Mathematical Logic

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Module - 1
Lecture - 7
Calculations and Informal Proofs

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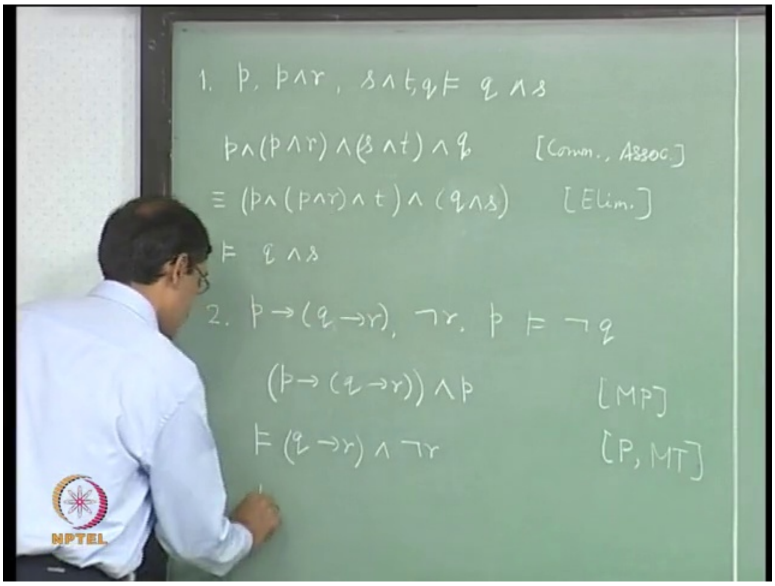

So, let start with say, p and r. Let us say, p and r. See, all the premises are given to us. We want to conclude this. You can at a time say that, all of them are given. I can put them together. Let us say all of them are; so the proposition we get by taking conjunct of all the premises.

Now, you can apply some associative rule or commutative rule, because you want to see that, you want to see them together. How can you get q and s together? You have to move all the other things; use commutativity and associativity. Once you have associativity, these parentheses can move and once you have commutativity, there order can change. So, we can just comment out, we are using commutativity and associativity. Using these you can write as, p and p and r and t, as one, and the other as, s and q. If you take commutativity you can write also q and s. Let us write that, but we do not want all these, we want only q and s not all the other things.

So, what would give us that? The law of elimination says: x and y, from there you can conclude x, from x and y, you can conclude y. That is called the law of elimination. Using that, we can go for this and get q and s. Just look at what we are doing here. In the first instant, we are telling this is equivalent to this, because the laws of commutativity and associativity. That gives us equivalences: x and y, and z is equivalent to x, and y and z. Also, we have commutativity, in the form x and y equivalent to y and x; that is a law. But in that law, we first consider x and y, or z even or propositional variables. Then those propositional variables are substituted here; we are using uniform substitution, is that so?

For example, here, when you come from commutativity to associativity, we take p and p and r and t, first. We take associativity, use commutativity. We use both the things simultaneously, with uniform substitution. Then from this step to this step, again, the same procedure: x and y entails y. So, x and y are considered as propositional variables, where you substitute x as this, y as this, then you obtain this. To see that all the laws and the meta-theorems are in work.

So, these laws, these mata-theorems like uniform substitution, we are not mentioning anywhere. Because of those only we are able to apply the laws. We will not mention anywhere. Since they are, it is permitted, and you proceed. That is the way we will proceed. Then such a thing is called a calculation. A calculation would look like this: some proposition you would start with, then you join with the next proposition by one of these two symbols: equivalence or entailment. Each of those sentences is together, they form equivalence or consequence. They are justifed by the use of laws and the meta-theorems, fine? But finally what do you get? You would be getting first sentence or first proposition entails or is equivalent to the last proposition.

If at least once we have used entailment, then it will be written us entails. If nowhere, you have written entails, then it could be equivalent also, because equivalent is stronger than the entailment. You know that x is equivalent to y if and only if x entails y and also y entails x, right? That is stronger. Therefore, at least one occurrence of entailment will produce only entailment, as the final result, all right?

Now, let us see the second one: p implies q implies r, not r, p entails not q. How do you proceed? p? that is why modus ponens, p, p implies q implies r, so this gives, modus ponens, q implies r. Now, q implies r, not r gives not q by modus tolens, its name is modus tolens, so, r.

You can write in a calculation. So, we start with p implies q implies r. Now, if there are many premises, you may not like to add them all together. At the beginning what we do is, as we go along whenever we need a premise, we just add it together. Then, in that case on the documentation line, we will be writing `P', that something else we are introducing here, which is a premise

For example, here we can start with one p itself, because we need it from the beginning itself. Now, our plan is to use modus ponens here, so, we just write, document it as modus ponens; which tells q implies r. Then what happens with q implies r? You want to add also not r, then here itself, we go to not r. From here you can plan, what is happening? In that case, you say this, not r; here, first thing is you are using a premise. Next thing is you are using modus tolens to conclude the next sentence. So, write modus tolens `MT', and this and then not q.

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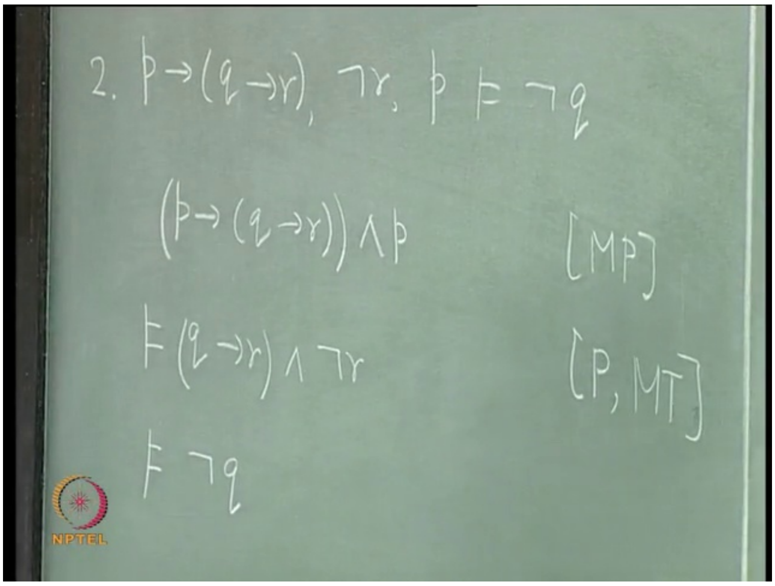

That is the end of it, Modus tolens says that x implies y, then not y this sentence not x. This is not difficult to see by contraposition: x implies y is equivalent to not y implies not x, now use modus tolens. It is a combination of modus tolens and contraposition.

Let us see the third one: p implies r implies not p implies not q implies q implies r. How do you propose to go about? This should be valid, right? That is what we want to prove. It looks a bit clumsy, yeah? Because from p implies r, you are getting all those things. But if you start thinking that way, then you can use deduction theorem. By deduction theorem you can start from this, the whole thing after the implication sign, that is an implication sign. You can again take this as an assumption and try to prove this. That is also in the form of an implication. You can take q again as a premise and prove only r. It is a serial implication.

That means by deduction theorem, we need to show that p implies r, then not p implies not q, q entails r, is it clear? There are three steps. First step is: p implies r, this entailment goes here, replaces the implication sign, p implies r entails this whole thing. Next step is: again that is in the implication form, conclusion, so you say p implies r coma not p implies not q entails q implies r. Next step, again bring q to the left side, so you obtain this.

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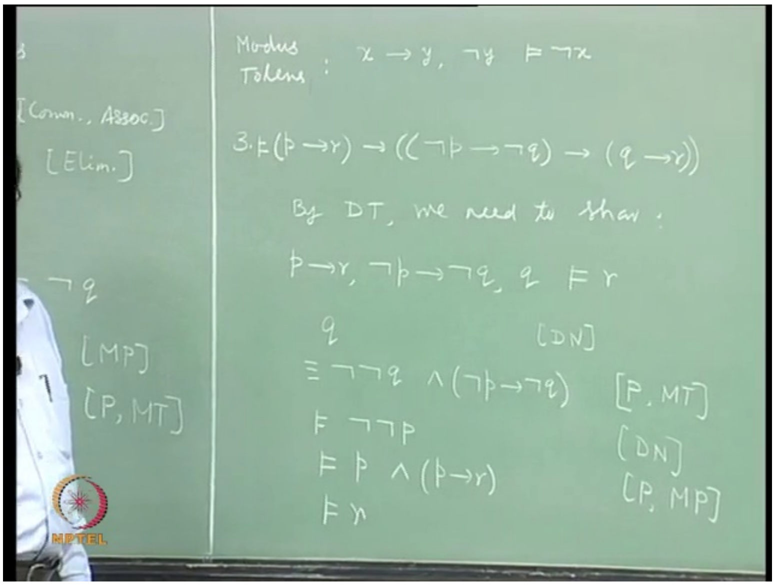

Now, can you show this? It should be easy, yes? Yeah? Suppose you want to use modus tolens. Then this should be in the form not not q, which is because q is equivalent to not not q by double negation. So, you can start from there. Let me start with q and I plan to use double negation. That is equivalent to not not q. And I introduce another premise: not p implies not q, justification is adding a premise. Then for the next step, I want to use modus tolens. Let me write modus tolens, that gives the entailment, what does it give? not p? Negation of that. Look at modus tolens, it says x implies y, not y entails not x. Then this is your x: not p, y is not q. So, not y, x implies y, that gives not x, so not not p, is that right? Then use double negation that gives p, and add the other premise. So, we have used another premise and plan is to use modus ponens; so that gives r.

But you can prove this directly. Let us have another calculation, we have not used equivalent substitution till now. See, the problem is, you have used modus tolens and double negation to bring to one form, where not symbol is not there, right? Now, not p implies not q; it can be replaced by q implies p, because of contraposition.

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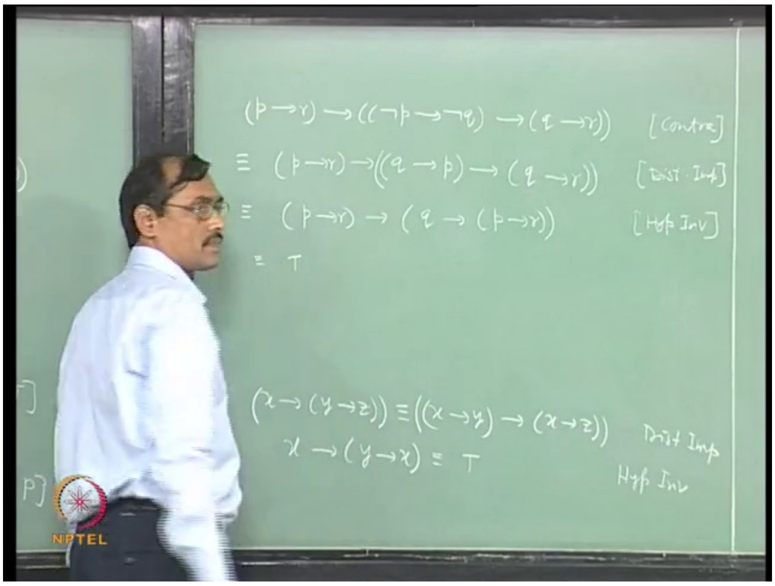

Let us try that avenue. You can start the calculation from p implies r as is given, not p implies not q implies q implies r. Our aim is to use contraposition here. So, write contraposition; that makes it equivalent to p implies r implies q implies p, implies q implies r. Next, this is distribution of implication. So, distribution of implication says, this is equivalent to x implies y implies x implies z. Now, it is matching with the right side, so, that we can replace by the left side. So, that says equivalent to p implies r implies q implies p implies r, Ok? Is that right?

This is the distribution of implication; that we have to use. Let us mention it here. Distribution of implication. Next, what to do? Our aim is to show that it is valid. Well this itself is valid; it is hypotheses invariant: x implies y implies x. So, you say that x implies y implies x is valid itself. So, that means this equivalent to top. This is your invariant. You want to apply that, now with x as your p implies r here and with y as you have, q here. We just write hypotheses invariant. And that is equivalent to top. In a calculation when you want to show something valid it will easier to show that, that is equivalent to top. Because, you are going through equivalences or entailment. Or you can show that top entails it; that requires some ingenuity; which one to introduce, we did not know. Top is a constant it can be written in any form, which is valid. So, which form will give rise to that one, will difficult to find out, but this way if you go, end with top; this way become easier, is it clear?

p implies q implies r. Next premises is s implies not p, next premises is, not s and t implies q q. You want to show that, this sentence implies r, any doubt? So, take a little bit of time to see how form the strategy.

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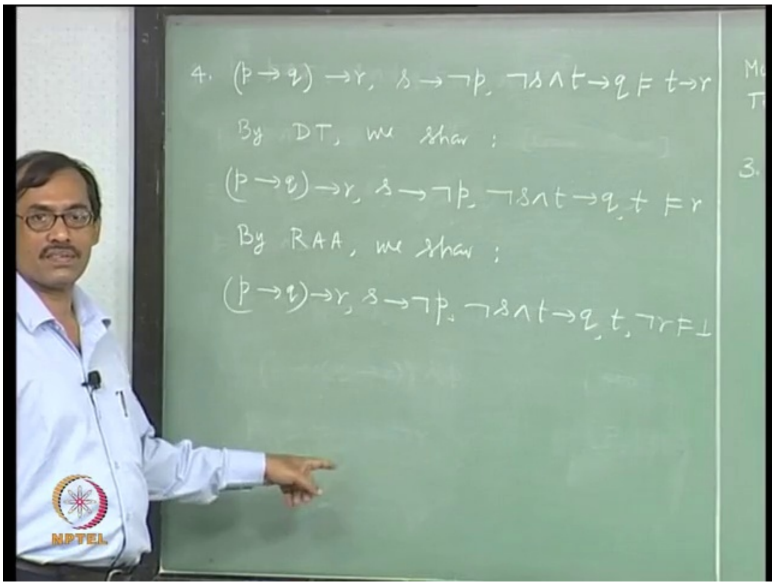

First thing you should see is, the conclusion is in the form t implies r; it is an implication. You can use deduction theorem. Then by deduction theorem, we show p implies q implies r, s implies not p, not s and t implies q, t entails r. It needs some some backward thinking. Backward thinking will be required here. How to get this r? We have here p implies q implies r. So, you must show first p implies q; in order to do that you will be able to apply modus tolens. How to get p implies q? q will be obtained from this; then not s and t should be proved.

Then not s an t, where from? p you can get not s, t is already there; is it? So, you need p and then you will get not s, then not s and t will give you q. Therefore, you have p implies q. Where from we get p implies q? We are getting only q. But you are getting q from the assumption that p, is that ok? So, you have got really p implies q by deduction theorem. It looks that the strategy should work. And you understand the backward reasoning; it says we want r. So, you should prove first p implies q. Now, how to prove p implies q? Assume p derive q. So, if you assume p, then along with this it gives not.

You have not s you have t, so from that you get q. Therefore, p implies q, is it? Then from p implies q you get r. Now, where to start? How to start it, t? Because, the problem is if you prepared the way, you have prepared here for the deduction theorem at a time. We have prepared this, then it proceeds, but here there is a problem. We prepare like this, it comes only to this and in between you have a sub-lemma, that is to be proved: p implies q. Now, to prove p implies q, you assume p and then come to q, in a deduction theorem.

So, that becomes a sub calculation inside the calculation; that is creating a problem. How to adjust the sub-calculation inside a calculation? Well, you can have something like indentation. Follow an indentation, prove it then come back by the deduction theorem. But that will be looking clumsy. Well, here only, reductio ad absurdum will help. So what we do, take negation of r with this, try to prove butttom. Let us try that. By reduction adabsodem. we show p implies q implies r, s implies not p not s, and t implies q, t not r entails bottom. It should be unsatisfaible. Ok? Now, here is another premise, which is not r, you can use it here. Then modus tolens, and continue. Strategy is clear now?

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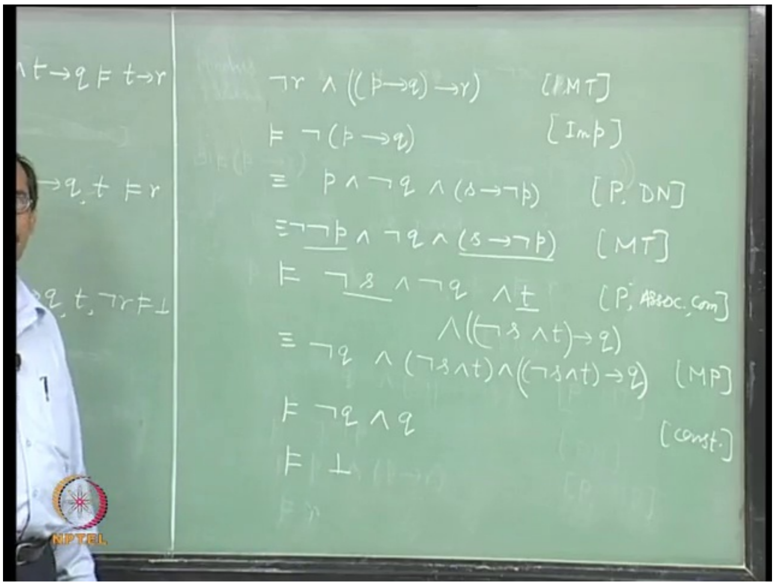

Let us start with not r, which is a premise. And then add also p implies q implies r. In the first step, you need not write premises, because any way you have to start with premises. So, two premises we have taken, our plan is to use modus tolens. not y, x implies y gives not of x, fine? So, let us write modus tolens with gives, entails, negation of p implies q. But how to use negation of p implies q here?

You have to use it some of these equivalent form. So, it equivalence is p and not q right, negation of p implies not q is p and not q. That is called us implications, so law of implication, that gives equivalent to p and not q. Now then you have to introduce some more premises, which one will take? Either this or this q is the p is also here. One of this you can use, which one to choose, which one you will use this or this, it is nondeterministic. You have to choose something then start with, this one s implies not p? s implies not p.

So, we are using a premise here, and then what else? p? modus tolens? You want use modus tolens, because s implies not p here. So, first you have to bring it to not y form double negation. That is equivalent to not p with another not, not q and s implies not p. Then you go for modus tolens. Modus tolens is not and an equivalence. So, this sentence, this is your not y this is x implies y. That gives not s not s and not q, not s is here, t is also here. You can use the premise t, this is a premise, but not only that you need not s and t implies q. Let us use an another permits. is that clear?

But then, we want not s and t together. You need associativity and commutativity of and, then, let us write that, for this bringing them together. So, this is equivalent to, we can keep it, we do not know whether we will need it or not. Let us continue, and see not q and not s and t and not s and t implies q. Next we want to use modus tolens: x and x implies y, there should be a bracket here, because of precedence rule. Then we have modus tolens, which entails not q as it is, and q. Now you see, it is coming up well; this is the law constants. That gives what? In fact, this is equivalent to bottom; this is clear?

The question is whether x entails bottom, x equivalent to bottom, are they same or not? No? Answer should be yes. Because, bottom entails anything, trivially. When you say x equivalent to bottom, it means x entails bottom and bottom entails x, but bottom always entails x. Therefore, they are same, x entails bottom or x equivalent to bottom, is it correct? You need only bottom here, anyway, it enatils is enough for us, but it becomes also equivalent. Finally, anyway it is entails bottom, because there is at least one entails symbol here.

Since, it is implication, you can simply start with p and r as a new premise; and your conclusion will be this one only, right? By the deduction theorem its enough to do this entailment. When you are making the strategy, you look at the conclusion. If it is in the form of not w implies v. There is one way. Not of w implies v is equivalent to w and not v. So, from all this premises you can infer w, you can also infer not v, then and them together and conclude this. That is all right. Or, there is another way. You can use reductio ad absurdum. That w implies is v, once taken as a premise, will give you bottom; that looks easier here. You do not have to do so many calculations.

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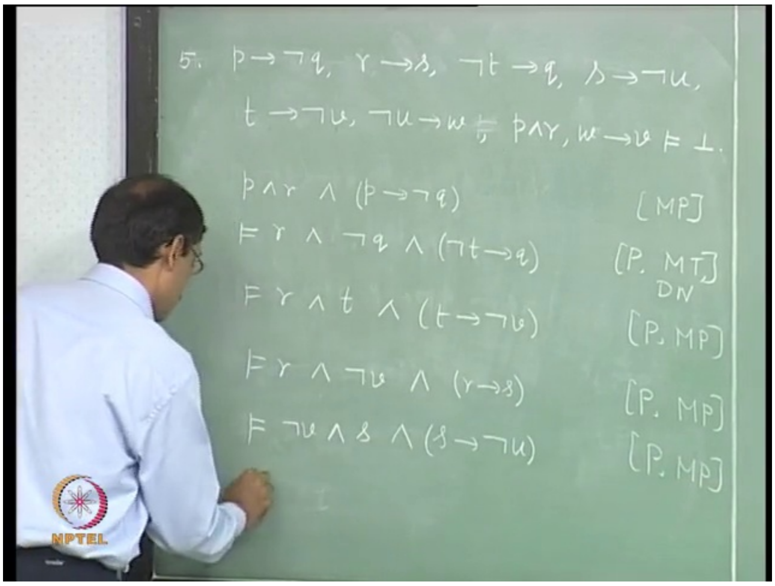

Let us take that view, so we add w implies v as a premise and try for reaching bottom. Ok?Just one application of deduction theorem and one application of reductio ad absurdum. You can think, where to go. See all these are implications, except one thing, which is p and r, from which you can use p, you can use r also. Let us take p if I take p, I would get not q; not q would give me t, t would give not v? Let be there, not v. You can take not w also, but does not matter, stop at not v then. So, from p, I can get not v what can I get from r? r gives s, s gives not u, not u gives w, w gives v. So, v, not v will give bottom, is it clear?

So, let us start. We start with p and r. So, this entails p. But our strategy says, we want to use also r. So, let us keep r throughout, that will also be help full will not eliminate now. Then from p, I want to use p implies not q. So, I can just keep r here as it is, and then manipulate all the others. Instead of entails p, I introduce the new premise p implies not q and keep my r as it is. Let us start with this: and p implies not q, this I can start with. You want to use modus ponens. That gives, r is kept as it is, and p, p implies not q, gives not q.

It is not only modus ponens; I have used commutativity, r comes in the beginning; associativity, I have not used parentheses anywhere; and is there. So, both these things are used implicitly, here associativity and commutativity. Fine. Now with not q our plan is to use not t implies q, so introduce another premises here. I have used the premise and then you want to use modus tolens. So, that gives r and not of not, at this stage if you want, you can put double negation here, not of not t will be equivalent to t. Fine? With t, which one? This one? Modus ponens. At this not t, on the other side, I could have used modus tolens, but it is on the left side. I cannot use that, but this one, I can use modus ponens.

So, introduce another premise: t implies not of v; the premise and modus ponens. That gives r and not v. Our plan was to keep not v there, start expanding from r. Then, for that, we need another premise, which is r implies s, we will use `P'. Then r implies s will give by modus ponens, not v. So, let us keep not v here, and r, r implies s will give us s. Then our plan was to use s implies not u, add a premise. Then we use use modus ponens to get not v and not u. So, our plan was to use not u implies w. We add a premise and use modus ponens; that gives not v and w. Ok? Next step, w implies v; so, add a premise and use modus ponens. That gives not v and v. There we are done. So, it is the law of constants, is that clear?

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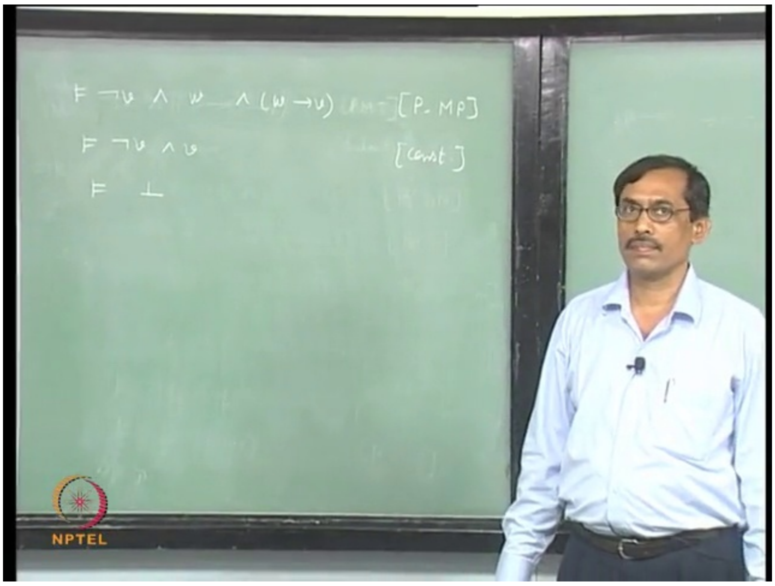

We had observed some discomfort in one of the problems, where we had to use reductio ad absurdum, because deduction theorem becomes a sub-lemma. And our proposal was, you can have some indentation, so that it is a sub-calculation inside a calculation, then conclude and carry on. But to do that sub-lemma inside a sub-calculation inside a calculation, you have to write a line by deduction theorem and so on. Because, we are not able to use deduction theorem as it is, inside a calculation.

This sequence of steps, which are joined by equivalent symbol or entailment symbol, prevents us to do that. So, the proposal is what we do, we forget equivalences, we just continue with entitlement. Do not write it, because everything is entitlement, we will not write it at all. But then how to use detection theorem?

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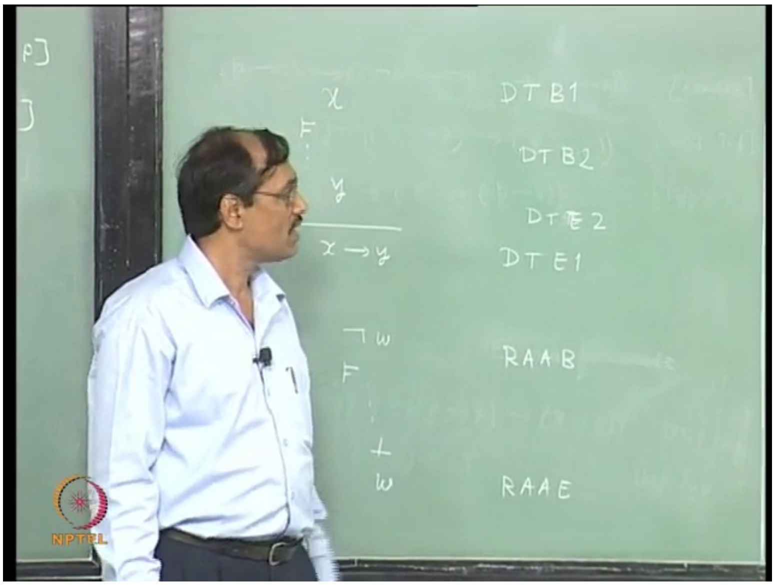

We can evolve some schemes. For example, suppose I have x, I have y. This x, I am taking as an extra premise. I see that my entailment relation, all these entailments, gives me y, then I conclude in the next step x implies y. But when I conclude this x implies y here, these extra assumption is removed. It is no more an extra assumption; without assuming anything, I could have concluded x implies y, by deduction theorem; that is how this deduction theorem should work.

You are assuming some extra premise x and then concluding y; then finally you say without assuming that extra premise, you could have got x implies y. Right? Because deduction theorem says sigma union x, x is an added premise, entails y. Therefore, sigma entails x implies y, fine? So, these things can be incorporated provided you do not have a calculation like this. But just continue the proof thinking that everything is an entailment. What do we do there, is, we have to mention this, that it is an extra premise. And then mention it here that my extra premise is removed, it is no more an extra premise. These two things you have to tell in some way. You have to write it somewhere. What we do is, when we take this extra premise, we write deduction theorem begins, here we are using the deduction theorem starting, then when we end it, we will write that deduction theorem ends. It will tell us that inside this you have used deduction theorem with this as the extra premise. Then We have concluded this by taking away that extra premise; right?

Similarly, for reductio ad absurdum, what do we do? You are taking another assumption not w; sigma union not w entails bottom, right? Therefore, sigma entails w. So, you can add one extra premise not w; which is an extra premise. Then finally with entailments, we get bottom. And the next step should be w, without this extra premise anywhere, by assuming this extra premise, whatever sigma is; those premises of sigma can be used here, also inside this. But then after this, once we derive bottom, we say that this extra premise is now removed, we say that all the other premises used entail w. Again here, what we do, we just mark it with reductio ad absurdum begins. Then when we conclude this w, after the bottom, we write reductio ad absurdum ends, right?

We are just thinking it loudly. How to implement it we will see; is that clear? But the thing is this is not clearly eradicating our sub-lemma, sub-calculation. We have to use there somewhere. So, that will come as a nesting. Inside this deduction theorem, there can be another deduction theorem; like this one, this one is again inside this. You write deduction theorem begins, but, it is a second loop. Then it should end somewhere else, inside that loop itself. If it crosses the loop, then it is wrong, it is a wrong use; because by that you have assumed both x and the other one, that is not possible. That means we can have loops like this, in nesting of structures of DTB-DTE. You can number them. Similarly, these also. When both are involved, that is not again any criss-cross. That should be nesting of this inside that or that nesting should be inside completely this.

Is the plan clear? So, such a thing, which is written in this fashion, where deduction theorem, reductio ad absurdum both are included, is called an informal proof. It is not very formal, because to make it formal, we have to give some meta arguments. We will call it as informal proof. Let us try the fifth one we listed; how the informal proof proceeds. Because, you know already how it is going, it will just go this same way, but now deduction theorem, reductio ad absurdum will be used inside.

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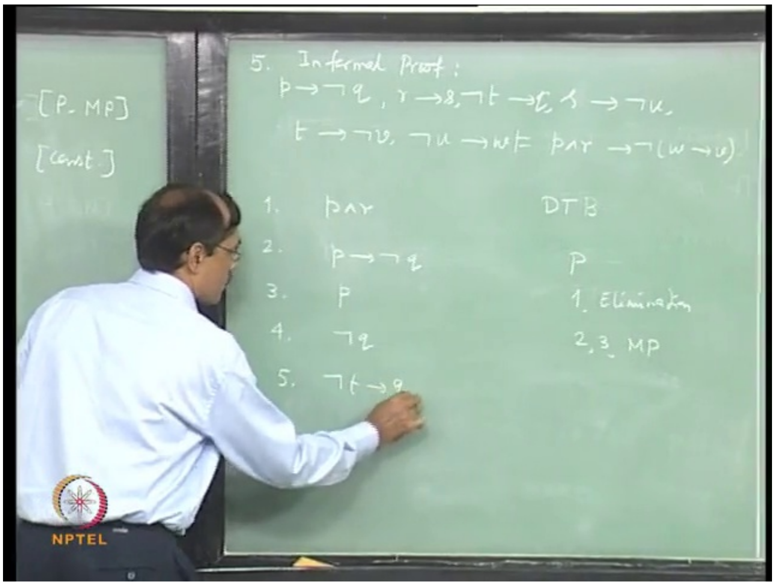

We start with the original, not with this form. We start with p implies not q, and r implies s, not t implies q, s implies not u, t implies not v, not u implies w. This sentence, p and r implies not of w implies v. What is our plan? Use deduction theorem. So, you start with p and r as the extra premise. Just look at the proof we have constructed; how this proceeds. We will have three columns, now in any proof first one will give the line numbers, so that we can refer them back; second one will have the correct proof; third one will have the documentation, just like our this MP, P, MT and so on.

So, we will be starting with p and r; its justification is extra premise, which is deduction theorem begins. Second line is, from p and r, we are using p implies not q as a premise. So, we can just add it again, p and r, and p implies not q; justification is, it is a premise. Or, there is an another version; you have already p and r. So, why do you write it? Earlier, you are writing it, because entailments, equivalence; it is there. Since, it is there already, there as a premise. So, simply add p implies not q; say it is a premise; then use both the lines. Ok?

So, you say p implies not q. Fine? Now, third line, from p and r, p implies not q, we wanted to conclude not q, by modus ponens. But we cannot, as it is, it has to be p. So, let us write p, p and r implies p. It entails p, elimination; so it is 1, eliminations. Then forth one is, from two and three, we apply modus ponens, to get not q. So we document, 2, 3, and modus ponens. Then fifth one will be another premise, you are using, r is already there, anyway, whenever we need, bring from 1, with not q, you wanted not t implies q.

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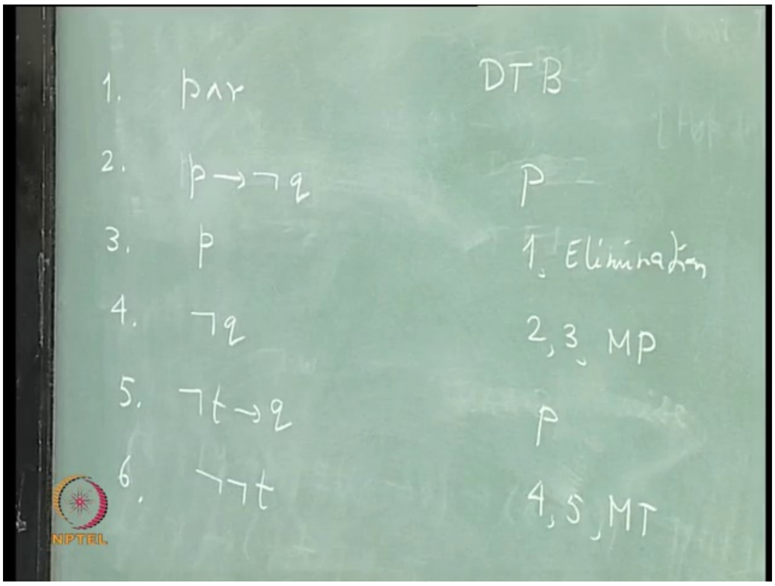

So introduce not t implies q, premise; and then sixth one is, not not t; 4, 5, modus tolens. Now you see, how it goes. You take it as an exercise, and complete the proof.