**Mathematical Logic**
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**Lecture - 4**
**Semantics of PL**

Because you have become matured, we will just have some convention so that, we can write the propositions by reducing some of the parentheses. It is something like giving precedence rules in programming languages. Even in mathematics you have used it from your child hood.

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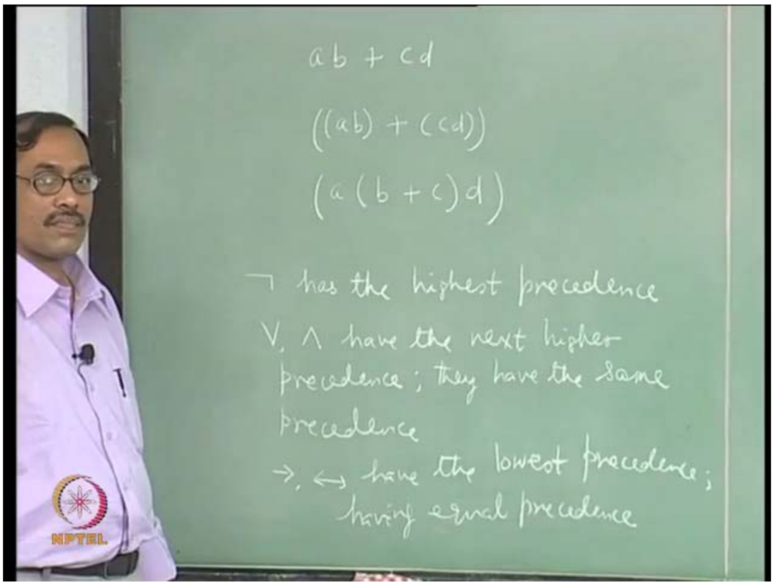


Suppose you write say, a b plus c d in usual mathematics then you would think it as these. If you put the parentheses it would look like this, but certainly not as and not like this. It is because multiplication has higher precedence than the addition, this is what you mean by higher and lower precedence.

Similarly, we will define some precedence for the connectives so that, number of parentheses will become less, it will not eliminate it all together, but it will be less in number. For the precedence we take: not has the highest precedence. Next precedence will be for these two connectives. Once you say both of them are the next higher precedence, what about comparing between them, we will put them at equal precedence. You see that parentheses are not all together eliminated (Refer time: 02:02), some might be there still. Let they have the same precedence. Similarly, the next two connectives, they have the lowest precedence. These having equal precedence means among themselves.

Let us see one example of this; it is not really readable, we need some parentheses. One parentheses we will keep here that will say that, even if they are the equal precedence we know how it has been formed. First q and r has been formed, next p has been added and r together. You can have another parentheses here, but we do not need it because implication has the least precedence.

Rather we need a parentheses here; well, we can have it this way, but without it there will be again problem because, even if not has the highest precedence, it will take with r not instead of this. Suppose you have inserted here, it would say that first r if and only if q will formed then not has been added. They will be different.

This is how precedence will be used and some less number of parentheses we can use after this. And let us come back to our relevance thing that, if a variable is not occurring in a formula or a proposition, then it does not matter what value you assigned to that variable.

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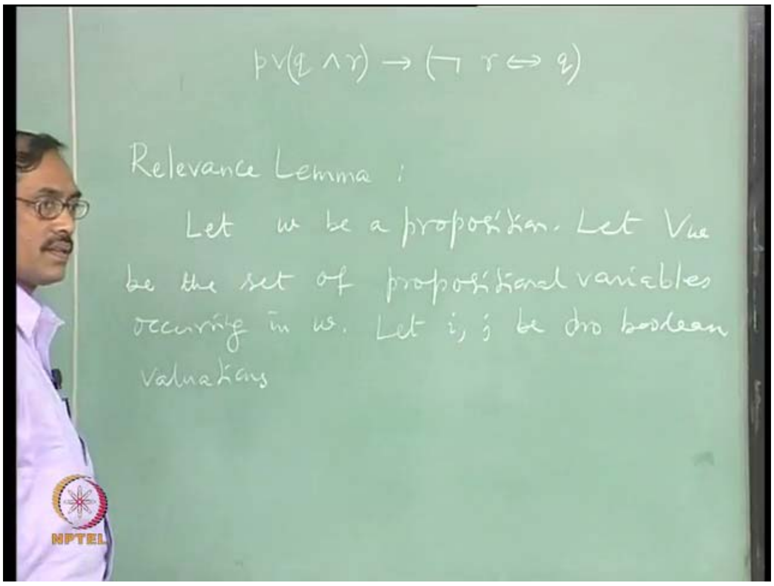


Let us formulate it first, it is called the relevance lemma, let w be a proposition, let Vw, V subscript w, be the set of propositional variables occurring in w. What do you want is, if some propositional variable does not occur in Vw, is not a member of Vw, then i of that propositional variable and j of that propositional variable, even if they differ, i and j should give the same value to w.

You put it the other way, if i and j both agree for all the propositional variables in Vw, then they should agree on the whole w. What we will be writing is, let i, j be two Boolean valuations that agree on Vw, this should be equal for each propositional variable in Vw, that which are occurring in w. Such that i of p equal to j of p, for each p in Vw is that scheme clear?

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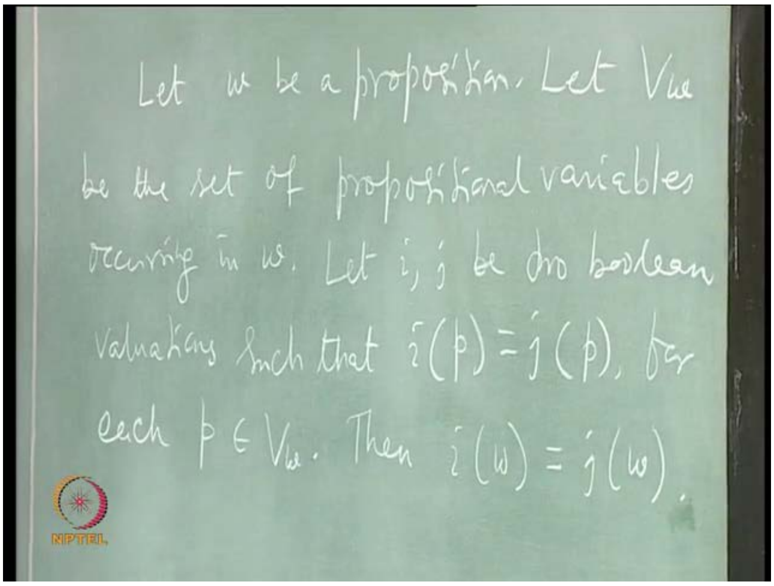


So, i and j agree on all the propositional variables occurring in w that is what we say. Then what should happen, these would agree on w itself. Then i of w must be equal to j of w, is the formulation clear? So that, if they do not occur, some propositional variable does not occur in w, it does not really bother us. Now once you understand the proposition, we should prove it; how do you prove?

Prove it by induction, by induction on? We make that standard, nu w means number of occurrences of connectives in w. First the basis case, when nu of w is equal to 0; in the basis step, nu of w is 0. In that case what could be w? There is no propositional variable occurring in it, what could it be? It is not a propositional variable? It is the connectives, nu w is the number of occurrences of connectives in w. If there is no connective, what could be w? It has to be atomic; atomic means what? It can be propositional constants top, bottom or it can be a propositional variable.

If it is top, it is a propositional constant i of top is always equal to j of top. If it is bottom then that is also same, it is equal to 0, instead of 1. Let us write that first, then w is equal to top or bottom or w is in, let us write, Vw, maybe we can write, but we must say w is a propositional variable, that is easy. If w is top, then i of w equal to j of w equal to 1, if w is bottom then i of,  j of w equal to 0.

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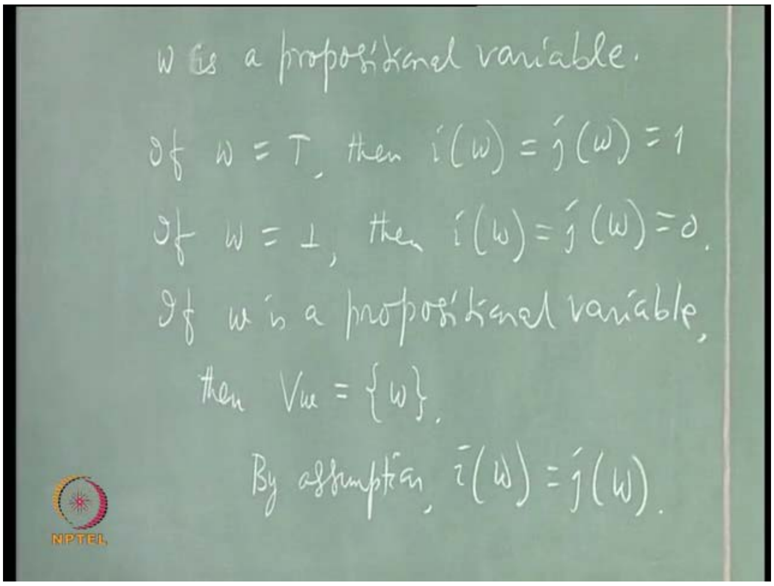


If w is a propositional variable, then what happens, then Vw has only one member which is w. Vw is equal to w itself and by the assumptions i of w equal to j of w, assumption of the statement, that i and j both are agree on Vw, in Vw there is only one member on that they agree. i of w must be equal to j of w, this clears the basis step.

Now in the induction step, we lay out the induction hypotheses. Suppose for all propositions u, with nu of u less than n, we have i of p equal to j of p, for each p in Vu implies: i of u equal to j of u; this the induction hypotheses. Then we start with, let w be a proposition such that, Vw, well, we need really for nu, nu of w is equal to n, number of occurrences of connectives in w is n.

Then, how to write w by definition of the grammar of w or all propositions we say, w must be, is: not x, or, x and y, or, x or y, or, x implies y, or, x biconditional y, for some x, y propositions with nu of x less than n, nu of y less than n. Now, you can complete it? Yeah? Let us see the first case suppose, w is: not x. Now, it has n, nu of w is n. nu of x is less than n, now apply the induction hypotheses, i of x equal to j of x.

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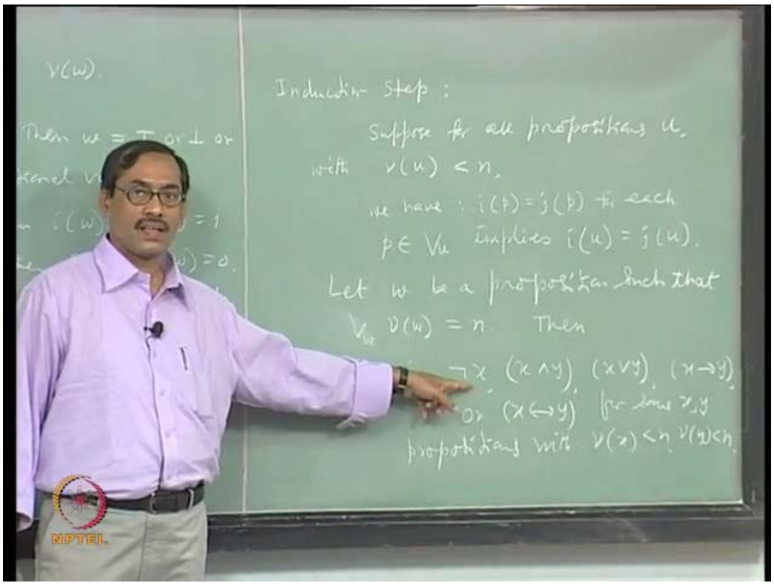


Because, all the variables that are occurring in x, that set, it is a subset of variables occurring in w. You can apply the induction hypotheses, now what happens, i of x is equal to j of x, then by definition of not, i of not x is j of not x, all the thing go similarly. In fact there is nothing more in it than telling that Boolean valuation can be extended uniquely, and that is because, we approve the unique parsing theorem.

Student: We are having the x and y have all the brackets in it without doing by this, precedence rule.

Yes, you are not using the precedence rule.

Student: Okay, so it is.

But even if you use precedence rule, it's unambiguous, because of the precedence rules.

Student: Within the same precedence or?

There will be brackets.

Student: Okay.

Still it is unambiguous but, we are not using unambiguity using the precedence, we are just using it as a short cut to write our original formula, original propositions. Then what it says is, if you consider an interpretation or you consider a Boolean valuation, you need not consider how it assigns the propositional variables, which are not occurring in a formula, in propositions. If that is so, then we are only concerned with the propositional variables that are occurring in w, all the events happen in Vw only.

Suppose there is w which is a proposition having two propositional variables, then we can consider only four number of possible truth assignments, we need not consider any more. In fact there are infinite number of truth assignments because, p3 which is not occurring in it can be given 0 or 1, p4 which is not occurring in it can be given 0 or 1 and so on.

It can be potentially infinite, however this allows, relevance lemma allows to confine ourselves to only those four, that is why the truth table method succeeds. All those truth assignments are the Boolean valuations having only Vw in its domain, are called interpretations. You can use the interpretations instead of the general assignments or Boolean valuations.

That is the clue from the relevance lemma that we can always confine to the interpretations, but there is a problem. You have a formula x which is not having an occurrence of a propositional variable p, there is another proposition which is y, where p occurs. Now you have w, which is equal to x and y. Previously that interpretation for x, which did not given any assignment to p, you had another interpretation for y, which has assigned something to p. Now for w, your interpretation should also consider the assignment for p, when you say it is interpretation, it is really interpretations of a given propositions, in that sense only we can talk of interpretations.

But usually we forgot it, we do not mention the propositions we say an interpretation; it means in the context whatever propositional variables are considered we take them all, that, say, a sloppy language used there. We have to be cautious about it, that is what it says. Let us see some examples, how the interpretations are used.

Consider a proposition which is w, which look like this. We want to evaluate it under various interpretations. Now first we have to find out which are the propositional variables. Look, neither p is a propositional variable formally nor q is a propositional variable formally, we have only p0, p1, p2 and so on.

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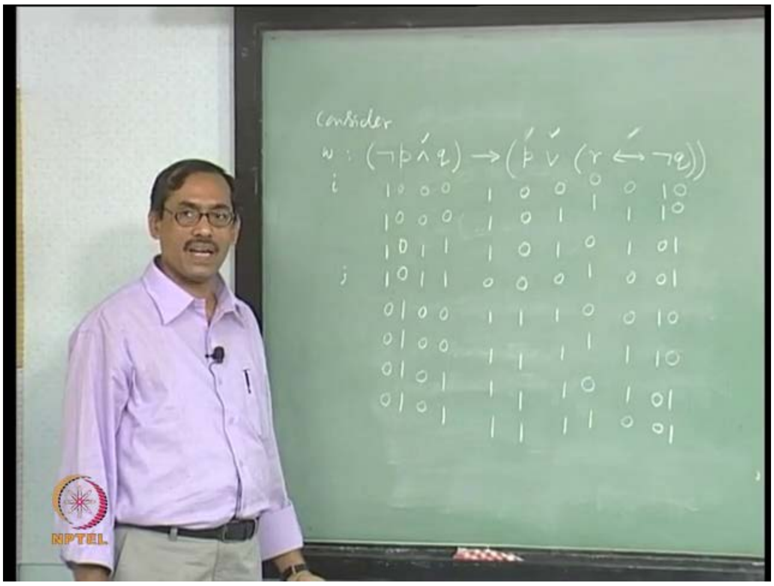


By writing these we are again making some convention, that instead of writing p0, p1, p2 and so on we will write p, q. r, etc. Because this is always difficult to write with subscripts, we will simply proceed with these different symbols.  Now, let us say p, q, r are the propositional variables occurring in this proposition. Since they are three in number, there will be 2 to the power 3, or 8 interpretations. All the interpretations can be written down by varrying our 0 and 1. Let us write them. Say for p, I have now 0, 0, 1, let me take one more 0.

And for q let me write 0 and for r,  now you are required to read the rows. First row says it is an interpretation which gives 0 to p, 0 to q, 0 to r; next row says it gives 0 to p, 0 to q, 1 to r, there is some difference. That is how these are eight possible different interpretations, in this context.

Now, instead of making a bigger truth table what we do, we will make it a shorter one. These are the repeated for the same propositional variable, whatever interpretation we are having. Let me repeat this column p under this p as it is, you have to be truthful; you may not repeat, but you have to make a bigger truth table later. I want to make the truth table shorter by repeating this. Similarly, for q, I will again repeat here under q, next I want to go for the evaluations of the connectives. Now not p this is one interpretation where p is assigned 0, its negation will become 1, I will write it under this not, under this connective, that is how it will go.

Now when you take not p and q, precedence says it is not to be interpretated as not of p and q, it should not be. What it means is not p first to be evaluated, then that along with q should be under together. That means this one, first column and the third column we have to follow the rules of and. That becomes 0 here, I will write below it 0 and becomes 1 when both of them are 1, otherwise it is 0.

Next, I have to do similarly here, now first not q. That changes the truth values. Now, if and only if between these columns and the last but one column, that is your not q. That becomes 1 when there are same, otherwise, that they are 0s. These become 0 and 1. That gives 0; 1, 1 is 1; 0, 0 is 1. Next, p will be added together with these if and only if. You have to consider this column along with this column and `or' them together. Or will be 0 when both of them are 0 otherwise, it is 1. That gives 0 here, just check them.

Next we have implications between this whole thing and this whole thing. This whole thing is written under this column and this valuation is written under this column. These two columns we have to take and perform the implications, implication is 0 when its antecedent is 1, consequent is 0, all other cases is 1. You have to search for 1, 0; it gives that 0 all others will become 1; 0 here.

That is how the truth table will go. What does it say us? Why the truth table? (Refer time: 24:30) Well, it tells if i is an interpretation, which assigns p to 0, q to 0 and r to 0 then, that i assigns this one proposition to 1; this is what it says. Similarly, if you say, let me say, this is i and this is j, this row, this says if j is an interpretation which assigns p to 0, assigns q to 1 and then r to 1, then it assigns the whole proposition to 0. Now you see there can be various possibilities of assignments of 1s and 0s to any arbitrary proposition.

It might quiet happens that after you complete the truth table you get everywhere 1, you may get everywhere 0 or you may get something like this. Some rows it is evaluated as 1, some rows it is evaluated as 0. You want to give different name for all these things.

If you go back to our definitions of models, it says one Boolean valuation i, is a model of a proposition w, if i of w is equal to 1. Due to relevance lemma, we can say that instead of Boolean valuations, only interpretations can be used. You may say model of a proposition is one interpretation, which satisfies the proposition. For example, if w is this proposition, which we have considered, then i is a model of w, i is an interpretation which assigns 1 to w, but j is not a model of w.

Now if we say w, then w has many interpretations which are models, and w is one proposition, one interpretation of which is not a model. For example j; j does not satisfy w. What we say is i satisfies w; there exits one such i therefore, w is satisfiable, it is satisfied under some interpretation.

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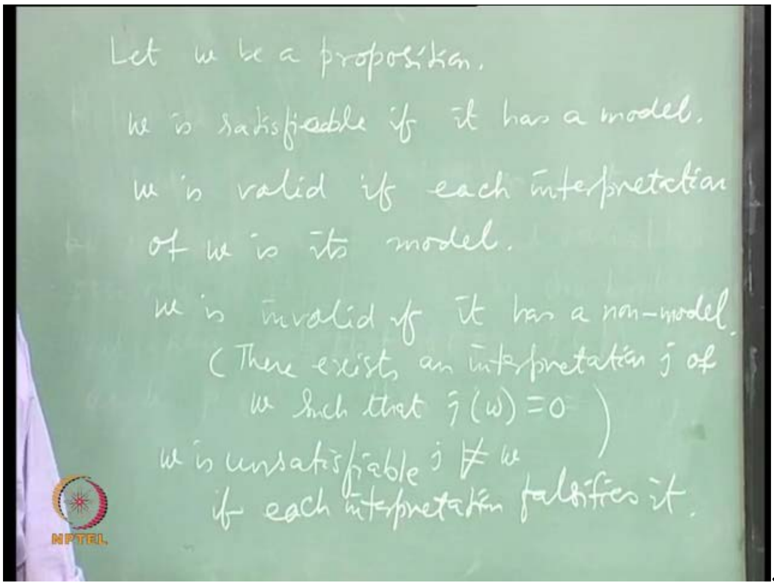


We just give definitions which arise from this consideration. We say that w is satisfiable if it has a model, which means there exits an interpretation, which evaluates w to 1. Now you are using high level languages, instead of telling i of w equal to 1 that is say it has a model. Then you say that w is valid, if each interpretation satisfies it, if each interpretation of w is its model. And this j satisfies w; you would have told that w is such a proposition, here whatever is taken, w is not valid. But you can give examples of valid propositions for example, p or not p it will be valid, but there is another basic thing, which is valid, can you tell me what it is?

Student: top.

Top, top is valid proposition; because, every interpretation of it evaluates it to 1 by definition. And bottom is invalid and also.

Student: unsatisfiable

Unsatisfiable, you say that w is invalid if it has a non-model, there is an interpretation which falsifies it, which does not satisfy it. If it has a non-model, which means there exits an interpretation j of w such that j of w equal to 0, that is what it says. Sometimes we write this to say that it is not a model, we write j does not satisfy w or j falsifies w. These are also some different notations and some different way of reading it, they falsify w or z does not satisfy w.

We say that w is unsatisfiable, if, what happens, if it does not have a model. It has a non-model and it does not have a model, are different.

You say it has a non-model means there is one which falsifies it. It does not have a model means each interpretation falsifies it, if EACH interpretation falsifies it. But there will be some, which are neither satisfiable nor valid, can be? Can there be some propositions which are neither satisfiable nor valid, they are unsatisfiable. If something is not satisfiable, then invalid, unsatisfiability implies invalidity. For example, you take bottom.

But there can be also some propositions, which are both satisfiable and invalid. For example this, it is satisfiable because i is a model of it, it is invalid because j is a non-model of it. Such propositions are called contingent propositions. Contingent means having some information content.

Valid propositions and unsatisfiable propositions  have no information content, though we are more interested in them. They do not have any information content because, either they are always true or they are always false. You get nothing by knowing that something happens, which is of that category. Like, your friend comes from outside, you ask him is it raining outside? He says it is either raining or it is not raining. He is true, he is valid, that his statement is valid, but it adds no information, it gives nothing to you. This valid propositions have no information content, but if he says it is raining. That means he is asserting one contingent proposition. That now creates a word, different world, which gives you some information.

That is why these models or interpretations, they are also sometimes called worlds, because of this. When you come to modal logic, you will use this terminology, always worlds, not interpretations. Let us continue with this. They do not have any information content; valid propositions and unsatisfiable propositions can be used for something else, to find out what else can be concluded from these given things, what else can be equivalent to these and so on.

If you go back to your definitions of equivalence, you will see that you can express it through models now. Recall, we say that u is equivalent to v if i of u equal to i of v for every Boolean valuation i. Now you can think of interpretations; that means u and v are equivalent, if their models coincide, each model of u is a model of v, each model of v is also a model of u.

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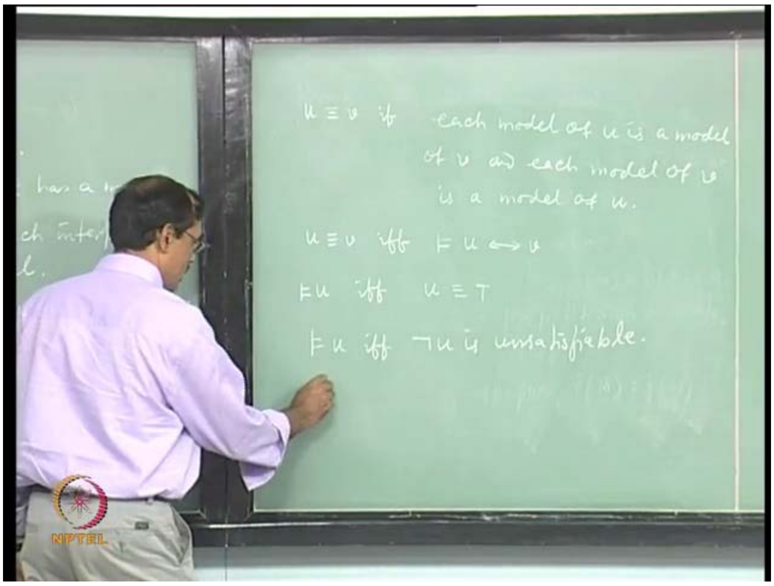


You can take it as a definition; now say, u is equivalent to v, if each models of u is model of v and each model of v is a model of u. Then it should be very clear that u is equivalent to v, if and only if, u if and only if v is valid. To write that it is valid we just have another symbol before it, the same symbol exactly, we are using for models, you are over-using the symbols. Once more we will over use it, then you will see why all those three will become the same.

To say that a proposition is valid instead of writing is valid on the right side, will just give a prefix of it, we will write this new symbol here, models, which we are writing for models also. We will read it as u if and only v is valid now, how do you see the truth of this statement? Suppose we use that each model of u is a model of v and each model of v is a model of u, how does it say u if and only v is valid?

Student: Consider any interpretation.

Consider any interpretation.

Student: Then either both u and v are models or they are not, for that interpretation.

Because u is equivalent to v, once it is a model of u it has to be a model of v it should evaluate those 0s also. It is not possible once it evaluates 1 to or u to 1, it has to evaluate v to 1 also and conversely. Both the things are same or you can get it from the truth table directly, use the same way. It will take only four lines though p and q are not propositional variables, you can still have a truth table because, under any interpretation, u gets either 0 or 1 similarly, v gets either 0 or 1. There are really four possibilities it is not a truth table, in the sense of starting from the propositional variables.

But it is just some four cases; we are showing it in a truth table, and then you can conclude over it easily. Now similarly, you can see that u is valid if and only if, u is equivalent to top can you see that?

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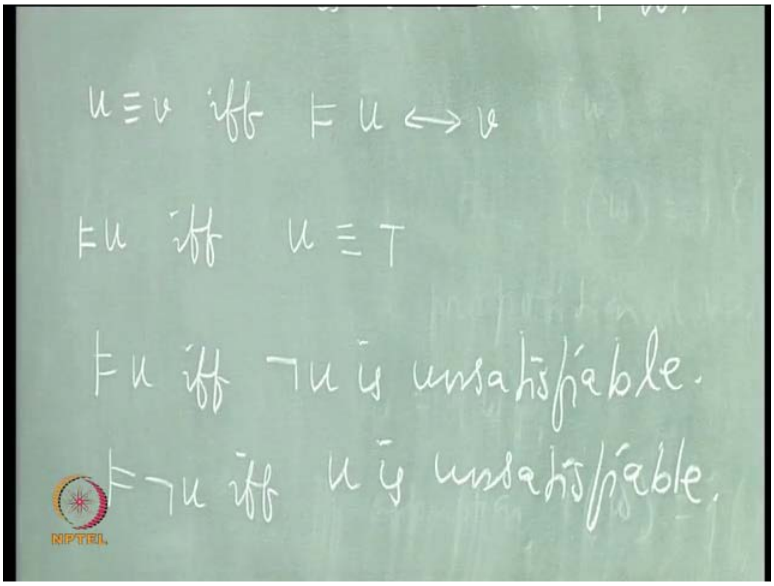

Once you say u is valid, you consider any interpretation, it is a model of u. Any interpretation is a model of this formula, of top, and conversely. Now you can say u is valid if and only if not u is unsatisfiable, can you see this also?

You start from the left side: u is valid now, consider any interpretation; you want to show that, it is not a model of not u. You start from left side. Now you want to show this, to show this consider any interpretation i. Now, because it is that i is a model of u, i is not a model of not u, for each interpretation it happens therefore, not u is unsatisfiable. Conversely suppose not u is unsatisfiable, now you want to prove u is valid start with any interpretation, that interpretation, since not u is unsatisfiable, evaluates not u to 0. u to 1, it is a model of u. Similarly, you can say not u is valid if and only if u is unsatisfiable.

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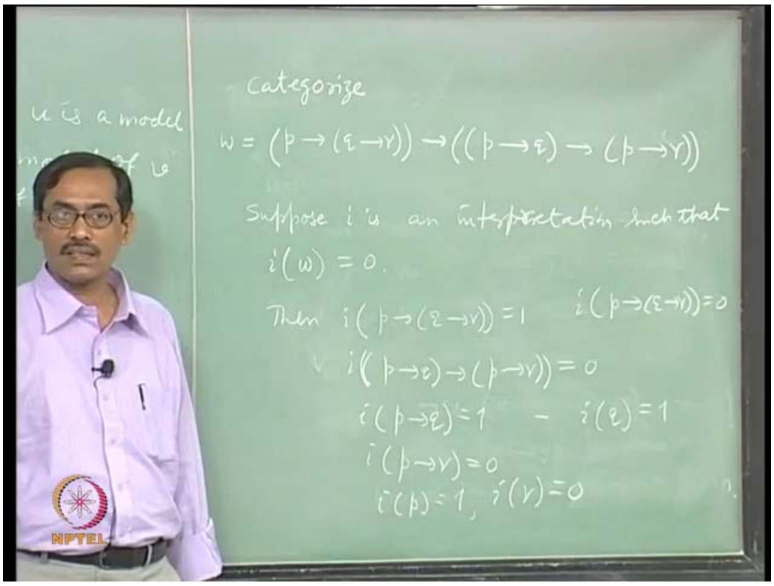


Let us take an exercise, whether this is valid and unsatisfiable or contingent. Yes, well it is not difficult for you, you have come through many more hurdles. Just eight lines, you have to make; assigning 0s and 1s; there are eight interpretations now, verify whether what happens for the interpretations.

Well, I will try to have a short cut here. Suppose i is an interpretation. let us call it w; i is an interpretation, which is not a model of w, interpretation such that i of w equal to 0.

Because implication is easier to evaluate for 0s, there is only one case when it can be 0, its antecedent is 1, consequence is 0 that is why I am trying it first. In that case its sub proposition which is p implies q implies r, should be 1 and the other sub proposition p implies q implies p implies r should be 0. Then i of p implies q implies r is 1, i of p implies q implies p implies r is 0.

Now again I take up this case first, this will give me three cases; this will give only one. Let me take this case; this says i of p implies q should be 1 and i of p implies r should be 0. Now again I take up this case; this is only one case which gives me i of p is 1, i of r is 0. Now i get p is 1, r is 0, if p is 1 and p implies q is 1, what can be q it can be , if q is 0, then it should have been 0, but it is 1. Therefore, q must be 1, this says i of q is 1.

Now i gives p as 1, q as 1, r as 0. What about these? p is 1, q is 1, r is 0. q implies r should be 0, but p is 1 therefore, this should have been 0. This says i of p implies q implies r as 0, but it is given to be 1. That is the problem, it cannot happen. It says this is not possible, i of w can never be 0 whatever interpretations i may be therefore, w is, w is valid because, there is no interpretation i, which falsifies w. Therefore, every interpretation satisfies it. Therefore, w is valid. Now, if one proposition is valid then it has to be satisfiable. Yes, why is it so?

Student: Because, all its models should satisfy it. So, there exists at least one.

Models always satisfy it.

Student: All its interpretations will act as models. So, at least one interpretation which acts as that one.

From all to some when you come, there should be at least one interpretation otherwise, you cannot conclude it. All interpretations are models that is why it valid, then you want to conclude, some interpretation is a model. This, you can conclude provided that all is not like wise, it might happen that a proposition is there having no interpretations. Then if all interpretations are models; but that is not the case. Every proposition has at least one interpretation. Every proposition has at least one interpretation. How?

See, a proposition, suppose, has a propositional variable in it, then assign that propositional variable to something 0 or 1. That shows that, that proposition has one interpretation. But, suppose there is a proposition having no propositional variables in it. For example, top what is the interpretation then?

Yes, there is no propositional variable that occurs in it. So? You take by default any assignment, any truth assignment, there are plenty of truth assignments. And top is equivalent to every valid proposition, you can think of top as p or not p, i of p equal to 1. Now this is a convention we are going to put to conclude that, the top and bottom have interpretations; that is the basic thing otherwise, you cannot show that there exists always an interpretation of a proposition.