Mathematical Logic
Prof. Arindama Singh
Department of Mathematics
Indian Institute of Technology, Madras

Lecture - 5
Consequences and Equivalences

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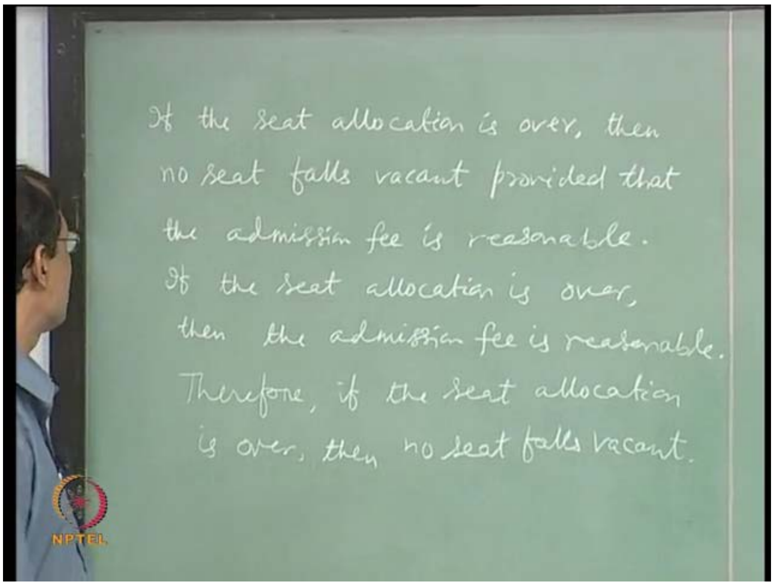

This is the typical situation in your seat allocation in JEE. It says if the seat allocation is over then no seat falls vacant, provided that the admission fee is reasonable. Now, the second premise says if the seat allocation is over then the admission fee is reasonable and our conclusion is coming after `therefore', it says if the seat allocation is over, then no seats falls vacant.

Now, how do we go around proving that the argument is correct or telling that or finding out that, argument is incorrect. In general, what we do is, think of a world (Refer time: 02:32), where all the premise are true, then in such a world (Refer time: 02:37) the conclusion must hold true. This is about that (Refer time: 02:42), wherever the premise are true, you have to consider now all such cases (Refer time: 02:46).

In terms of our interpretations of propositions, what we find is you think of all those interpretations, where the premises are true. There is assumed to be one, all the models of the premises and you see each of the interpretation is such that it satisfies all the premises at a time. You need to consider here that the set of such premises has a model not only one sentence, one proposition, but a set of propositions. First we define what do we mean by a set to be satisfiable or having a model.

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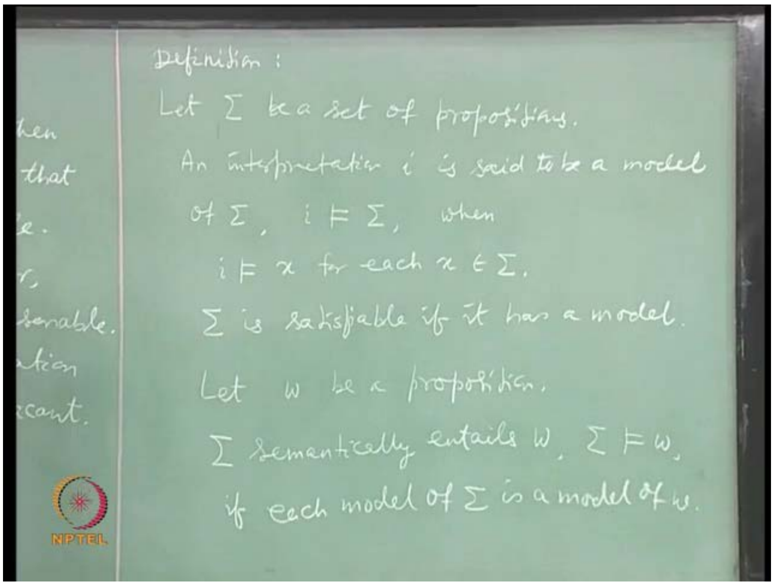

Suppose Sigma is a set of propositions. We say an interpretation i is said to be a model of Sigma and we write it as, i satisfies Sigma as earlier, when that happens? i satisfies x for each x in Sigma. It is simultaneous satisfaction of each proposition with the same interpretation. In such a case, when a model exists we say that Sigma is satisfiable. We say that Sigma is satisfiable, if it has a model, all these will be a definition.

When such a scenario comes, when you have a set of premises and there is a conclusion, you need something else, not only satisfaction of these, but you need that, such a model of Sigma should also be the model of the conclusion. That we formalize as follows taking w to any proposition, it may be in Sigma, may not be in Sigma, we say that Sigma semantically entails w, write it as Sigma entails w, if each model of Sigma is a model of w, is it clear? is it going along with our intuition? That imaginary world where all the premises are true, in such a world the conclusion must hold, all worlds are now only interpretations.

Now if such a thing comes, such an argument comes how are you going to proceed, we have a formal definition now. First thing is, we have to write it as an entailment as a consequence. Now this says, sometimes we omit this semantically adjective, Sigma entails w, but Sigma entails w means the argument is correct, that is what we are writing cryptically without introducing what is a consequence. What we do is, we again introduce cryptically the notion of consequence. We say that the consequence Sigma entails w is valid, to say that Sigma entails w; the adjective the consequences, we will say that we do not know till now, whether it is valid or not. Once it is valid, we can write this way; it is overloaded in that sense.

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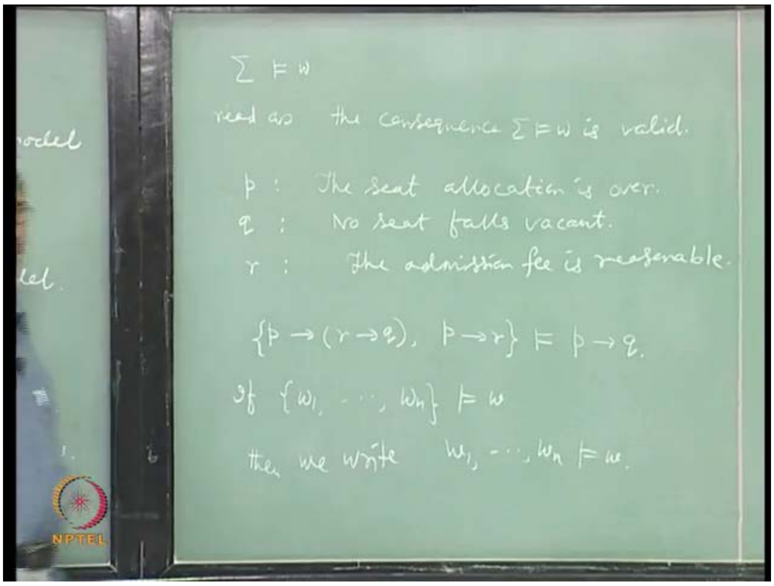

Whenever Sigma entails w we also read it as, the consequence Sigma entails w is valid. There is a philosophical over turn, that creates a problem because, we are not defining what a consequence is, we are telling, defining, only validity of a consequence. Then to refer that it is a consequence, we just put the adjective, the consequence Sigma entails w, which means we do not know whether, it is valid or not. Once it is valid, we can write that as Sigma entails w, without the adjective.

Now first thing is given an argument, we want to write it as a consequence, in that form, which is to be validated or to be seen that it is not correct. Now how do you do it? See, mathematicians are basically lazy people, they do not want to take a big expression, they would write a symbol for it, that helps clarifying the concepts. What we do here is, first identify the atomic propositions write them in symbols and then see, how we can translate to Sigma entails w.

First thing is you identify that; here, it is the seat allocation over one, second is say no seat falls vacant, we can say seat falls vacant also, then negate or something. But it does not matter because, always it is coming as no seat falls vacant, everywhere it is occurring like that. We can start from the beginning: no seats are vacant. Next, the admission fee is reasonable, that is all.

Let us write them say, p stands for the seat allocation is over, next q stands for no seat falls vacant and r stands for the admission fee is reasonable. How do we symbolize the first premise, if p then q provided that r, provided that means, if, it is another if? p implies, then, this if this, r implies q it is q if r? if r then q. Then, shall we write p implies r implies q, sometimes you may write it as r implies p implies q, we will see that they are equivalent. What is the next premise, what is the next premise? p implies r that is all about the premise. We have the set if premises as this which is Sigma, we want to check whether this entails the conclusion.

The conclusion is p implies q, again we are lazy people we do not want to write the braces, what we do? if it is a finite set we make it a convention, not to write the braces. If that means, if Sigma is some finite set of sentences w1 to wn and we have the consequences of this, you would write it as w1, to wn entails w. Here you can just omit the brasses that is what it says, but that is only convention, just to write less. Now, how to validate it? Whether this is valid or not you have to go for interpretations and see what are the models of this. One is you can take the truth table, let us construct the truth level, then we come to shorter ways.

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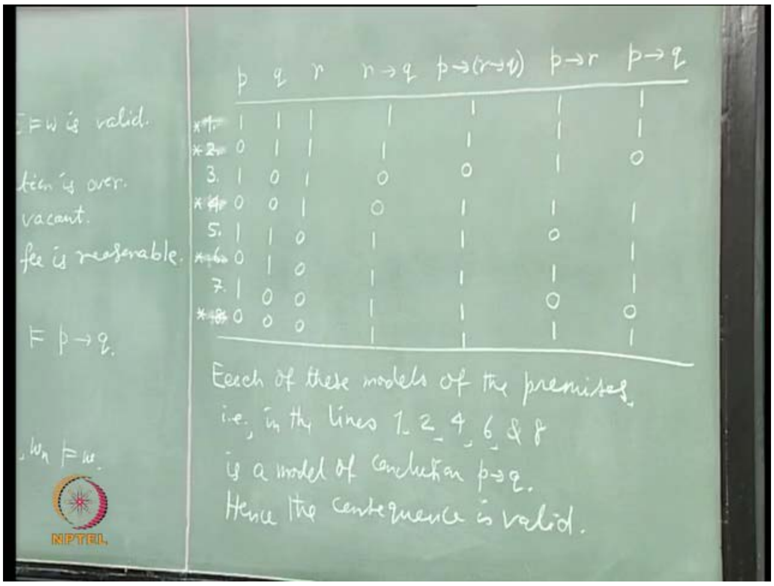

We have here, three proportional variables and then, three sub propositions r implies q, p implies r, p implies q and another p implies r implies q. Let us take r implies q, then p implies r implies q, next p implies r, next p implies q and we construct the interpretations. There will be eight lines, these are the eight interpretations. Now, you have to evaluate all these connectives. r implies q that becomes 0 only when r is 1, q is 0, in this case it will be 0, this case also 0. Now where else, next we go for p implies r implies q, that will be 0 when p is 1, r implies q is 0, this case, that is all, all others are 1.

Next we go for p implies r, that will be 0 when p is 1, r is 0, all others 1. Next we go for p implies q, that will be 0 when p is 1, q is 0, here is 1, third line and the last but one line, all others are 1. Now then, what we are required to do, find out all the models of all the propositions simultaneously. Take one interpretation, take one row, which satisfies both the premises at the same time, evaluate it to 1.

This must be 1, this must be 1, this is 1, interpretation next, this is another interpretation, fourth one is 1, more? Sixth one, then the eighth one. Let us number them. We say it is 1, 2, 3, 4, 5, 6, 7, 8; out of these we have 1, 2, 4, 6 and 8, these are the interpretations which satisfy Sigma. Now you have to check whether, all these interpretations also satisfy w, the conclusion. So? You do not have to consider the interpretations which are not satisfying at least one of the proposition in Sigma, you do not have to consider them.

Now then, in the first interpretation, p implies q is 1, in the second interpretation p implies q is 1, in the fourth interpretation p implies q is 1, in the sixth interpretation p implies q is 1, in the eight interpretation p implies q is 1; the requirement is satisfied. Each of these models, each of these models of the premises, that is in the lines 1, 2, 4, 6 and 8 is a model of conclusion, hence the consequence is valid.

Now you can look at it in a different way, instead of looking at the all the models of premises, you can try to see where the conclusion is falsified.

In all those cases the interpretation must falsify at least one of the premises if that is correct then also Sigma entails w. That means here I will be considering all those lines which are not starred ones, that is 3^{rd}, 5th and 7^{th}. In the third one, p implies q is 0 and in the seventh one p implies q is 0. I do not have to consider the sixth one, where it is 1; we take only those where p implies q is falsified. Now what happens, it can come in certain other scenario, that the unstarred ones, will have a 0 here.

It is not the stared or unstarred, we have to start from wherever p implies q is 0, it so happens, here is that, they are within this, that is why Sigma entails w. In general that may not happen; the procedure says we should look for 0s in the column for p implies q and then try to verify whether, in each of those rows wherever p implies q is evaluated as 0, at least one of the premises is also evaluated to be 0.

You start from third 3rd line, where p implies q is 0, we find that one of these premises p implies r implies q is 0. Next in the 6th line, we find p implies q is 0 and one of the premises p implies 0, p implies r is evaluated 0 therefore. Sigma entails w.

That can be going as an alternate definition to this, in fact that follows from the definition, it says if i falsifies w, then i falsifies at least one of the premises in Sigma. Then you say, Sigma entails w. Now from the definition, you can see easily that if Sigma is unsatisfiable, what will happen to Sigma entails w? If Sigma is unsatisfiable?

Student: Then Sigma may or may not entail w.

Yes?

Student: Then Sigma should entail w.

for which w?

Student: Any w.

Any w.

Student: Yeah.

Is it clear? What is the reason?

Student: Sigma itself has no a model, so.

Ok.

Student: It is always, if you look it as the first few of the propositions implying what we want. So, if all the propositions together itself is not now satisfying.

Ok.

Student: So, it is like 0 implying anything.

Not like, give the argument correctly.

Student: Any model of Sigma is no model, a model of w.

That’s what we want to verify. Now Sigma has no model? So? Vacuously, every model is a model of w. You see it in a different way, it will be easier to do the the alternative way. You say, when is, Sigma does not entail w? If you find one model, find one interpretation or there exists one interpretation, which is a model of all the proposition in Sigma, but not a model of w. But you cannot find any, because there is none; that is a easier way of looking at it. Let us write it down.

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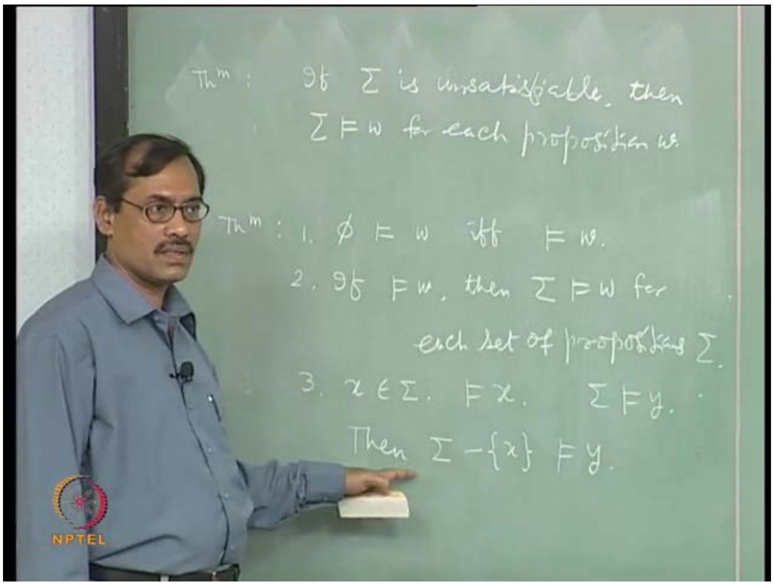

So it says: if Sigma is unsatisfiable then, Sigma entails w for each proposition w. Well, what is the proposition? What do you want to propose? That suggests?

Student: When empty set entails w, if w is valid.

Is that okay? And what is the reason? Well, let us carry the suggestion and see what is happening. We suggest that it should happen this way. What is the model of an empty set? I would claim that every interpretation is a model. Otherwise, you have to find one interpretation which is not a model of one of the propositions in the empty set; but there is none. We can’t find a non-model of some proposition in the empty set, because, empty set has no proposition in it. Therefore, every interpretation is a model of it, any interpretation satisfies the empty set.

If this is given that empty set entails w then any interpretation is also model of w; therefore, it is valid. Conversely, if every interpretation is a model of w then every such interpretation is also a model of the empty set. Or, you start from any interpretation which is a model of the empty set. Now that happens to be a model of w because, w is valid therefore, empty set entails, in fact, once w is valid every set will entail w. You may write this way, if w is valid then Sigma entails w for each set of propositions Sigma.

One of this is the empty set. For the converse of that. This shows why valid propositions are really redundant. They have no information content because, it can be conclusion of any set of propositions, but we will see it in a different way now.

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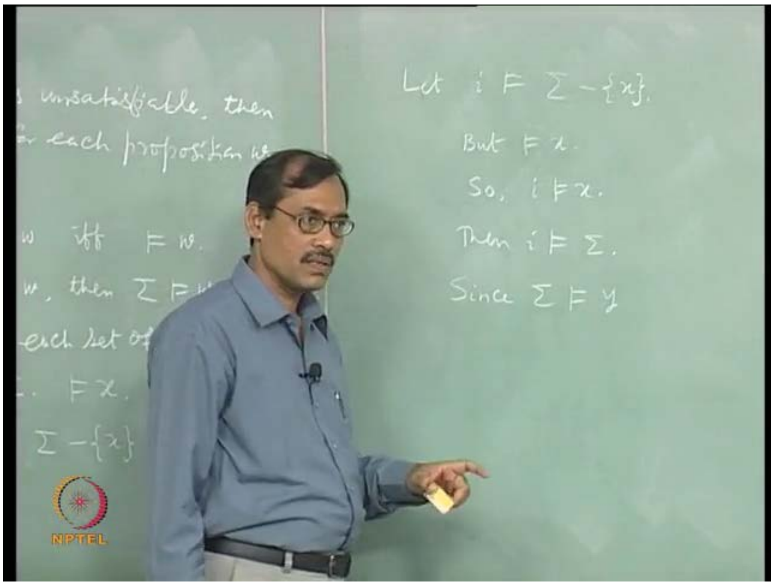

Let us say x is a proposition in Sigma, it is given, and x is valid, and also Sigma entails y, then we may say Sigma minus x also entails y. It shows the redundancy, that any valid proposition from the set of premises can be deleted, the same conclusion we will get as earlier. Can you see how does it happen? Let us see what is to be done, we are given with x is a member of Sigma, x is valid and Sigma entails y, anywhere we can use them. Now suppose you want to prove this, Sigma minus x entail y, then how to prove this? You have to start with an interpretation of? interpretation of, which is a model of Sigma minus x. Suppose we start with that.

Let i satisfy Sigma minus x, remember our aim is to show that Sigma minus x entails y, that is, i should be a model of y, that is what we want. Now look at this i, i is a model of Sigma minus x, but x is valid, i is also a model of x. Once x is valid, any interpretation is its model, here is one interpretation i that has to be a model of x. Now, i is a model of Sigma minus x, i is a model of x, i is a model of Sigma. Now use the hypothesis: Sigma entails y; Sigma entails y means every model of Sigma is a model of y.

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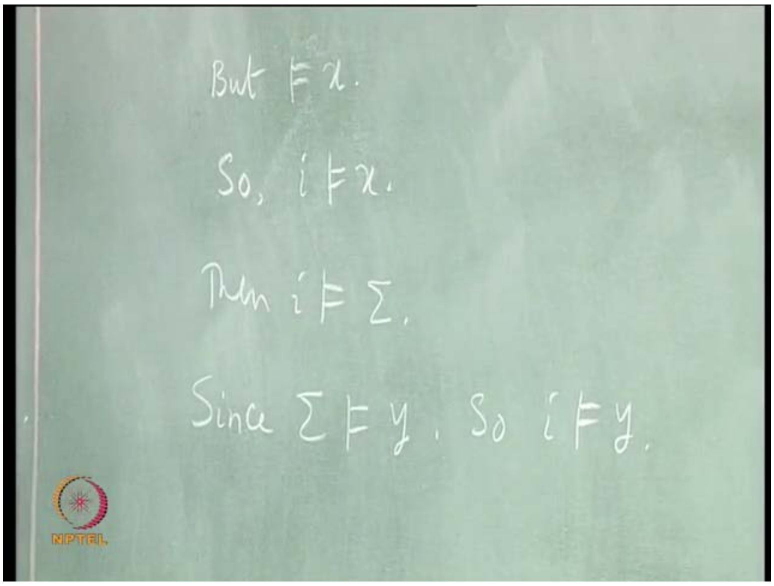

i is a model of y and there we stop, that is what we wanted to show, every interpretation that is a model of Sigma minus x is also a model of y.

See, what I propose is this. I take u entails v, if u implies v is valid. This is the reason we want to be getting interested in validity, though they are redundant. Because, these entailments can be seen as also valid propositions. Of course, not the same propositions, it is a different proposition, but it can be reduced to checking for validity. Now how do you prove this? Suppose you start from this side, suppose u entails v, you want to show that u implies v is valid. To show that u implies v is valid, we have to verify taking any interpretation that it is a model of u implies v.

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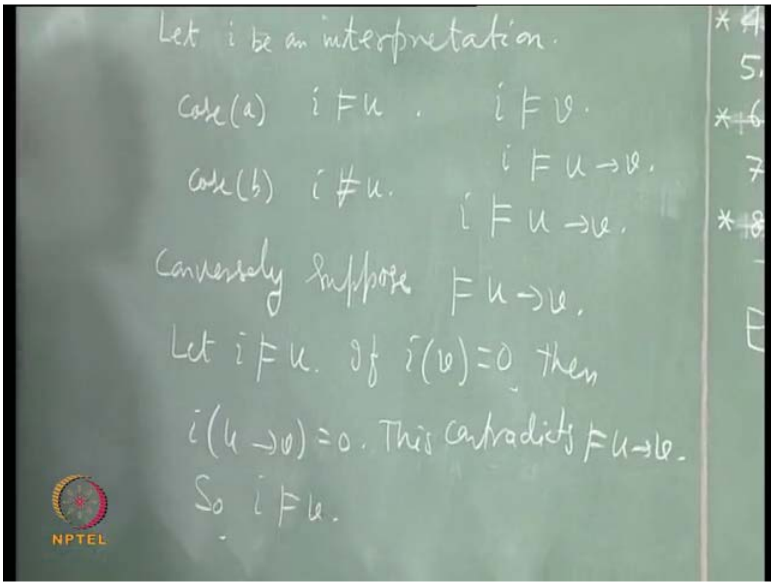

Let i be an interpretation. We want to show that i is a model of u implies v. Now, you have to exploit u entails v, but u entails v says: if something is a model of u, then it will be a model of v. There will be another case where it is not a model of u, break it into two cases now. Case a is: i is a model of u and case b is when i is not a model of u. Now in case a, if i is a model of u, utilize the fact that i entails v. Therefore, i is a model of v; this gives i is a model of v.

If i is a model of v, then i is a model of u implies v because, u implies v is true when v is true. This case says i entails u implies v. In this case, if i does not entail, does not satisfy u then, by definition of implies i satisfies u implies v. If i of u equal to 0 then i of u implies v is 1, whatever may be. In this case, we directly get i is a model of u implies v and that is only one part, if u entails v then u implies v is valid whatever be the case; it should be easier. Because, this breaks into two cases that should give only one case.

Conversely suppose you have to start with a model of u and so that, it is a model of v; using u implies v is valid. Suppose u implies v is valid. Let i be a model of u. Our aim is to show that i is a model of v. Now i of u is 1, i of u implies v is 1; therefore, i of v is 1 nothing more to do, that comes from the definition of implication. It can either be 0, or you can go the other way. If i of v equal to 0 then what happens? Then i of u implies v equal to 0, this contradicts the validity of u implies v. i is a model of v. Once you see, there is a connection between entailment and the connective implication, validity of that, then you can connect equivalence with validity. Because, u is equivalent to v means u if and only v is valid and u if and only if v can be broken down into two parts, u implies v and v implies u.

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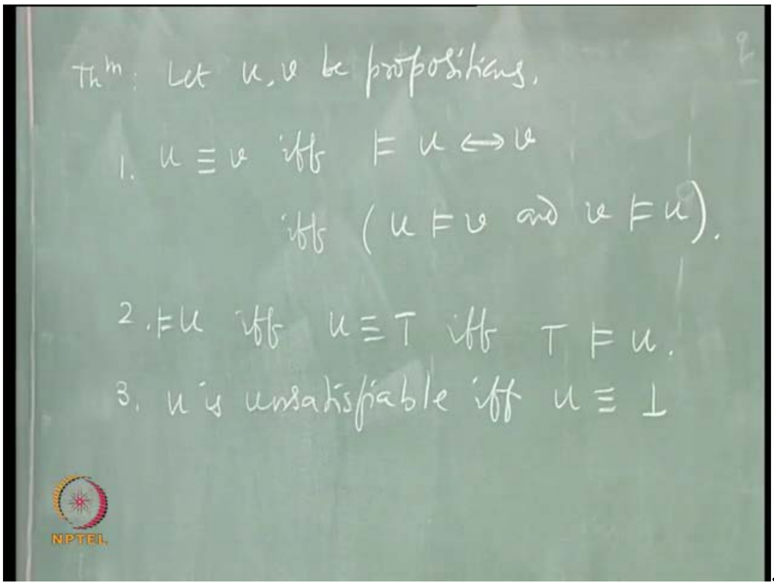

You may say that u is equivalent to v if and only if u biconditional v is valid, if and only if, u entails v and v entails u. u is valid if and only if u is equivalent to top; yeah, if and only if top entails u, no you are giving only one side.

Student: u entails top always?

u entails top always, whatever u may be. Only this part is relevant here in this. This is also clear once you say u is valid means every interpretation evaluates u to 1. Take any interpretation, you want to show u is equivalent to top, but what is top? Every interpretation also evaluates that to 1. Under any interpretation u and top are interpreted the same way, are evaluated to 1. They are equivalent and conversely. Then you can do for unsatisfiable, u is unsatisfiable if and only if u is equivalent to bottom.

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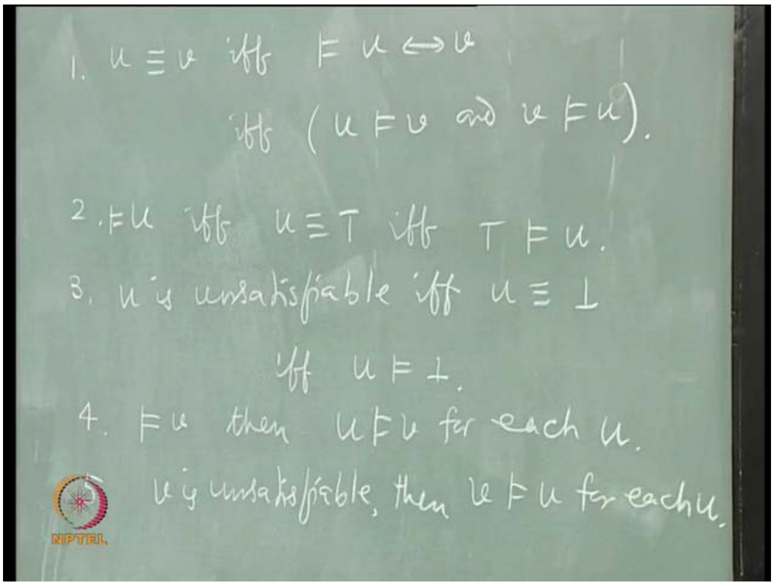

And again as earlier, you can take only one part. This says u entails bottom; let us write that again v is valid, then u entails v, for each v each preposition u; and if you say v is unsatisfiable, then v entails u for each u.

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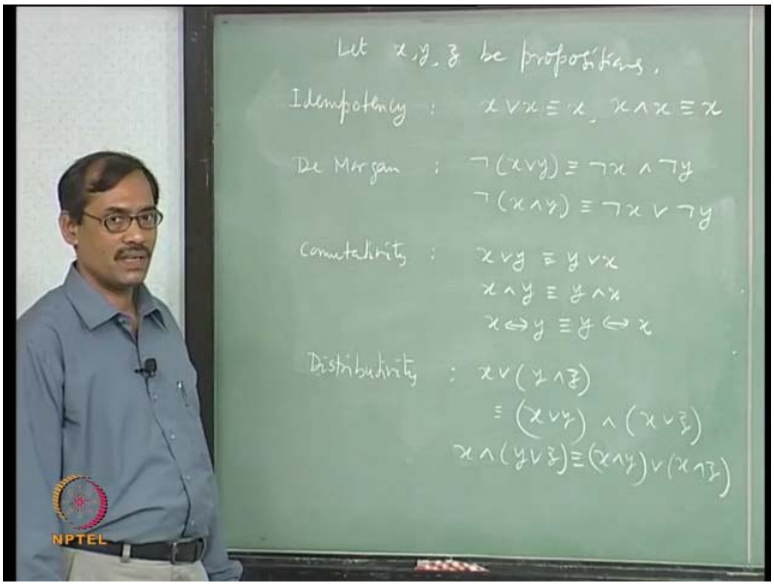

Now you can utilize earlier interpretations and models for finding out various tautologies or valid sentences or even valid consequences. For example, law of idempotency: you take x any proposition or it with x itself, you would get x, right? These will help you in calculation, when you calculate or compute something, how is it going to valid or not, to check it, this type of equivalences will be helpful.

Let us mention some of them. These laws, you take any proposition. Idempotency says, x or x is equivalent to x, x and x is equivalent to x. Similarly we have De Morgan, which says, not of x or y is equivalent to not x and not y and not of x and y is equivalent to not x or not y. They are easy to prove from truth tables. You have commutativity, which says x or y is equivalent to y or x, x and y is equivalent to y and x, x biconditional y is equivalent to y biconditional x.

Similarly, distributivity. There are so many distribution laws, we will give only one or two. This says x or y and z is equivalent to x or y and x or. In such a case you say or distributes over and. Like your multiplication distributes over addition, or distributes over, and also distributes over or, unlike the arithmetic. We say x and y or z is equivalent to x and y or x and z, even implication distributes over itself. Yesterday you have seen x implies y implies z will be equivalent to x implies y implies x implies z. There are so many other distribution laws.

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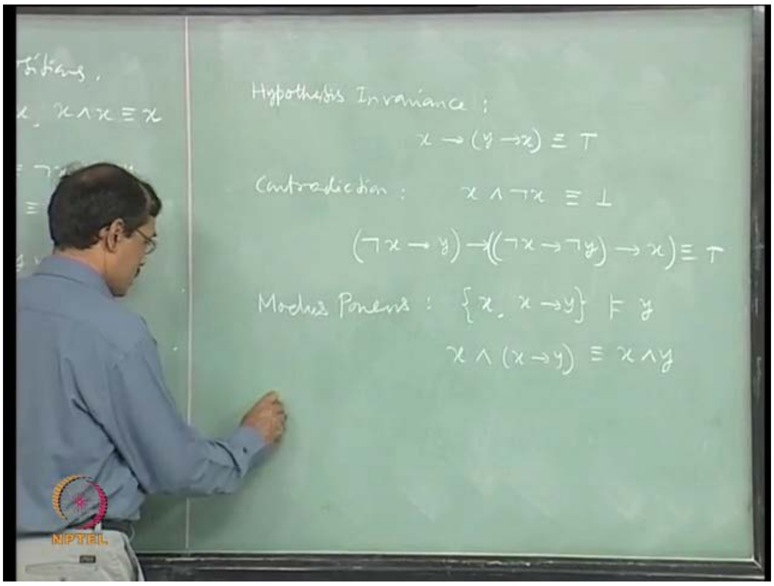

Something called hypothesis invariance; it says x implies y implies x is valid and that distribution of implication also, you can take as one of the laws. Next we check, say, law of contradiction, it says x and not x is unsatisfiable. Right? And there is one more, it does not look like, but we will say why it is called contradiction. It says not x implies y implies not x implies not y implies x, this is valid. If you read this implies as your entailment, once it is valid you can see that as entailment; it says assuming not x if you derive y and assuming the same not x, if you derive not y then x must be correct, and not x is also unsatisfiable, x is valid, right? Then x must be true.

That is why it is also called law of contradiction, intuitively, it means that assuming not x we get y, assuming not x we get not y therefore, assuming not x we have got a contradiction. Therefore, not x is incorrect, x must follow, that is why. There is something called Modus Ponens, this is very helpful. It says x, x implies y, from that we can conclude y as a consequence. Once you write it as one equivalence, you can also keep your x, that means, x and x implies y is equivalent to x and y. In this consequence you are not showing x, but x is still there once you assume this, therefore, you can write, it is equivalent to x and y.

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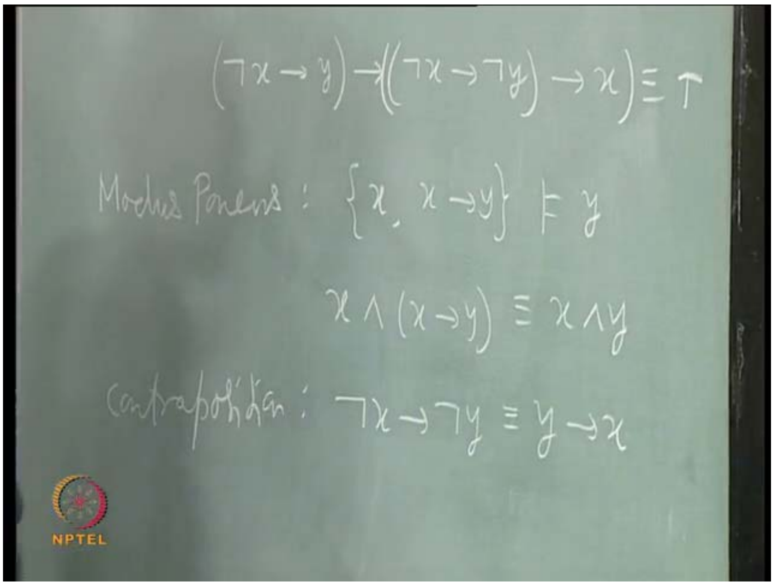

Then there is something called contraposition; it says not x implies not y is equivalent to y implies x.