**Mathematical Logic**
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**Module - 1**
**Lecture - 6**
**Five Results about P L**

We start with very innocent looking expression, that, something like you take, a function, which is differentiable is necessarily continuous, right? Now, I give you a function, which is twice differentiable. Then immediately you conclude it is also continuous, because when you said twice differentiable it means first derivative exists, second derivative exists. Now, the other one, you said that first derivative exists then it is continuous, right? That means you have one added premise, here, second derivative also exists.

Now, if some premises are added, then the earlier conclusion still holds, right? That is a property, which is violated in many logics, but that holds in propositional logic and in mathematics, in general. We start with that; that is called the property of monotonicity for the entailment relation.

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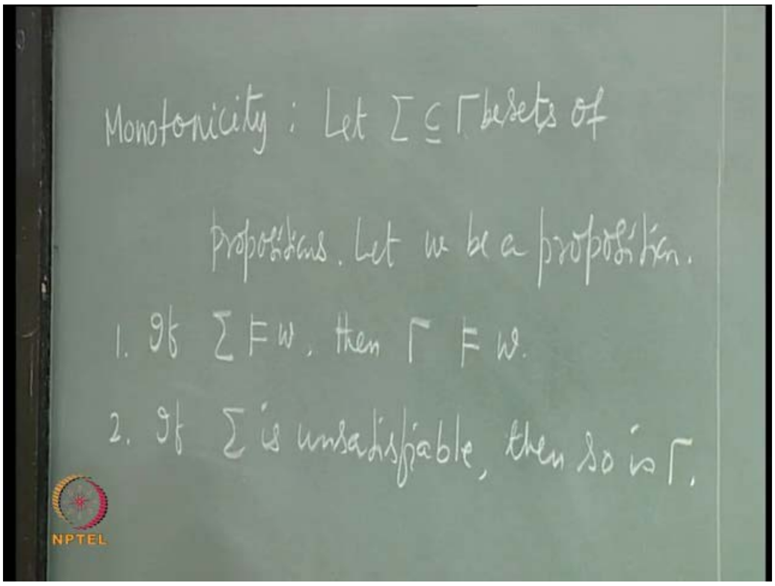


We start with the first result, which is called monotonicity. We start with two sets of propositions, where you say that sigma is a subset of gamma. There are more premises, now let w be a proposition. What do you see is that, if from sigma follows w then from gamma also follows w, this is one form of the monotonicity? There is another form, which says if sigma is unsatisfiable then, so is gamma. Essentially, what we do, assume that sigma entails w, which means every model of sigma is a model of w. Now, you want to show that gamma entails w.

Let i be a model of gamma, since gamma includes more propositions possibly, then i is also a model of sigma. A model of a set means, it should satisfy all the propositions at a time. So, all the propositions in gamma are satisfied by i. Therefore, all the propositions in sigma are also satisfied by I. That is, i is also a model sigma. Once, i is a model of sigma, by assumption, sigma entails w. So, i is a model of w. That is it; we wanted to show that.

If you start with any interpretation, which is a model of gamma it has to be a model of w, that is what we have shown, is it clear?

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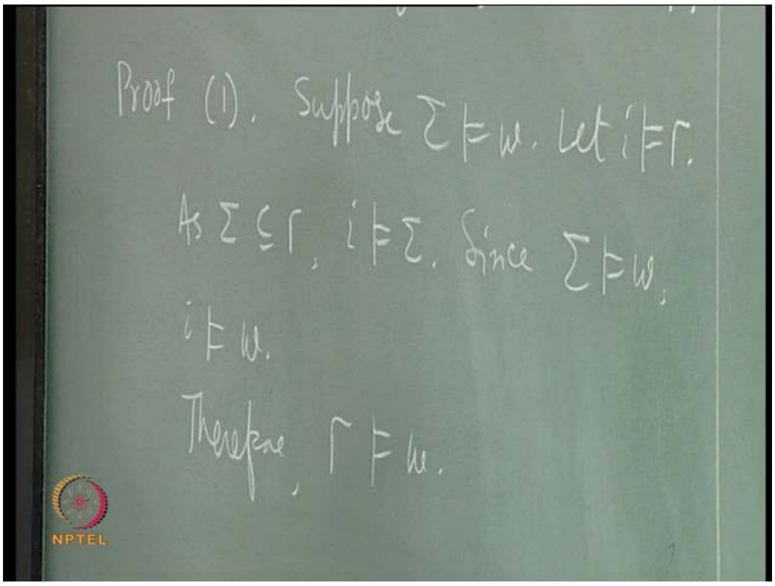


Let us write it. Suppose sigma entails w. Let i be a model of gamma. As sigma is a subset of gamma, i is also a model of sigma. Since, sigma entails w, i is also a model of w; and the proof is complete. We have started with any model of gamma, we have ended with that as a model of w; therefore gamma entails w.

Once we have done that, the second one should be immediate, as a corollary. But, we have used the crucial fact that each model of gamma is also a model of sigma. If sigma is unsatisfiable there is no model of sigma, therefore, there is no model of gamma. Because, otherwise if there is a model of gamma then that would have been a model of sigma, where as sigma has none, is that right?

You go to the next property, this is easy. And next one is also easy, which says that when you prove something, say, sigma entails w. What you do essentially is, assume sigma, assume negation of w, derive a contradiction. As a proof method you always follow it, proof by contradiction, of which originally told as reductio ad absurdum; reduce it to absurdity, right? That is the name. How do we formulate it now? It says, or should we write the complete expression?

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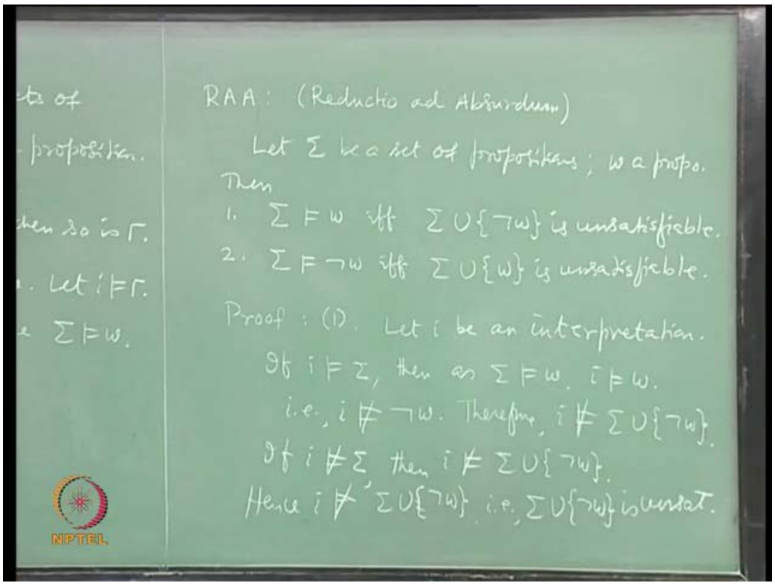


This says, sigma entails w if and only if sigma union not w is unsatisfiable; is that right? You have to write what sigma and w are. It has also another form, which plays with this w and not w. We say that sigma entails not w if and only if sigma union w is unsatisfiable. But again you are using the same thing. What you are proving, in some sense you are proving by contradiction. What you are proving is by contradiction. You are proving it by a proof by contradiction.

It is not wrong. Can you avoid that? you want to show if sigma entails w when sigma union not w is unsatisfiable. You are assuming sigma entails w, how to show this is unsatisfiable? You take any interpretation i that will evaluate it to zero; it will not be a model of sigma union not w. That is what you want to show, because you are starting with models. You are getting the problem? Start with interpretations, so that it is not a model, right?

Now suppose i is one interpretation of sigma union not w. You want to show i is a not a model of sigma union not w. You have information on sigma entails w, so that means if i is a model of sigma, it has to be a model of w. So, it cannot be a model of not w. Therefore, it is not a model sigma union not w. Is that clear? But these are for all interpretations, which are models of sigma. There can be some interpretations, which are not models of sigma. If that is so, then what happens? Provided sigma is non empty, assume as not. Then it is clear from the beginning, w is valid if only if not w is unsatisfiable, if sigma is empty. So, you forget the empty case is it clear now?

Let us write the proof. Let i be a model of, let i be an interpretation. If i is a model of sigma we have now two cases: i is a model of sigma or it is not a model of sigma. If i is a model of sigma, as sigma entails w, i is a model of w. That is, i is not a model of not w. Therefore, i is not a model of sigma union not w. Is it clear? That proves one in the first case, when i is a model of sigma.  If i is not a model of sigma, then i is not a model of sigma union not w. Because, sigma union not w is a super set of sigma. It is not a model of sigma union not w. In either case, you get i is not a model of sigma union not w. Therefore, sigma union not w is unsatisfiable. That proves one side of that if only if statement; there is one more, which we have to start from sigma union not w is unsatisfiable. Then prove that sigma entails w, right?

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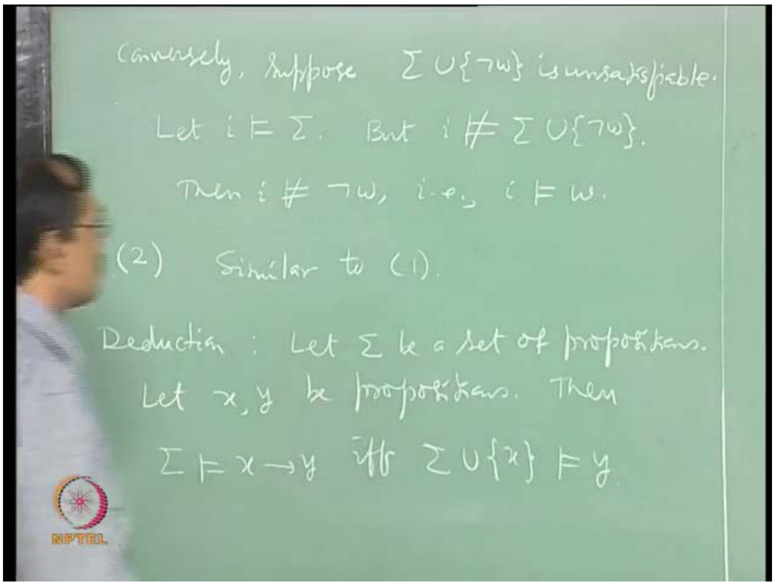


So, conversely suppose sigma union not w is unsatisfiable. You want to show that sigma entails w. To show that sigma entails w, you have to take a model of sigma, show that that is a model of w. We start with a model i of sigma; our aim is to show that i is a model of w. Now, what happens, i is a model of sigma, but sigma union not w is unsatisfiable. So, that means i is not a model of sigma union not w. What does it say? i of every proposition in sigma equal to 1, but i of not every proposition in sigma union not w is 1. Where it is going wrong? At not w. That means, i of not w is zero. This says i is not a model of not w, that is i is a model of w. That proves it.

See, if you have the ideas it is not difficult; see what is going on. But only problem you are facing is writing it correctly. That is why I am writing it again, you have to reproduce, this is the way we have to write. There is no other way you have to use the meta symbols. We have to use the correct arguments and so on; that is all; they are simple otherwise.

Now, what about the second one? you have to prove it again. Replace w by not w, not w by w in the above proof, you would get it. So, you just write similar to one.

Now, if you see reduction ad absurdum and compare with monotonicity; there are two statements similar in monotonicity. Can you see what the second one is telling? It says sigma is unsatisfiable, then gamma is unsatisfiable. Instead of sigma, suppose i take sigma union not w, right? If sigma union not w is unsatisfiable, then gamma union not w is unsatisfiable. Then i get the first one by reductio ad absurdum.

That is why they are listed under the same monotonicity; you see from this to this, is not so quick. Because, this is for arbitrary sigma not especially in the form sigma union w or not w. But you can always find out one and put it in the form not w provided, they accept the law of double negation, if nothing is negated then how to bring it? Well, you can take w is equivalent to not of not w. So, it is in the form of not some x, right? That is why both the things are same due to reductio ad absurdum. We will give another common proof procedure, which we use in mathematics that is also simple; but we have to formulate it.

Suppose you want to prove one conditional statement. Say, sigma entails x implies y, if x then y. Then what do you do? Do you just assume sigma and produce if x then y? What you do is, you take sigma along with that x, assume also x, and prove only y.  But these two are different things. One says sigma entails x implies y another says sigma union x entails y, is that right?

These two are same, that is what we are going to prove, that is called the deduction theorem. You take a set of propositions, which will be our premises and let us take two other propositions. Then what it says is sigma entails x implies y if and only if sigma union x entails y. If our formulation is correct, then you can prove, how to prove this? There are again two parts it is an if and only if statement.

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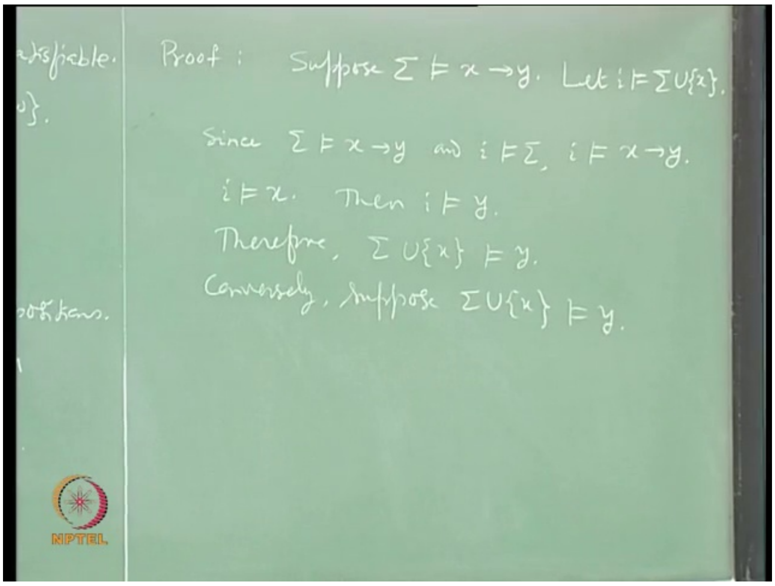


Let us assume sigma entails x implies y, try to show that sigma union x entails y. Suppose sigma entails x implies y. Our aim is to show sigma union x entails y. That means, you have to start with a model of sigma union x and end with showing that it is a model of y. So, let i be a model of sigma union x. Since sigma entails x implies y and i is a model of sigma, i has to be a model of x implies y. i is a model of sigma, because it is a model of a bigger set; i of everything in sigma union x is 1. So, i of everything in sigma is also 1; it is a model of sigma.

Now what do you have got? i is a model of x implies y, but i is a model of sigma union x. So, that itself gives i is a model of x. Now, i of x equal to 1, i of x implies y equal to 1, what can be i? It cannot be zero, anyway. If it is zero then you would get i is a non-model of x implies y. Then, i is a model of y. That proves the first part. You just have to write it. Therefore sigma union x entails y.

Now, for the other part. Here, we have to show that sigma entails x implies y. This is what you have to show, so we start with the model of sigma.

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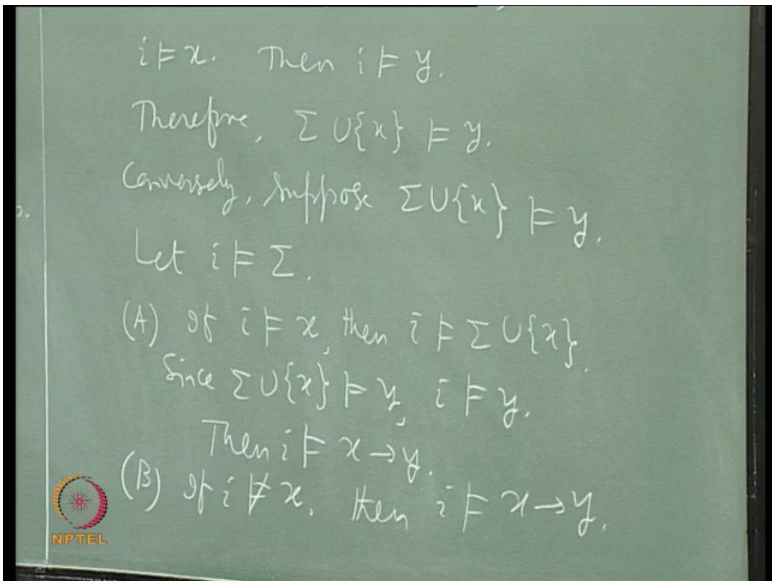


Let i be a model of sigma. Our aim is to show that i is a model of x implies y. How do you proceed? There will be two cases now, because we have x there already, to use that we have to consider first if i a model of x, first; next, if it is not a model, what to do. Now, case A. It says that if i is a model of x, then you have i as a model of sigma union x. Because, of the assumption sigma union x entails y, i has to be a model of y. So, i of y equal to one. If i of y equal to one, i of x implies y equal to one. That is what you have seen earlier, so then i is also a model of x implies y.

What about the second case? If i is not a model of x, then clearly i is a model of x implies y because if i of x equal to zero, i of x implies y has to be 1, there it ends, right?

Let us start using these, suppose we revisit the example what we did in the last class. You want to show p implies r implies q and p implies r entails p implies q; let us see this example. To show this, you can always use the truth table, that we have done yesterday. Let us see deduction theorem, how is it useful. By deduction theorem, since this is the one in the form of x implies y, we can use deduction theorem, it is enough to show by deduction theorem, that p implies r implies q, p implies r, bring p to this side, entails q; is it? Imagine what is happening in terms of the truth table. In the truth table you add eight rows, now here you have only four rows to show it. Because, already which have been assigned to 1, you have to consider only those interpretations, which are models of the premises.

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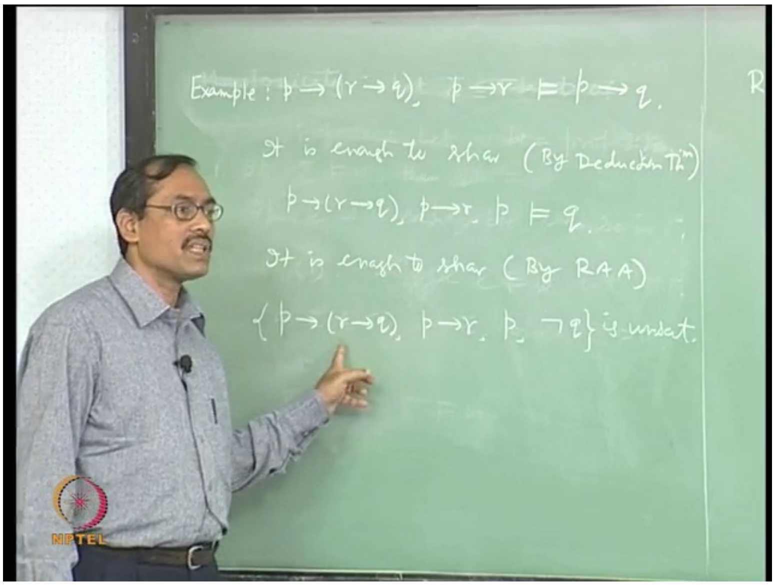


p is assigned to 1, half of the truth table goes, now itself you construct the truth table and job is over, is it clear? It will be easier now. Let us see, if p is 1, p implies r is 1, r has to be 1; you can argue that way also. Again, another half of the truth table is gone, only two interpretations. You have to prove: give p 1, r 1, q 0, or 1. Now, with q 0, this becomes 0, which is not possible. Therefore, q has to be 1, right? It is easy to see now.

Suppose you want to use reductio ad absurdum, now onwards what will you do? Again you can say it is enough to show by reductio ad absurdum that p implies r implies q, p implies r, p and not q is unsatisfiable. Now, how do you proceed?

That again is easy. The same way you can argue. Take an interpretation in order that you are trying to make it satisfiable, you will be assigning p to 1, q to 0. Because, not q is 1, and then argue p is 1, q is 0. Since p is 1, p implies r has to be 1. So, r has to be 1; now r is 1, q is 0, p is 1; this is zero; it is unsatisfiable.

So, any interpretation, which is a model of all these three cannot be a model of this. That is what you have shown; therefore, this set is unsatisfiable. It is not a particular interpretation you are taking. Your argument is: take any interpretation, which is a model of all these three. Then it can never satisfy this, therefore it is unsatisfiable.

One of our laws says that p or not p is valid, ok? This is called the law of a excluded middle: either p or not p. Nothing is in between. There had been many objections to this law; but that is a separate matter. In propositional logic, it holds,  that we know that; because p can either be zero or one. Whenever p is one, you get the or as one; interpretation; if p is zero, then not p becomes one; so or is one. Right? So, this statement, or this proposition is always evaluated one. Therefore, it is valid, right?

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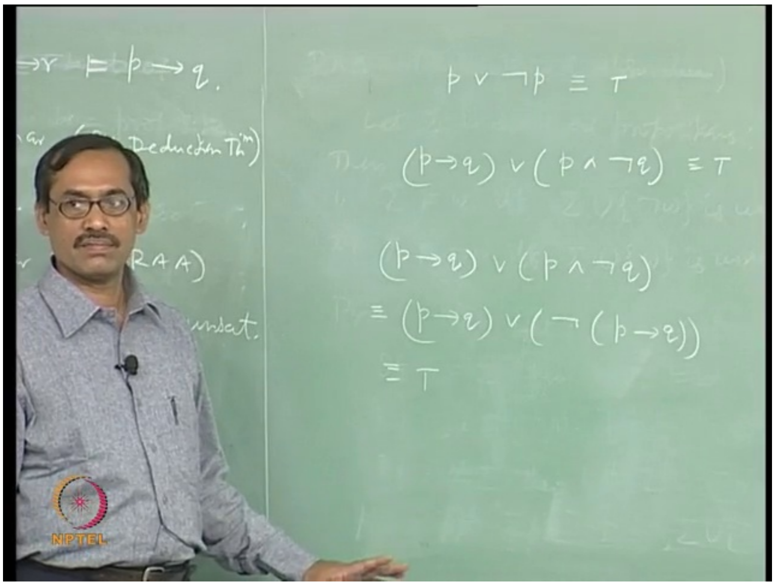


Now, from the validity of p or not p, I can just write that this is also valid. Independently you can see this is valid. But I am not going for that. What I am asserting is that from the law of excluded middle, I can derive that this is also valid. OK? What does that mean? There is something in my mind, yeah? So, you have read it? Well, negation of p implies q is p and not q, that is the law of implication. OK? There is another law; that means, I am using two laws: one is excluded middle and other is negation of p implies q is p and not q. This also you use almost everywhere, in proving theorems. Suppose p implies q does not hold. Then what do you do? Assume p assume not q and proceed. So, p implies q is equivalent to, its negation is equivalent to p and not q. Verify it by the truth table.

What we are asserting is p implies q, if we consider it is negation, it is p and not q. So, that means, I can write first this as p implies q or p and not q, is equivalent to, p implies q or negation of p implies q. This is what I am stating, right, since these two are equivalent.

Now, what I say? This is same thing, since I have a excluded middle, x or not x is valid. This is also valid, what about this argument p implies q in place of p and then telling that it is correct? There is another substitution here p and not q is equivalent to not of p implies q. So, this substitution is also allowed and finally this is equivalent to top, since I have the law of excluded middle. There are two kinds of substitutions involved here. First one is p and not q, is equivalent, to not of p implies q. Which is our equivalent principle we have followed in geometry, equals can be substituted for equals. That is what we have done here, right?

The other one is different, this one is telling something like if x plus y whole square is equal to x square plus y square plus two x y. Then three plus four whole square is equal to three square plus four square plus two into three into four. So, this is another kind of substitution in place of these propositional variables. We can substitute any proposition, but this needs justification, as you have told. How can we say that?

Well, we have to prove two substitution theorems, that equals can be substituted for equals. Then uniform substitution can be used, right? Let us look at uniform substitution first. So, this is the problem in logic, whatever we have assumed till now, you have to prove. Now that you have assumed the substitutions can be made. You have show that here substitutions can be made by using our definitions of interpretations and models.

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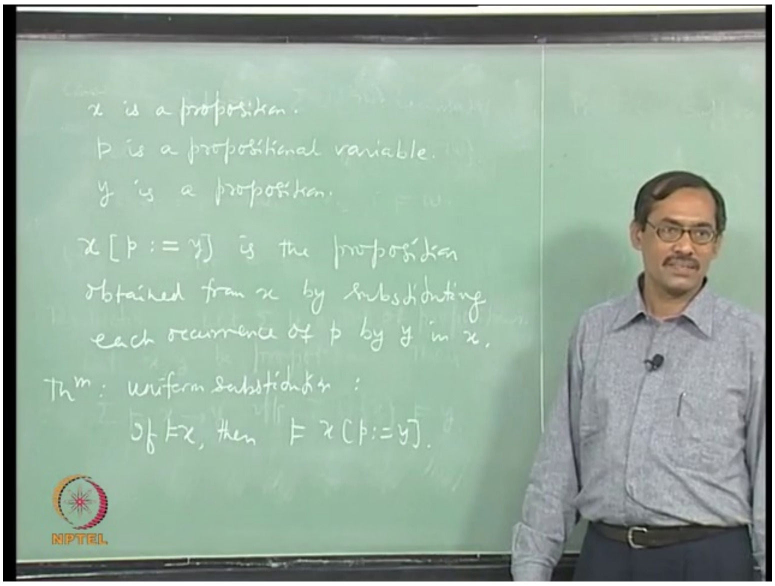


So, let us introduce a symbol say, x is a proposition, p is a propositional variable, y is another proposition, possibly different from x, where it can be same as x also. Now, you say x, p replaced by y is the proposition obtained from x by substituting each occurrence of p by y in x. That is why it is called uniform substitution. You can read this symbol as p is uniformly replaced by y in x, to get the proposition x, p equal to y. One example is here, what we have done in place of p. We have substituted p implies q, right? That is what we have done.

Then what the statement should be? This is uniform substitution. It says, if x is valid, then x, p equal to y, is valid. Fine? Now, how do you prove this? that happens, you know? When you become addicted to two wheelers, you will never try to walk. Since you are you are using the term interpretations, models and so on, you are forgetting the truth table. Just look at the truth table for x, x is valid, because the truth table for x. Now, somewhere there p is occurring; p is having zero or one, it is a propositional variable. Instead of p you have substituted y. So, truth value for y can be zero or one, even if it is one it does not matter, always still, it is there in the truth table. So, a portion of the truth table is still there, but everywhere in the x column it is one. So, everywhere this also will be one, right? Is it clear?

But we want to give another syntactic proof, proceed by induction. That is also possible and that induction can be done many ways, on the number of occurrence of connectives, as usual. You can say how many p's have been replaced by how many occurrences of p, is there in x. That is also well, but through semantics, it is clear. So, there is nothing to do. Well, an example is here; what we have done in place of p; we have substituted p implies q, right? Since p or not p is equivalent to top, p implies q or not p implies q is equivalent to top. That is how it will be used. Next, let us go to equivalent substitution. So, again we have to introduce a notation for equivalent substitution.

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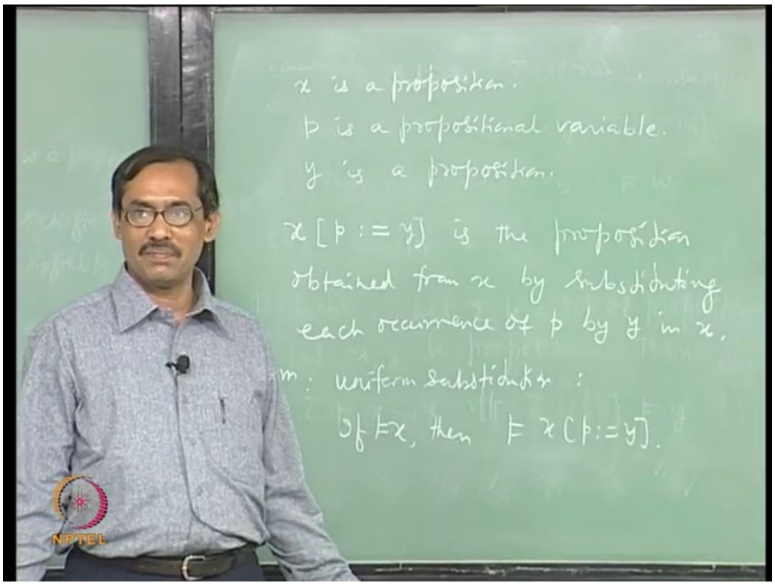

Suppose x is a proposition. Well, we can take some more; say, y and z are also propositions and say, y is equivalent to z. We are concerned with equivalences. Now, denote by x where y is substituted equivalently, so, give a subscript here e, that symbol equal to z. So, this is obtained from x by replacing, now you are liberal, we do not say each occurrence as we have done in that uniform substitution, that is uniform, this is not uniform, this thing. So, you say by replacing some of the occurrences of x by `y by z', so this `some' maybe nothing maybe all. Let us call this replaced version of x as x y equivalence replacement with z. y is equivalently replaced by z; a precondition is y should be equivalent to z. Otherwise, you cannot substitute, we cannot use this notation. Well then, what do we get? What do you have there? You can write validity or entailment some notation you get. Let us write first entailment. If u entails v, then u, y equivalently replaced by z should entail v, y equivalently replaced by z. You may go for the second one if u is equivalent to v then u, y is equivalently replaced by z, should be equivalent to v, where y is replaced equivalently with z.

As a special case, we can get: if u is valid then u, where y is replaced by z is also valid. Because, v you can take as top, a professional constant; there, if you replace, nothing will happen. Right? Now, again how do you prove? Consider the truth table for u and v also, both at the same time. A truth table where u occurs, v also occurs. When u entails v is evaluated you are taking some of the portion of the truth table, where you have only models for u in all such cases. You find that they are also models of v. Now, in u there might be some y, so they have been evaluated to either one or zero, something, but since y is equivalent to z, in place of y if you look at z column there also zero or one correspondingly.

So, it does not matter in the truth table whether it is y or z, it is the truth values, which are important in the truth table, fine. The truth values of y and z are the same, therefore the truth value columns for u where y is replaced by z or v will be same, as you want v that is all. That justifies this statement. Is that thing answering your question? So, if p or not p is top, sorry, we come from where? This one. p and not q can be replaced by not of p implies q; still, they will be equivalent.

These two theorems really give us some way to use our previous laws; whatever laws we have mentioned. Some others are there; they can be used now to prove some more equivalences. We can show some equivalences some consequences to be valid by using those laws.