Real Analysis
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Lecture - 10
Sequences of Real Numbers (continued)

We discuss in the last class the concept of what is meant by limit superior and limit inferior of a sequence of a bounded sequence, let us just recall that once again.

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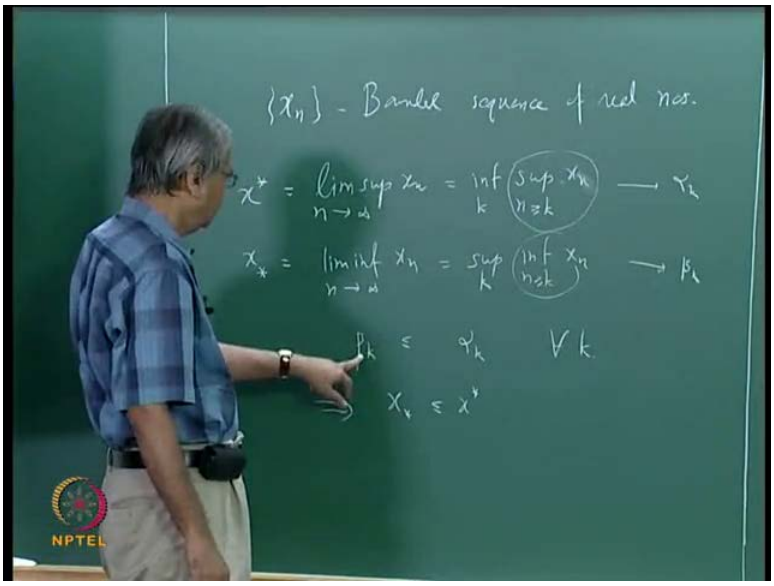

Suppose x n is a bounded sequence bounded of course all sequences are sequences of real numbers, so I need not say that separately. But, if you want you can make sequence of real numbers then we define what is meant by limit superior of x n limit superior of x n this is the notation lim sup and what was the definition of that we have taken that first of all supremum over n bigger not equal to k of x k.

Then infimum over k, infimum over k, let us denote this by x super script star x super script star and then we also define what is meant by limit inferior of x n as n tends to infinity this is a standard notation lim inf. There the roles are inter change that is this is supreme over k infimum over n bigger than or equal to k, sorry this should have been x n, x n and let us denote this by x let us say sub script star.

This is Rudins notation what we have seen is that the sequence may or may not be conversion that is limits that is may or may not be exists, but once the sequence is bounded these two numbers will always exists limit superior and limit inferior will always exist for every bonded sequence. Now, let us see a few properties of these two numbers first of all what we have seen is the following that is if this number supremum over x n, n bigger not over k we had remember we had denoted this number as alpha k. Similarly, infimum of x n for n bigger not equal to k this number we had denoted as beta k and what we had seen is that alpha k is below the degree decreasing sequence that converse is to x star which is infimum.

Beta k is a below notated increasing sequence and that converse is to this number and of course since these are a supremum infimum over the same set we will we should always have beta k is less not equal to alpha k beta. So, what we can say is that beta k is less not equal to alpha k for all k, but we can say something more that it is suppose I fixed a particular beta k then beta k is less not equal to alpha k. So, in other words since beta k is less not equal alpha k for all k what we also must have is this x subscript star is a limit of beta k and x super script star is limit of alpha k.

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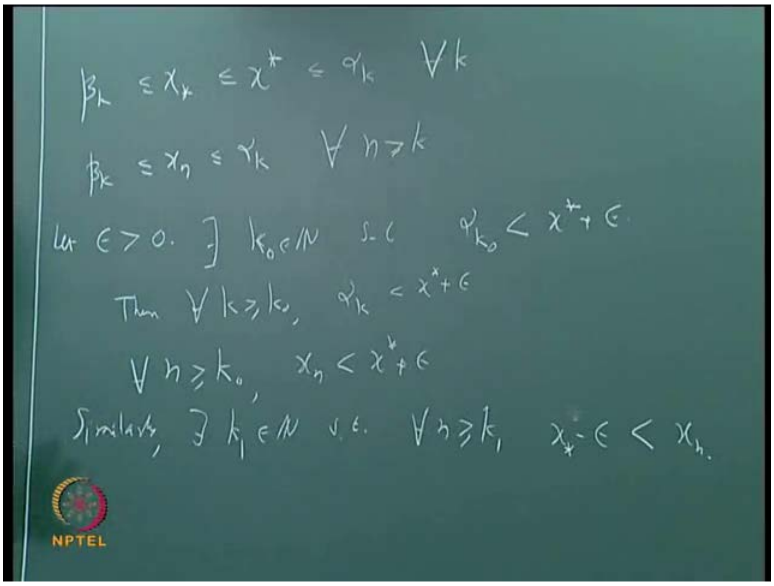

So, is it clear that should imply that limit inferior is always less not equal to limit superior that is true in fact we can say something more. Since x star is suprememum over beta k all of these x star will be bigger not equal to beta k and similarly all of this alpha k will be bigger not equal to x star, so what we can say is this that is for all k.

That is beta k is less not equal this x subscript star this is less not equal to x super script star and this less not equal to alpha k for all k and because of the way in which we have defined. This numbers alpha k for all n bigger not equal to k all essence will lie between this two numbers, let us also recall that that is we can say that beta k is less not equal to x n this is less not equal to alpha k for all n bigger not equal to k. Now, let us, let us see something more suppose we are given some epsilon bigger than 0 let epsilon then we can say that since alpha. Since, this x super script star is an infimum over all these self again infimum mean greatest lower bound.

So, if I take any number bigger than that that is not going to b a lower bound that is not going to be a lower bound, so what does it mean that there will exists some number we will exist some number which is bigger than that. In other words for every epsilon there will exists some alpha k which is bigger than x star sorry which is smaller than x star plus epsilon.

So, relatives then you can say there exists for example for suppose I call k not exist let us say k not in n such that alpha k is less than x star plus epsilon alpha k strictly less than x star plus epsilon. Now, if let me call alpha k not alpha k not is less than x star plus epsilon but, alpha k is a decreasing sequence, so if the if this happens for one k not all subs all for k bigger not could k cannot alpha k are less not equal to this alpha k not. So, they will be less than x star plus epsilon there will be all less than x star plus epsilon, so we can say, so then we can that then for all k bigger not equal to k not alpha k is less than x star plus epsilon.

But, again remember this for n bigger not equal to any k x n is less not equal to alpha k at alpha k is less than x star epsilon, so using this we can say that for all n bigger not equal to k not in fact this is what I wanted to have. For all n bigger not equal to k not x n is less than x star plus epsilon alright. So, what did we prove that for given any epsilon they will exists some k not they will exists some k not such that for all n bigger not equal to k not x n is less than x star plus epsilon. What can we say similarly for this number they will see for example that cannot may be different there will be we can say there exists some k one for example, such that we can say that similarly you can say that.

Similarly, there exists k 1 in n such that I will skip this steps such that for all n bigger not equal to k 1, for all n bigger not equal to k 1 x star minus epsilon less than x. Now, this is a small observation, but we have proved something very important, here see given any sequence this x sub scripts star is less not equal to x star always. But, suppose those two numbers co inside let us say for some sequence those should numbers is co inside then what does it mean, that means this. That means for an epsilon I can take that is a n not which is maximum k not and k 1 and for that n not for that n bigger not equal to that n 0 k all x n will lie between x and this is this should have been x.

Suppose x star equal to x, suppose limit superior is equal to limit inferior and suppose we call the common value x then this argument shows that we can always find the number n 0. Such that whenever n is bigger not equal to n 0 x n lies between x minus epsilon to x plus epsilon, in other words the sequences converges, so what did we show.

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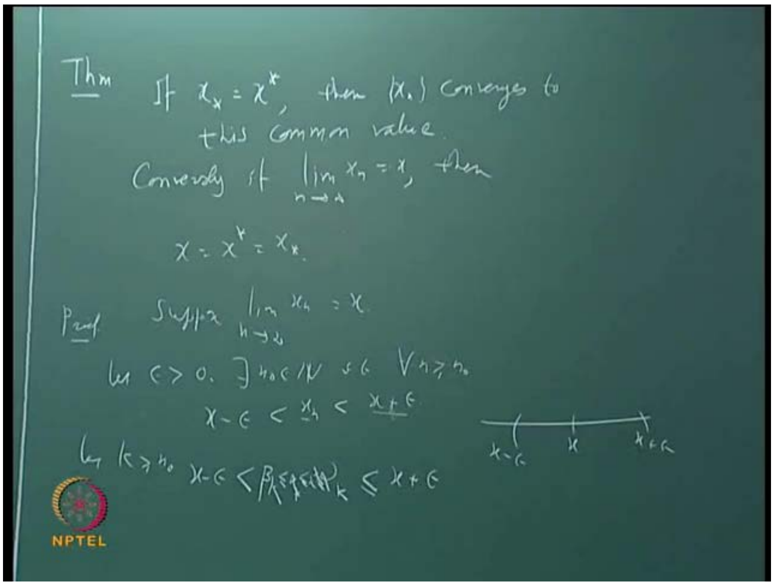

Let me write this is theorem, if x star is equal to this then x n converges to this common value in fact we can say that x n converges to this common value I have already told you what is it. Now, what about the converse x n is an conversion sequence then can we say that these two must be co instance let us say let me just write that we will conversely if limit x n as end touch to an infinity is equal to x. Then x is equal to these limits superior this is equal to limit inferior, now this part we already proved this part we let us just prove the other part, suppose this happens.

Suppose limit of x n as n equals to infinity is equal to x let us take some epsilon bigger than the 0 let then we know that for this epsilon since this happens there exists n 0 in n such that for all n bigger not equal to n 0. We must have mode x n minus x less than epsilon mode n x n minus x less than as well I giving this for x minus epsilon less than x n less than x, x plus epsilon anyways for all n bigger not equal to n 0 x minus epsilon less than x n less than x plus epsilon.

Now, suppose I take some k bigger not equal to n 0 remember all the x n are lying let us say this is x all the x n are lying between this x minus epsilon and x plus epsilon all n x n bigger for n bigger not equal to n 0. Now, suppose I take, suppose I take k bigger not equal to n 0 then what can I say about alpha k and beta k see for n bigger not let us, let us take this in equivalent for all n bigger not equal to n 0 I know that x n bigger than x plus epsilon let us say k is equal to n 0 suppose k is equal to n 0. So, for all n bigger not equal to that k x n is less than x plus epsilon what is alpha k alpha k is supremum over of x n for n bigger not equal to k.

So, what can you say about that if each of this x n less than x plus epsilon that alpha k should also be less than x plus epsilon alpha k, so this implies that and not only for n 0, but this will be true for every k bigger not equal to n 0. So, alpha k is less than x plus epsilon, alpha k is less than x plus epsilon for all k may be less not equal to similarly x n is bigger not equal to x minus epsilon all x n are bigger than x minus epsilon and beta k. What is beta k, beta k is infimum over all those x n each of this is bigger than x minus epsilon, so the infimum should also be bigger than x minus epsilon.

So, what we get, here that is x minus epsilon is less than beta k, now look at this inequality I will insert that x that x subscript star and x super script star between those beta k as an alpha k. So, what will, what I can say is it x minus epsilon, so let me just take that, so this implies, so I can say this is less not equal to x sub script star this is less not equal to x super script star and that is less not equal to alpha k, I shall re write this once again we will continue here.

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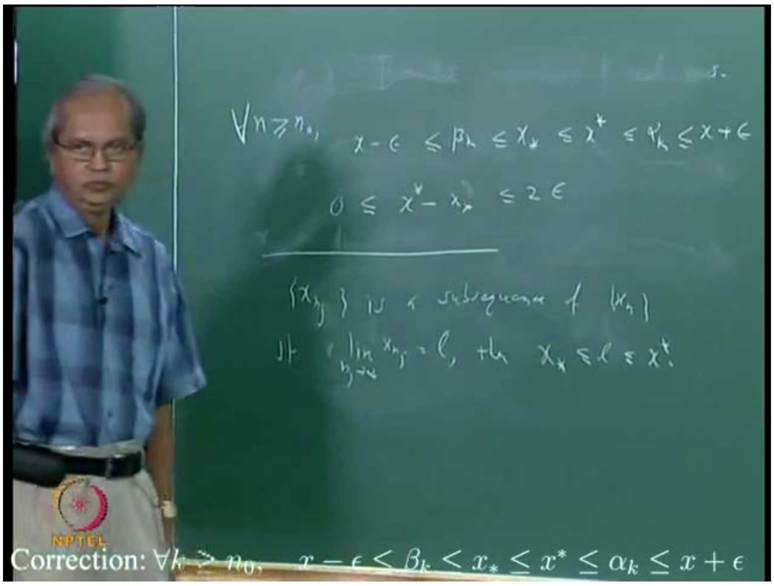

So, what we have proved this for all n bigger not equals to n 0 x minus epsilon less than less not equal to beta k less not equal to x sub script star less not equal to x super script star this is less than equal to alpha k and this is less than or equal to x plus epsilon. So, does it follow from, here that if I look at the difference between x star these two numbers limit superior and limit inferior then that difference is less not equal to 2 epsilon because the whole thing is like that means this two number limit superior. Limit inferior both of those numbers lie in this interval x minus epsilon to x minus epsilon, so the difference between them must be less than two epsilon.

So, what we can say we already know that this is limit superior is bigger not equal to limit inferior, so combining all this will get this inequalities 0 less not equal to limit superior minus limit inferior and this less not equal to 2 epsilon this should be. Now, the argument is usual epsilon was arbitrary what inequality I have written here is true for every epsilon. So, only way in which this can happen is that this must be same this must be the same.

So, whenever sequence converges whenever sequence converges its limits superior and limit inferior equals the limit of the sequence and conversely. If the limit superior is equal to the limit inferior then the sequence must converse and it must converse to that common value, so this is a good way of deciding whether a bounded of course if a sequence is not bounded it is to converges. We already shown that every convergent sequence is bounded, so there is no question there if a sequence is bounded you can always calculate its limit superior and limit inferior and then decide that is convergent or not.

So, this field should always work in most of the sequences, now let us also say something about the sub sequences let us say that x n let us say that x n k is a sub sequence of x. Then you can say that if this I can this is bad notation let we use x n z because k we have use earlier for something else let us say x n z is a sub sequence of x. Then what we have observed, here is that for n bigger not equal to k x n lies between beta k and alpha k x n lies between beta k and alpha k, so what follows from here is that if this n z if this index n z is bigger not equal to that k.

Then that x n z should also lie between beta k and alpha k x n z should also lie between beta k and alpha k, so in particular if this x n z is an convergent sequence if x n z is convergent sequence then that also should lie between beta k and alpha k and for all k for all k. In a similar way again we can prove that that limit must lie between this two numbers, so if you if a sequence has a convergent sub sequence then their limit of that convergent sub sequence must lie between limit inferior and limit superior.

So, let us say that if x n z is, if x n z let us if limit of x n z as z n, z turns to infinity is equal to let us say l then l lies between this two numbers limit inferior less not equal to l less not equal to limit superior. We can also observe one more thing and that is the following that is that means if any if you take any subsequence of us sequence and that subsequence is convergent that is limit must lie between limit inferior of the sequence and limit superior of the sequence, so if you take this interval.

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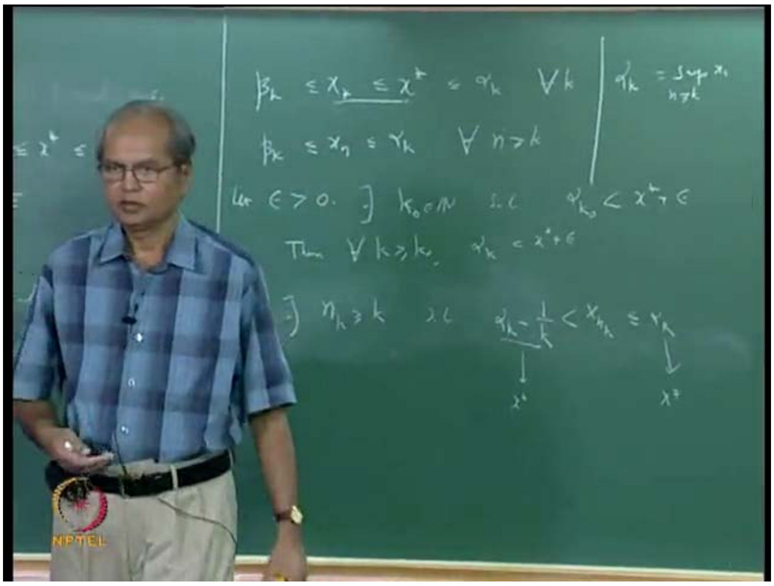

Let us say if you take this interval x, x subscript star to x super script star limits of all convergent sub sequences must lie in that of course is a sequence is convergent those two are going to coincide. Then obviously we have already shown that if sub sequence will also converse the same limit, but if the sequence is not convergent then sub sequences is may converse to different limits. But, all of them live must lie between this two numbers all of them must lie between this two numbers, further we can show that we you can always find a subsequence which converges to this number as well as to this number at that is something you can see again.

Further, you can you can follow in a similar way for example suppose see, here we have seen that given any let us just recall as we got what we did here given any epsilon bigger than 0. We have shown that there existed k not since that alpha k cannot be less than x star plus epsilon, now use the fact then, so for all k bigger not equal to k not alpha k is less than x star plus epsilon. Now, alpha k is a supremum for x n let us recall it what was the alpha k, alpha k is a supremum of x n for n bigger not equal to k.

Now, here I have use only what happens if you take x star plus epsilon, but suppose you take alpha k minus epsilon if you take alpha k minus epsilon then you can always, it will be in the there exists always some x n bigger not equal to k, such that that x n is bigger than alpha k minus epsilon. So, what it means is that let me just write, here given any alpha k you can say that there exists say suppose I call that n k bigger not equal to k there exists n k bigger not equal to k such that alpha k minus epsilon is less than x n k. Of course, this n k may depend on this epsilon also, this n k may depend on this epsilon also and of course this is less than alpha k x n k is less not equal to alpha k because alpha k is a supremum for all n bigger not equal to k.

So, what we can what it mean since that for it each came I can find some number from this some number from the sequence which is arbitrarily closes to alpha k which is close to alpha k. So, and since this is true for every epsilon I can make some choice of epsilon, I can make some choice of the epsilon for example I could have started with taking epsilon as for example like could have started with taking epsilon as something like let us say. Let us 1 by k then that means for a each k I can find some x n k such that alpha k minus 1 by k is less than x n k less not equal to alpha k.

Now, we know that alpha k converges to this x super star x super script star what about alpha k minus 1 by k that will also converge to this x super script star, so that means what should happen to this subsequence x n k then use this sand witch theorem. So, there should also converges to this limits superior, so what did we prove that there exists a subsequence there exist a subsequence which converges to the limit superior that is.

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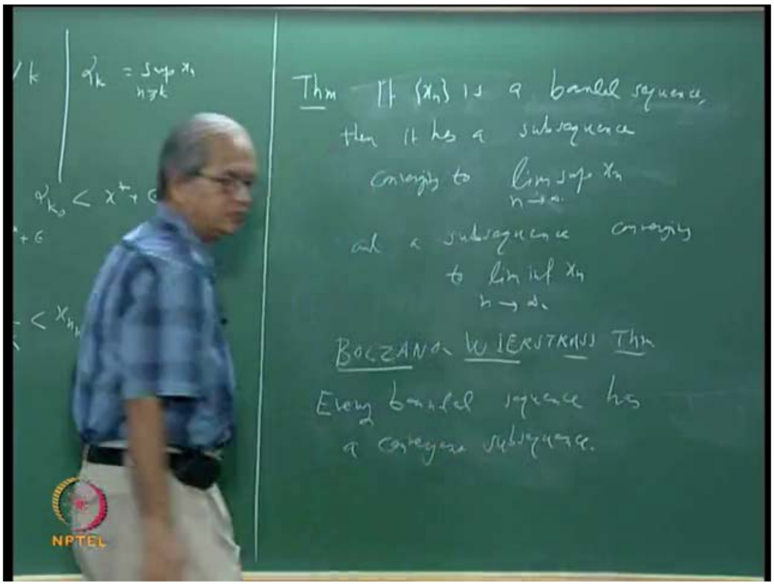

So, let us just write that, here if x n is a bounded sequence then it has a subsequence converging to let us say limit superior of x n, so limit superior of x n as n tense to infinite. In a similar way, we can also segregate also has a subsequence converging to limit inferior of x n also then it has a subsequence converging to limit and a subsequence converging to limit inferior of x is that whatever you sets for. So, if the limit superior and limit inferior are different that means we can find two sub sequences converging to two different limits and that will also been that sequence is not converging.

But, which is something we have seen earlier also, but this has one very important consequence and what is that it means every bounded sequence has a convergent subsequence. This is a very well known this known this is known as a Bolzano Weirstrass theorem very fairly well known theorem Bolzano Weirstrass theorem of course it does not say like this it says it every bounded sequence has a convergent sequence, has a conversion subsequence.

One of the important theorem in elementary analysis in fact you can you can see that we have proved, here something more we have said that their existence subsequence and we have also said something about its limit of course every convergent sub sequences its limit. But, lie between limit inferior and limit superior that is something we have seen already and important thing to realise, here is again that all this thing follows basically from the l u b epsilon the whole idea of defining a supremum. The proof that every monetarily increasing sequence which is bounded above has a limit all those thing are used in this development.

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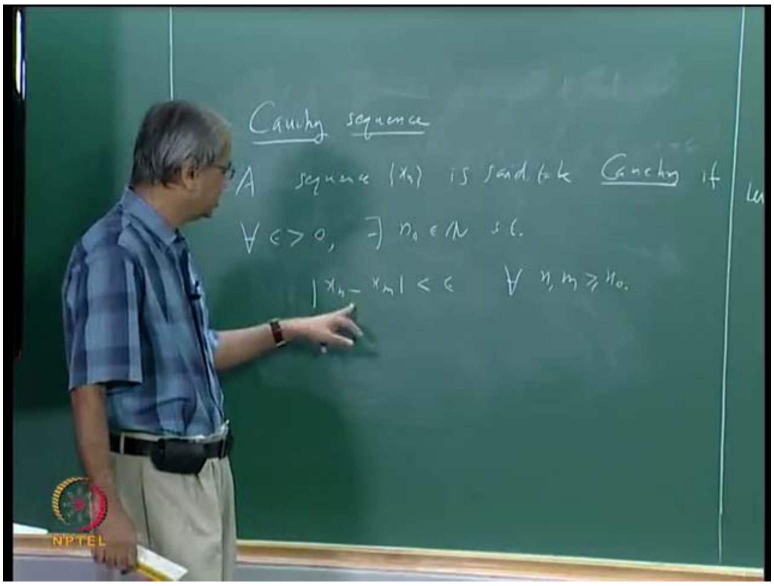

Now, let us go to another important concept what is known as Cauchy sequence again let say that suppose x n is given sequence and will say that a sequence x n which is say that this is a Cauchy sequence. A sequence section is said to be Cauchy if what is tabbed it if you are given any epsilon bigger than 0 if for every epsilon bigger than 0. Again there should exist some n 0 in N such that if you take any two numbers n and m bigger not equal to n 0 then the difference between corresponding x n and x m should be less than this epsilon.

That is then such that mode x n minus x m is less than epsilon for all n and m bigger not equal to n 0 what is this mean that given any epsilon see there is no idea of limit involved. Here, what it means is it given any epsilon the terms of the sequence x n and x m close come arbitrarily, close to each other for large values of n, for large values of n we can make x n and x difference between x n and x m arbitrarily small.

But, taking n and m big enough roughly speaking when we say the sequence is convergent it means that for large values of n all x n go close to that value x where as for Cauchy what it means is that for large values of n and m each of the elements x n come close to each other. Now, intuitively it is clear if all of them go close to someone point x there should also come close to each other that is what is express by saying that every convergent sequence is Cauchy every, let us just quickly see the proof of that.

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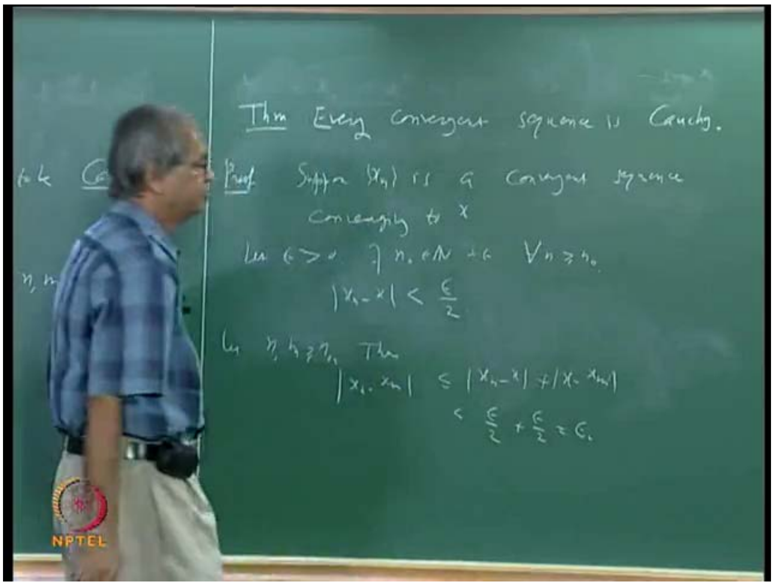

So, theorem every convergent sequence is caution, so every convergent sequence is Cauchy convergent, so suppose with take a convergent sequence x n we should show them Cauchy. So, suppose x n is a convergent sequence, so it converges to x convergent sequence converging to x then we has to show that this happens we have show that this happens. So, let us take epsilon bigger than 0 you will see that the proof follows simply by the element properties of this absolute value function, so we can say there then there exists n 0 in N such that for all n bigger not equal to n 0 what we have is small of x n minus x is less than epsilon.

But, I can as well take epsilon by 2 whether it is epsilon and epsilon by 2 or epsilon by 3 it does not matters you can take any small number, basically for any small positive number you can find some n 0. Now, take n and m both bigger not equals to this n 0, let n and m bigger or equal to n 0 then look at mode x n minus x m what is to be done is clear just add and subtract x. So, this is less not equal to mode x n minus x plus x minus x n and since each for this is less than epsilon by 2, now the obvious question after this is what about the converge can we say that every Cauchy sequence is convergent.

The answer is that it is true for real numbers that is important property of real numbers that every Cauchy sequence of real numbers is convergent and that is express by saying that real line is complete. We shall, we shall discuss what is meant by completeness little later when we discuss matrix spaces this property is not held by all kinds of numbers for example if you take a sequence of rational numbers you can find that sequence of Cauchy sequence of rational numbers.

But, not converging to rational numbers, but that is the real line has this property that every Cauchy sequence of real numbers is convergent there are various ways of proving this. One can prove for example that if a Cauchy sequence, first of all we can prove that a Cauchy sequence is bounded that is very easy let be, I will give that you as an exercise.

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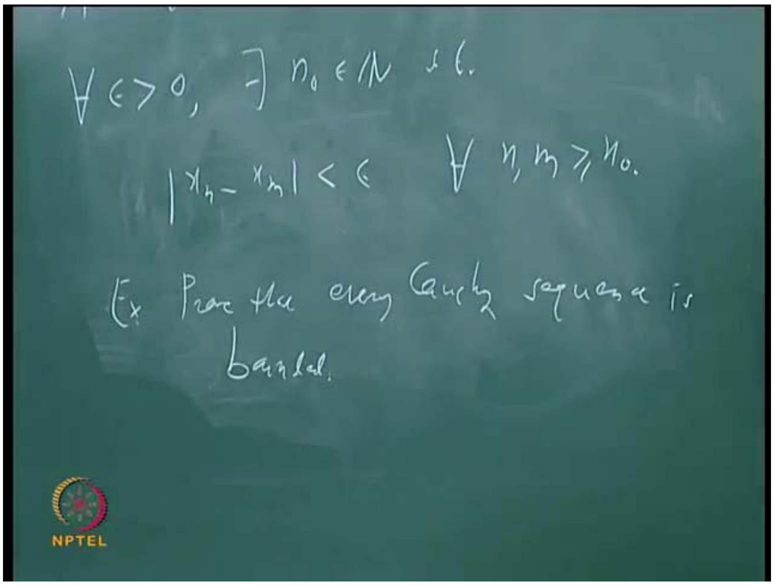

Exercise, prove that every Cauchy sequence is bounded, every Cauchy sequence is bounded then once we know that a sequence and Cauchy sequence is bounded it has a convergent subsequence. One can show that if Cauchy sequence has a convergent subsequence then the sequence itself is converges, so try to prove this on your own this way because it has a convergent subsequence means what suppose that sequence converges to. Let us say some number x it means all those terms of the subsequence go close to x, but it is a Cauchy sequence as if in case of Cauchy sequence, we know that all the terms go close to each other also that mean every term must go close to that number.

This is the idea of their proof you can use this idea and prove that use this method to prove that every Cauchy sequence converges. But, let us use something else that also I have already mention you when R K Kalmon professor, Kumaresom that role of l u b axioms that also contains vote direct proof for this that this how to use l u b axiom directly to show that every Cauchy's sequence is conversion.

So, you can also look at that proof, but it is better to give the proof directly by using this idea of limit inferior and limit superior let us try to do it in that fashion the idea. Here, is simple see this mode x n minus x m is less than epsilon for all n and m bigger not equal to n 0. So, suppose this sequence is I will just give an idea and you can write the proof on your own see suppose I will take m is equal to n 0. Suppose I will take m is equal to n 0 will it been from will it follow from that it will mean that mode x n minus x m 0 is less than epsilon that it will mean from that.

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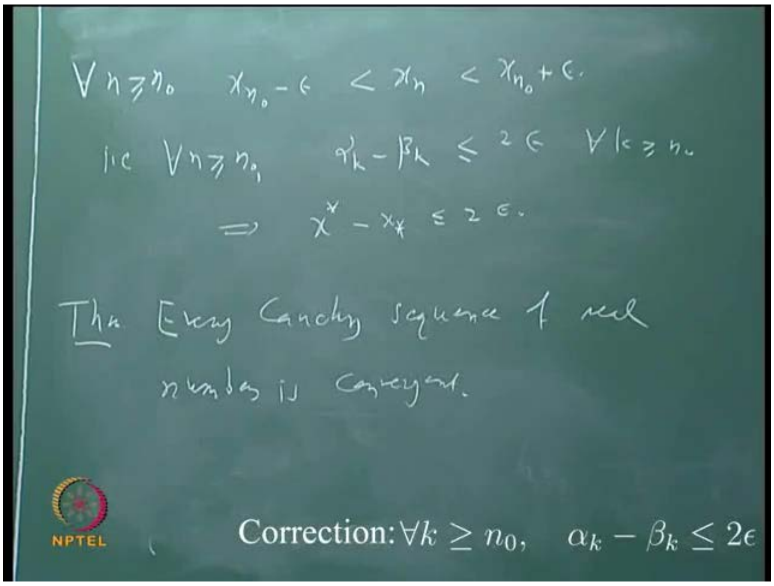

For all n bigger not equal to n 0 x n 0 minus epsilon is less than x n less than x n 0 plus epsilon then what can I say about this beta case is an alpha case from here. Suppose you take k bigger not equal to that n 0 then the difference between alpha k and beta k must be less then can you say from here that for this will imply that is for all n bigger not equal to n 0 beta k minus alpha k must be less than two epsilon.

Remember all of x n are lying between x n 0 minus x, so beta k for k bigger not equal to n 0 beta k must less than x n 0 plus epsilon, sorry alpha k must be less than x n 0 plus epsilon and beta k must be bigger than x n 0 minus epsilon. So, difference between beta k minus alpha k must be less than two epsilon will it follow from here there is a difference between limit superior this is this is true for all k bigger not equal to n 0, for all k bigger not equal to n 0. So, will it follow from here that see this beta, I should have written alpha because alpha k is going to be bigger alpha k minus beta k right so alpha k minus beta k less not over to epsilon for all k bigger not equal to n 0.

What does this mean this means limit superior minus limit inferior less not equal to 2 epsilon and again, now the usually depends since epsilon was arbitrary since epsilon was arbitrary this means that, sorry again this should have been since epsilon arbitrary. This shows that limit superior and limit inferior co initiate and we already shown that when that happens that sequence must be convergent. So, what did we prove that every Cauchy sequence of real numbers is convergent, real numbers is convergent? So, as far as the sequences of real number is concerned there is no difference between convergent sequence and the Cauchy sequence every convergent sequence is Cauchy and every Cauchy sequence is convergent.

This is quite useful in several other concepts which we shall see subsequent, now if we want to give an example of Cauchy sequence in case of real numbers it has be an example of a convergent sequence. If you want to go example of a sequence which is not Cauchy it has to be an example of a sequence which is not convergent, so since these two concepts coincide in that for the sequences of real numbers there is no point in discussing separately. The examples of sequence which are Cauchy are not Cauchy etcetera, so we shall not do their anything like that. So, let us, now proceed further with the concepts which are again closely related the concept of this sequence and that is what is called series.

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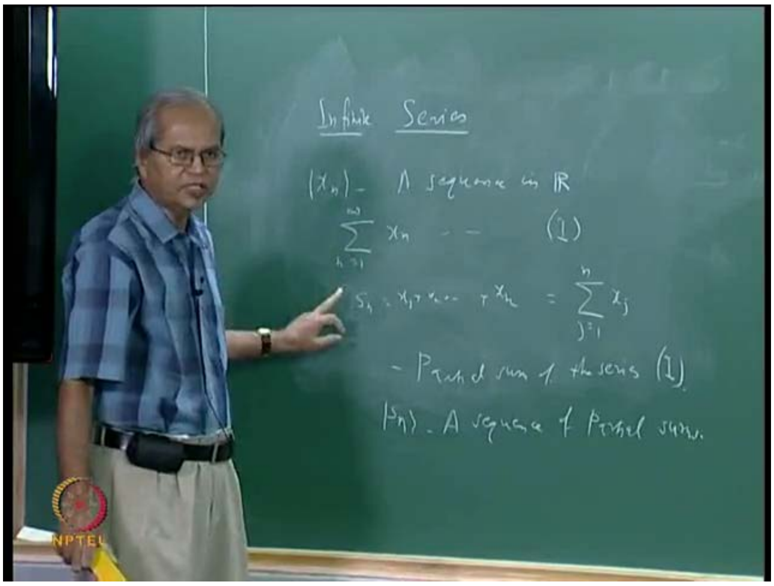

Sometimes also called infinite series, what is that infinite series one can say that infinite series is this basically a sequence it is a sequence, so suppose x n is a sequence a sequence in R. Let us say x n is a sequence in R then we just look at this express sigma x n, n going from one to infinity for the time being we shall say that this symbol may or may not having any meaning this may or may not mean anything other sudden. See if we take only finitely mille terms it will, it means the addition of those real numbers, but in general addition of infinitely real numbers has no meaning in general.

But, at the suddenly conditions we can give a meaning and that is the, that is what the theory above of infinite series deals with. So, let us say suppose we take the first n terms of this sequence and denoted by some new number suppose I call s n as x 1 plus x 2 excreta to x n that is take first n terms since we are going to use this fairly often. Let us use this notation sigma x z, z going from 1 to n sigma x z is going from 1 to n this n is called a partial sum of this series we say n is a partial sum of this series I will give some name and notation for this series.

Suppose I call this series one this is called partial sum of this series 1, so this is a new sequence that means what it is given a sequence x n you form a new sequence which you call a sequence of partial sums, which you call sequence of partial sum. After this whatever you want to define about the series is defined in terms of new sequence this is n, this is the sequence of partial sums remember that x n and s n determine each other.

Once you know the sequence x n you find you can find as sequence s n by this method converge is also true if you know the sequence of partial sums. You can easily find the sequence x n how is the term s 1 is any way x 1, but if you know s 1 and s 2 you can find x 2, if you know s 2 and s 3 you can find x 3. So, in general what we can say is that, so s n plus 1 will be x 1 plus x 2 etc up to x n plus 1.

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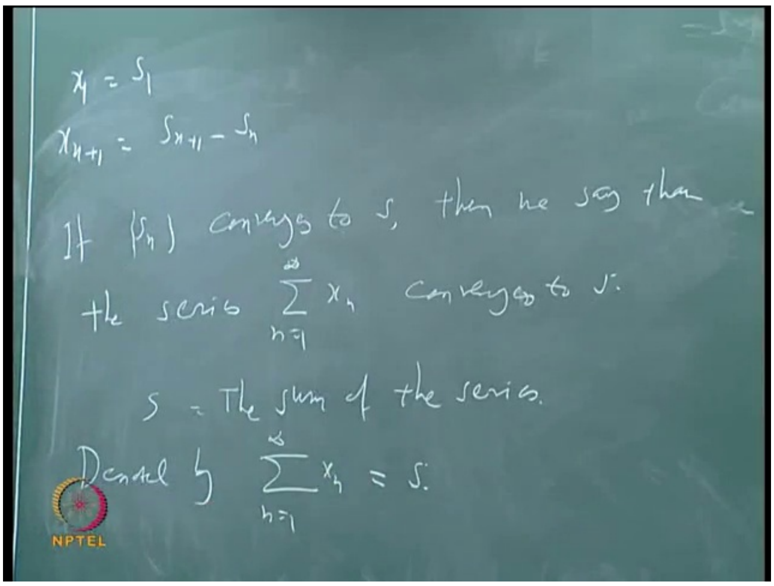

So, we can say that x n plus 1 is s n plus 1 minus s n x 1 is s 1, so if you know s n you know x n and if you know x n you know s n, so both are determined by each other. So, we just look at this sequence s n and if this sequence converges we say that the series converges, we say that the series converges and whatever is the limit of this sequence we call that the sum of the series.

So, if s n converges to s, if s n converges to s then we say that the series sigma x n, n going from one to infinity converges to s and this s is called the sum of the series denoted by this denoted by sigma n going from 1 to infinity x n is equal to s. So, only in this special case this expression sigma x n, n going from 1 infinity has been it only when the series converges we can talk of what its sum sometimes what are called convergent series is also called some books also like to call summable series convergent.

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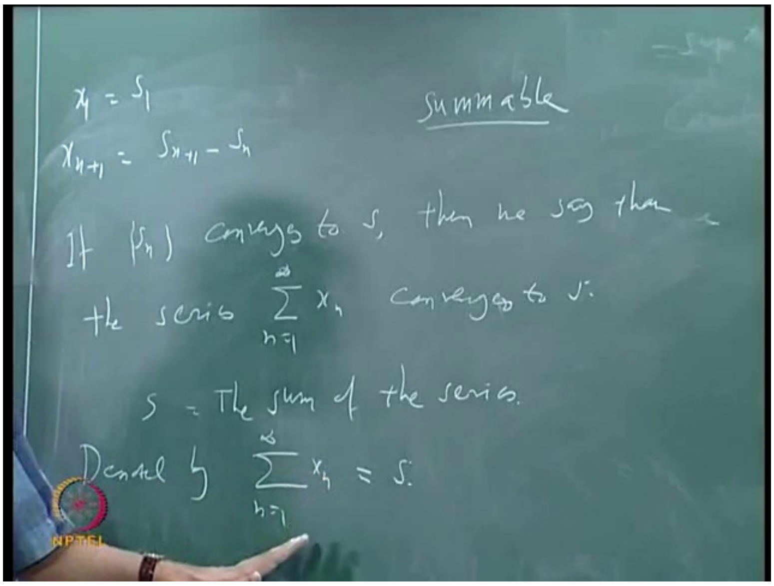

In case of series summable is same as convergent you can see there is any way because it means that a series can be summed you can find the sum on the series that is a new sum I have got. You can make a few observations, here since everything that we are going to talk about a series is going to be in terms of this sequence of partial sums it does not matter for example whether you start from one or 0 or two or any such thing.

So, one more thing just as we are defined that what is meant by convergent in a similar way you can if the if the if this is not a case if the sequence s n diverges we shall let the series diverges. If the sequence s n if the sequence of partial at least not convergent it means it is a divergent sequence we will select the series is divergent and in which case this has no meaning with there is no such thing as this sum on the series is equal to s. Now, we know that given any sequence if you let us say remove of finite number of terms from that sequence the remaining sequence is also going to be convergent. If you add a finite number of terms to that sequence either is the beginning or somewhere in the middle that new sequence is also going to be convergent.

So, what follows from that about the series whether a series is convergent or divergent will not matter if you add a few terms to the series or if you remove few terms from the series the whole behaviour depends on what happens for the large values of x the convergent or divergent of series depends on how x n behave. The values of n that means again our technology for n bigger not equal to n 0 you should above to say something about x n for n bigger not equal to n 0 that is equal to determine the convergent or divergent of a series. That is why it does not matters whether I start from n, n equal to 0 to infinity or n equal to 1 to infinity or 2 to infinity or minus 3 to infinity starting point does not matter.

It does not matter in the sense, it does not matter to decide whether a series converges or diverges, it does not matter does not mean that the sum will not change some will change. That is if you, if you, if you starting from x 1 plus x 2 you start form x nor plus x 1 extra that x nor go this get added to each of this s n some will become that original sum plus x not that will change.

That is the sum will change sum will depend on where you start, but where the series converges or diverges that will not depend on what is the starting point or whether you take a few take a way of few terms. So, either adding or removing a finite number of terms to a series does not ultra its behaviour as for as the convergent is concerned alright so with that we will stop for the we shall see methods of deciding how the series converges or diverges in next.