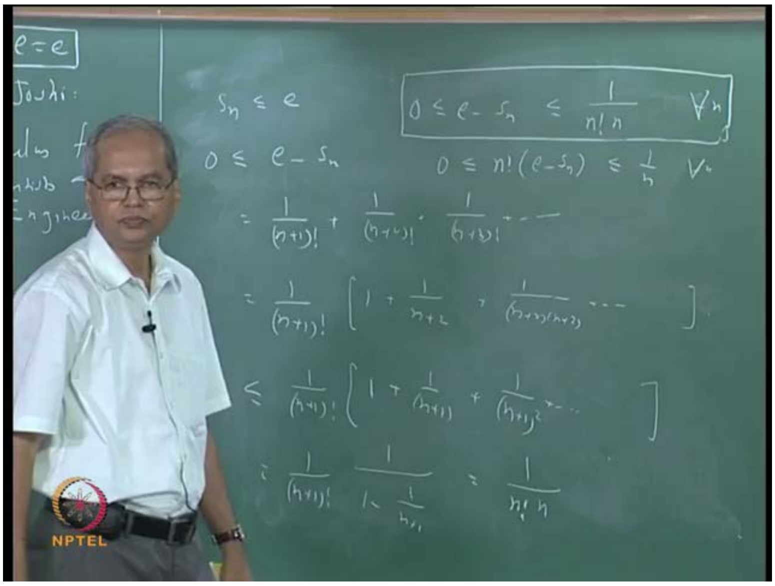

I can this, this from here it also for long that 0 is less than or equal to n factorial into e minus s n. This is less than or equal to 1 by n for all that is also true right. I am just multiplying by n factorial whole thing.

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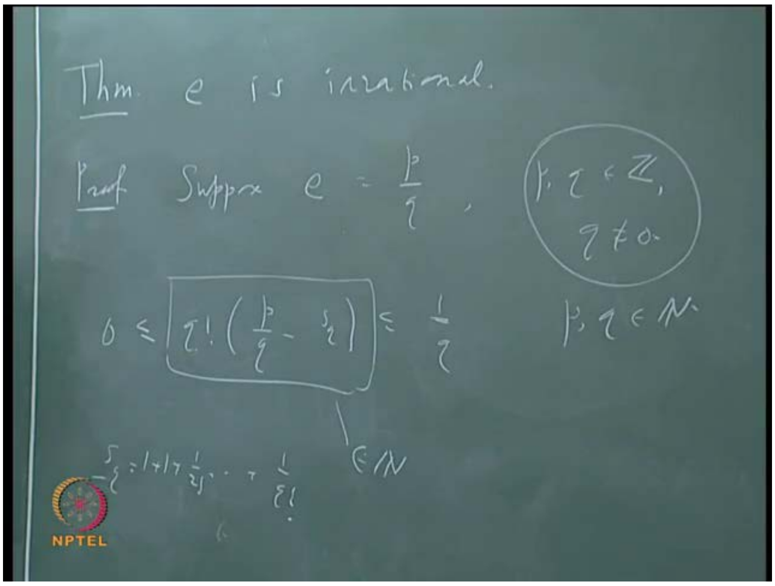

Suppose, I use that for n factorial then what will it mean. It will mean that 0 is less than or equal to n factorial into not n factorial q factorial into e minus s n, e minus s n is e is p by q minus S minus s q and less than or equal to 1 by n. Say 1 by q alright, now just look at this, can something like this happen. What can you say about this number here, see let us look at this q factorial multiplied by p by q, should that be integer, right because it is this will be q minus alternate by p alright. Then sq minus q factorial should that be an integer because what is s q remember, sq is 1 plus 1 plus 1 by factorial 2 etcetera up to 1 by factorial q.

When you multiply that by q factorial, is it clear that what you going to get is a integer in fact a positive integer. So, this number q factorial into p by q minus s q, this must be member of n this must be member of n alright. So, what does it mean that there exist some natural number between 0 and 1 by q. Is that possible right, there is no natural number between 0 and 1 by q that is not possible. So, that is the contradiction, this is a contradiction. It is contradiction to what, contradiction to this assumption that e is rational.

We assume that is referent that is how we got with started with this e is equal to p by q. So, that to that e must be rational alright. Now let us now affect so far we have discussed the cases when the series consisted of non negative terms and all the consequences of taking the non negative terms. Now, how does one discuss the convergence or divergence of a series when the terms are not necessarily positive of course, one thing is that we can discuss absolute convergence.

It would take the absolute value then all terms becomes non negative. Then we can apply, but that initially tells you about whether the series is absolute convergent or not it can happen that series converges, but it is not absolutely convergent. How does one go about testing those things, of course, the actual fact is there are very few methods of testing that kind of converges and we shall discuss one such case ok.

That is the case what is known as Abel' s test. Abel's test is a apply to what are known as alternating series. Let me again repeat there are no book tests for testing the convergence of totally arbitrary series of positive and negative containing both positive and negative terms. So, this is one of those tests alternating series means what the series changes in alternate series changes in alternately. So, the convenient way of representing that is you assume that all a i are positive, all a i are positive. Let us let us say a n bigger than 0 for all n. Then assume that the series is like this, a 1 minus a 2 plus a 3 minus a 4 etcetera or which is same as Abel's series is not sigma a n, but it is sigma minus 1 to the power.

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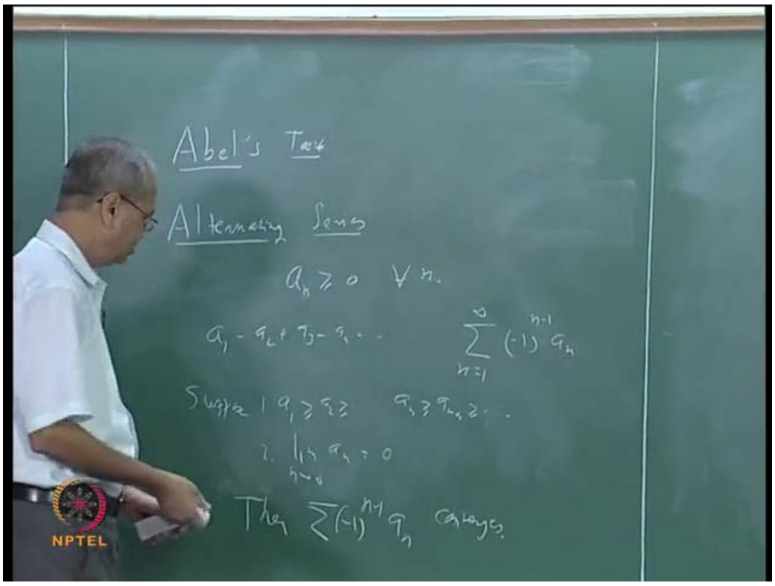

Since you are starting a 1. So, minus on to the power n minus 1 into a n going from 1 to n this is these series ok. That is called alternating series, what does Abel's test say that suppose you have a alternating series like that, that means the terms of the series is like this minus 1 to the power n minus 1, that is the series changes the sign alternately. Suppose the following thing happen, suppose this a n are ((Refer Time: 46:27)) decreasing. Suppose, a 1 bigger than or equal to a 2 etcetera.

In general a n is bigger than or equal to an plus 1 ok. Of course, remember this a n is not terms of the series terms are the series are plus or minus a depending on what is the value of n. This is one requirement and limit a n is 0 limit a n, n tends to infinity that is 0. Then this series converges not sigma n this series converges. We have seen that one of the ways of showing that a series converges is that, it sequence of partial sums converges. In fact that is the only really speaking, if you do not know any other tests etcetera and to show that sequence converges one of the whether showing it is a ((Refer Time: 47:50)) sequence. How does one show that a sequence is ((Refer Time: 47:56)) that again the usual, that is you should show that for the large values of n and m. Difference between s n and s m can be made small.

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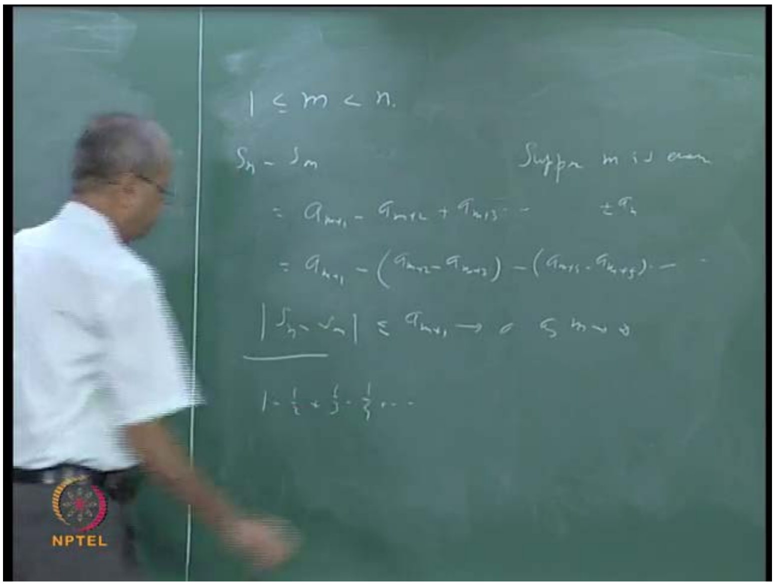

So, suppose we take something like this, that is one less than or equal to m less than n and look at s n minus s m s n minus s m. Now, s n minus s m will be what it will be the corresponding series of the terms starting from m plus 1 m plus 2 etcetera. Only thing is we do not know whether m, m plus 1 is positive or negative, but that depends on m is even or odd. Let us to begin, let us assume that m is even. Suppose m is even suppose m is even, then s n minus s m will be the first of the a m plus 1, then minus a m plus 2 etcetera. Then again plus a m plus 3 etcetera.

It will go up to let us say plus or minus a i, do not bother whether it is a plus I will come to that little later ok. Now, what can we say from here is that, this will be less than or equal this will be equal to a m plus 1, minus this a m plus 2 minus a m plus 3 right. Then minus next a m plus 4 minus a m plus 5 etcetera. If the last term is depends it may be either minus a n minus 1 minus n or it may be just minus a depending on whether n is even or not. In fact that does not matter, what is important here is that each of these terms are non negative right that follows because of ((Refer Time: 49:55)) decreasing. So, each of these terms are non negative.

So, that means you are subtracting some non negative quantities from a m plus 1 which is positive a m plus 1. So, does it follow from that that is this means that mod s n minus a m mod s n minus s m, this is less than or equal to a m plus 1. This is less than or equal to a m plus 1. Only problem is that only we have taken when m is even, we already taken the case m is even if m is odd, what will happen this will be minus a m plus 1 plus a m plus 2 etcetera. Now, how to deal with that case what I will say is that afterward you want s n minus s m. You look at s n minus s m, then it will be again a m plus 1, minus a m plus 2 etcetera. Again you can do the same thing right. So, which ever you take it really speaking it does not matter.

So, even if the, what I want to say is that you can always show that plus or minus of this quantity will be less than or equal a m plus 1 in all cases. So, whether it whether m is even or odd this will be the case is that clear. So, this means mod s n minus s m is less than or equal to a m plus 1 always. Now, what can we say about this quantity, a m plus 1. You look at this last condition, here we assumed that limit of n th term goes to 0 n th goes to 0. So, this tends to 0 as m tends to infinity, but we are assumed m is less than n.

If m and less than m and n both tends to infinity which means you can given any epsilon we can always find some n 0 large enough. Since that where ever m and n both are bigger than n 0 this a m plus 1 is less than epsilon, but in that case mod s n and a m will be less than epsilon. That is same as saying that, this is a ((Refer Time: 51:59)) sequence and that is same as saying the series converges is that clear. So, that is the test for convergence of alternating series. One of the well known examples of this, where this is applicable is, this series one minus 1 by 2 plus 1 3 minus 1 4 etcetera.

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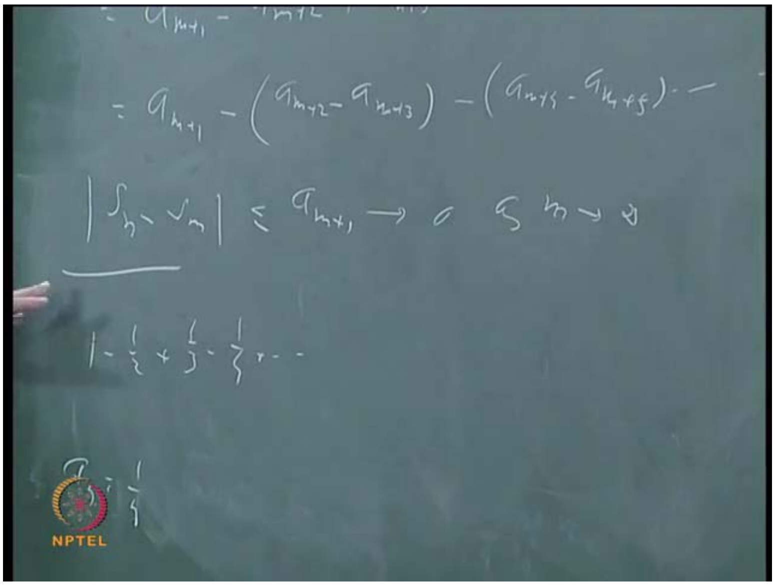

You can see here, we have taken a n as 1 by n a n as 1 by n. So, this is a decreasing sequence, it converges to 0 and that is why this series converges. At the same time it does not converges absolutely because if you take absolute values it will be 1 plus 1 by 2, 2 plus 1 extra it means sigma 1 by m and we have already shown that series diverges. So, this is a example of a series which converges, but does not converge absolutely. I think we will stop with that. We shall make a few observations about this series of this type. Then we shall proceed to the next topic to the next class.