Mathematical Logic

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Lecture - 39
Axiomatic System FC

So, for the first-order logic we had seen the semantics, then came to calculations, informal proofs, and then developed the proof by resolution technique. Then, next in the sequence should be our axiomatic system. And the axiomatic system, we will call it as FC, first-order calculus; and this will be an extension of the propositional calculus we already had. That means the three axioms and the inference rule M P will be kept as they are. And we have to add certain more things because of the quantifiers. Again, for the quantifiers, we will take only one quantifier for each x. Because the other one can be introduced with a definition by using the not, De Morgan's rule we have, right? So, like if you have there is x X, we can introduce it as a definition with not for each x not X, fine. Just the way we have introduced the connectives by definitions the same way we can go for the quantifiers. But we have to take care of the special predicate: equality predicate; that might give you some axioms and inference rules. Something we have to introduce there; so these are the extra things we need for FC. So, let us start with FC.

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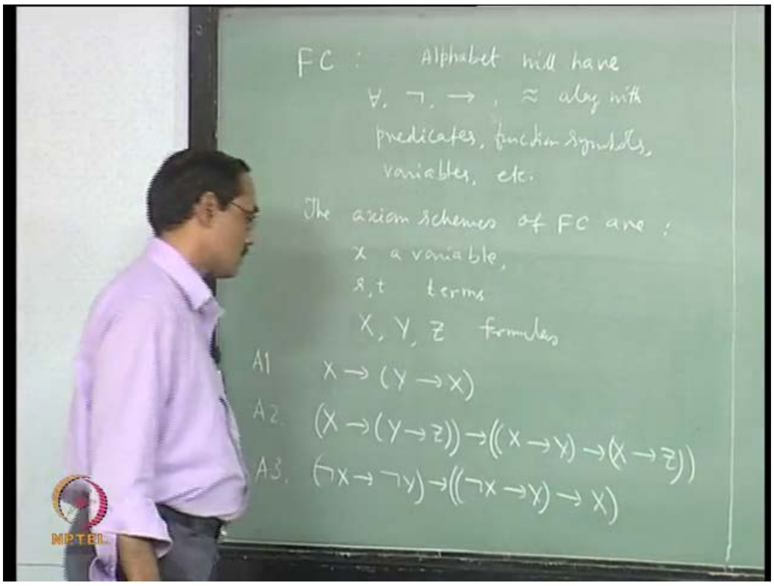

Again, our axioms will be axiom schemes. We are not using there exists symbol. That means we have the alphabet; we will have only this symbols "for each" and then "not" and "implies"; these two are the connectives and this is only the quantifier. And then we will also use the equality predicate. All the other things like predicates, terms, variables, they are as they are; only restriction will be up to this. Yes, along with predicates, function, symbols and so on. So, we will start with the axiom schemes. The axiom schemes of FC are as follows. Here, we will be using x as a variable then we have s, t for terms and X, Y, Z for formulas. With this we will have the axioms; first one will be our earlier axiom for PC. That will be X implies Y implies X. Next, A2 is the distribution of implication, which is X implies Y implies Z implies X implies Y implies X implies Z, just like your PC. Then we have A3 which is for the negation sign. So, not X implies not Y implies not X implies Y, so that should give us X, Law of Contradiction. Then we have the other axioms peculiar to FC.

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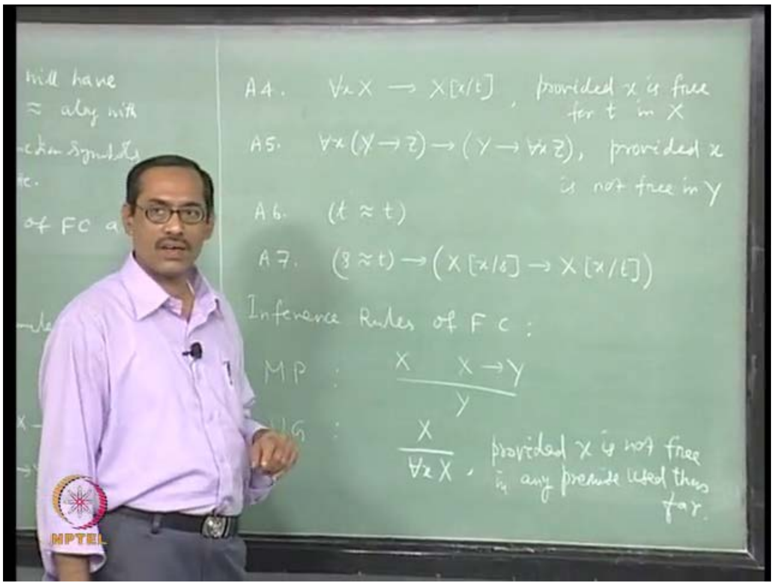

First, we will take the quantifier form for each x X implies X x by t. Now, you are having something like universal specification; but then this term t, there are some restriction on it, fine. That this; what was the restriction? Can you recollect with this universal specification?

Yeah. So that means, x should be free for the term t, x is not only free, x should be free for t. So, write "provided x is free for t in X". Next we have another for implication sign itself. For each x, let us write Y, Y implies Z implies Y implies for each x Z; here again this is not a valid formula for every x Y. But, it is valid when x does not occur in Y, right? So, let us write it “provided x is not free in Y”. Then we should have something for the equality. So, let us write, say, t is always equal to t, reflexivity. And next, we will have s equal to t implies, if you have X s, then we should get X t, fine. So, “substituted” is remaining in the formulas; that is what we want. That should do. Then we will have the inference rules of FC. As earlier, we will take modus ponens, which says X, X implies Y, therefore, Y. And then, universal generalization, that is, from X we can conclude for each x X. But, there is again a problem; always you cannot infer, from P x you cannot say for each x P x, right? It will allow something wrong. So, what should we have is, x should not be free in the premises, right. In fact, when you come across the proofs, if it is not occurring at all in any premise, that might be a bigger constraint. What we need exactly is, till now whatever premise has been used, X is not free there. Then there is no problem, right? So, we will just write “provided x is not free in any premise used thus far” right. That much should be sufficient.

Then as earlier, we will be introducing theorems, proofs, right, consequences, and so on. First thing is, we should define what a proof is. As earlier, a proof will be a sequence of, a finite sequence of formulas, where each of the formulas occurring in that sequence should be either an axiom or it follows from earlier formulas by application of inference rules. For MP, when you apply, you need two such formulas. Earlier two formulas. When you apply UG, you will need one formula, right. It should follow from that; follow from means whatever is in the denominator. Numerator must be there in the proof already, denominator should be, we will be telling that it is derived from that or it follows from that. That is what a proof is.

Next we should define what a theorem is. As earlier again a theorem is the last formula of a proof, fine. We are going so formally; it looks funny; but that is what it is. So, a theorem is simply the last formula of a proof. Next, we will be introducing consequences or rather provability of consequences. For consequences again, we will take sigma is a set of formulas and X is a formula. We will write sigma entails X in FC now. This FC often we will omit, once we know that we are working in FC. So, sigma entails X if there is a derivation of X using formulas from sigma. The formulas in sigma are called the premises. And what is a derivation? Just like your proof, you will be having a derivation. Again, a derivation is a finite sequence of formulas, where each formula is either an axiom or it is a premise. It is a formula from sigma or it is derived from earlier formulas, right? Similarly, this symbol can be used for the theorems; without any thing on the left side, which means now a derivation is a proof without any premise. It allows that.

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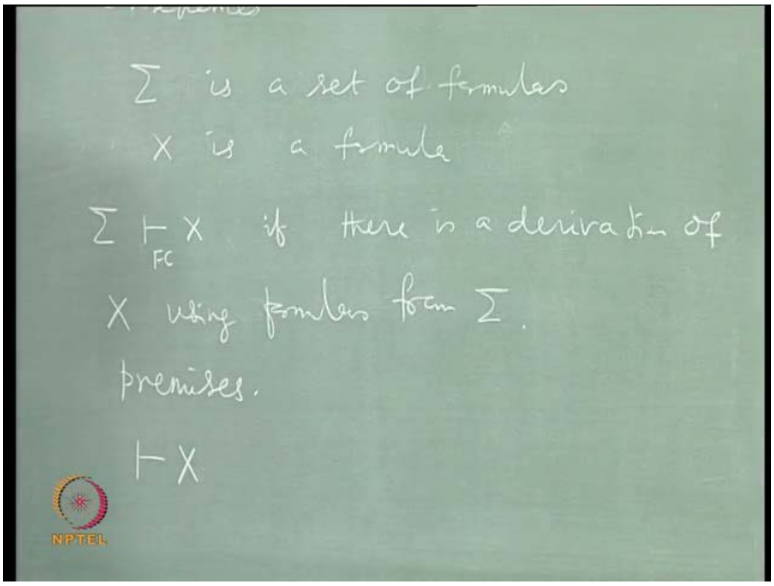

Then, we will introduce also inconsistency. We will say that sigma is inconsistent if for some formula X we have both sigma entails X and sigma entails not X. That means, we have a derivation for sigma entails X, we have a derivation of sigma entails not X. Otherwise, we will say that sigma is consistent. Here really inconsistency is nicely defined; consistency is not, because consistency says that it cannot be derived; whatever X you choose, either X cannot be derived from sigma, not X also cannot be derived or one of them can be derived other cannot; it is difficult to show, right? But inconsistency, maybe we can show it; we can demonstrate it by having two derivations: one for sigma entails X, one for sigma entails not X. We will see some examples, how the proofs should go there.

Now, this is a familiar theorem, yes? The first theorem in PC we proved was this: p implies p. You can use the same proof exactly here; also that will prove for each x X implies for each x X, right. But, here with quantifier axioms and quantifier rules we can have a different proof also. Let us try that. How it goes? First is, you may say for each x X implies X; which axiom is it? See, provided x is free for t in X. So, I can take t equal to x itself, so x is free for x in X. Now then this becomes X itself; there is no change; x is substituted by x, so it is simply A4.

Next, we would go for, for each x X implies X implies for each x X implies for each x X. Where from it comes? Just check. See, Y is your X.

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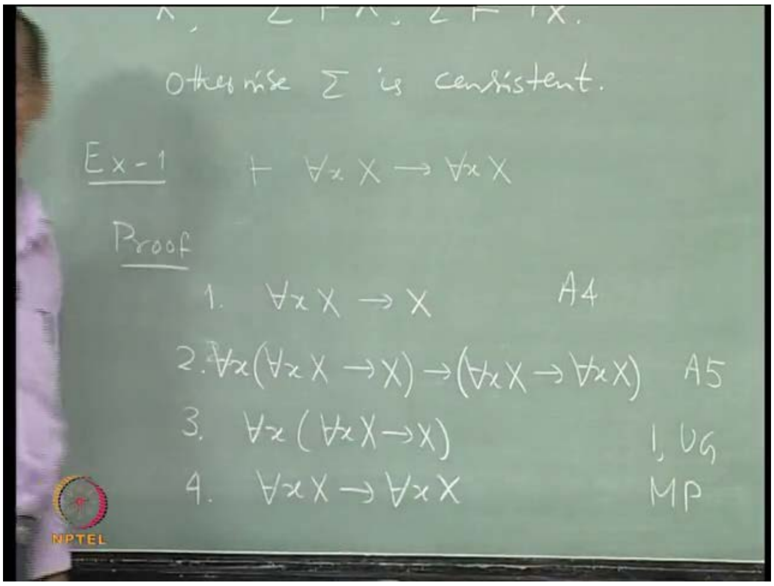

Student: Y Is for each x X.

Y is, you have to take for each x X. What we need is this, suppose I take for each x X, this is X; then I would get for each x X, here, for each x Z, here, right? The condition is x is not free in Y. Suppose I take Y as for each x X, then x is not free there, right. But, it needs one for each x here. That is the only thing, right. So, let us put it. Fine. Now, its all right; this is A5. Next, we add this one, so that we can apply MP; and this can follow from this. Why? UG. Because there is not a premise used. So, it says the restriction on UG is that the variable x should not have been free in any premise used till now. No premise has been used, right. It can always be used. So, third line; we may say for each x, for each x X implies X. This comes from 1 by UG. Then, fourth is by modus ponens for each x X implies for each x X.

Then, let us take some more examples to see something which is not in PC. This is really coming from PC; this theorem. It is p implies p, though we have a different proof. instead of using just p implies p form, fine.

This is your familiar rule of renaming. You have, say, P X or P y. Here for each y P y should entail for each x P x, right; y has been substituted by x, fine.

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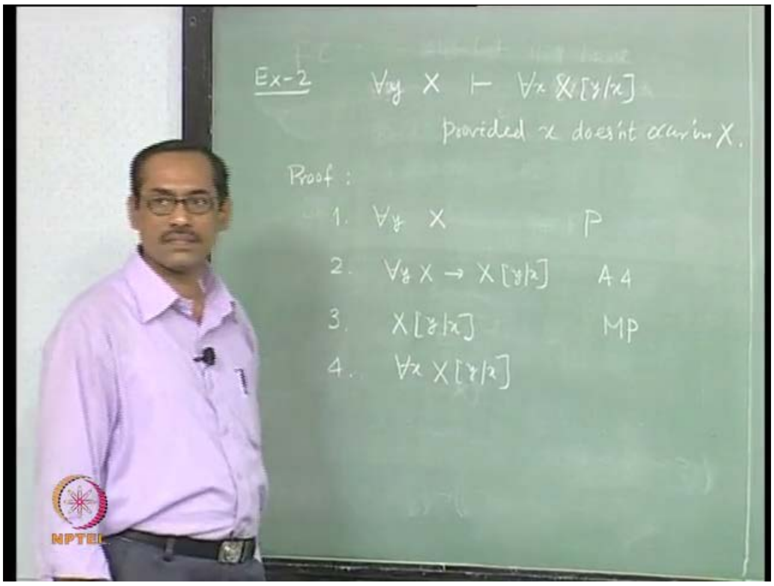

Student: Y is not already.

Yeah; so what is the condition?

Student: x.

Student: X should not be free.

x is not free in X.

Student: Sir, it can be.

Right. You may say better; or, let us say, x does not occur in X then?

Student: X should not occur.

x does not occur in X. Suppose this is given; fine, it should be all right? Let us see a derivation. Again, we will not be fussy about writing derivation or a proof; we will just write a proof; you may write a derivation. Now, we can use this as a premise; that is what it says and then finally, you should have for each x X y by x. So, let us write that as a premise first.

Student: A4.

Which one we should choose? A4, in which form? For each y X implies?

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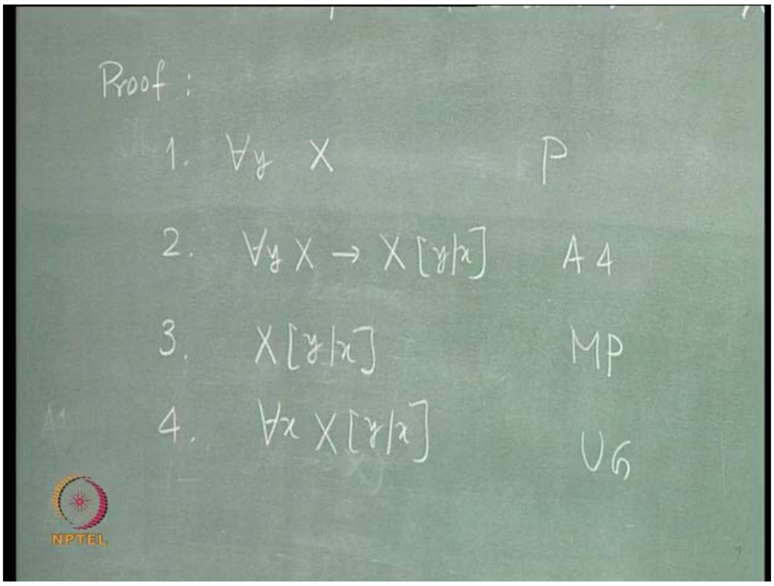

Student: For x X by x.

X y by x; because that is what we want, right. So, x is your term t, which satisfies the condition required there; the variable y should not be free.

Student: Small.

Should be free for x, right. It is free for x because x does not occur at all; fine. This is A4. Then you have modus ponens, which gives X y by x. Then UG, which is the natural thing. And UG is applicable? Constraint is, x should not occur free in for each y X. That is the only premise used; right; it is not free because it does not occur.

You can interchange the quantifiers; this should be easy to go. All that you need is, get X, and then go on generalizing, right? So, we start with our premise for each x for each y X, second we go for for each x for each y X implies for each y X; x is substituted by x; that is all right; this is allowed by A4. Next, we conclude for each y X, by modus ponens. Next, again for each y X implies X, A4, next modus ponens. Now, slowly generalize. So, six, for each x X; UG. It is allowed, because in the premise, x is not a free variable. x is not free; there is only the premise we have used. Then again, for each y for each x X; UG, well.

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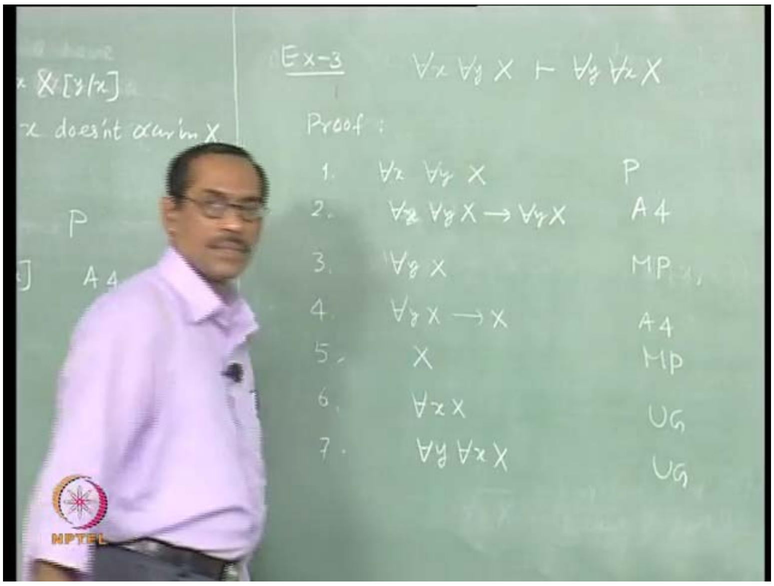

I can start with the premise always; for each x not X, that is a premise. Then we can say not X, because this is an axiom, yeah? A4 is that; right? Then modus ponens gives not X. Now, from not X we want to infer not for each x X. No axiom says that; right? No inference rule is there. But, we can always infer from for each x X, X. Can you see some relation between these? See, you have not X; you want to infer from this not for each x not X. No, it is not for each x X; you want to infer this; right. Look at the other side; can you infer this? For. Yeah? That is A4.

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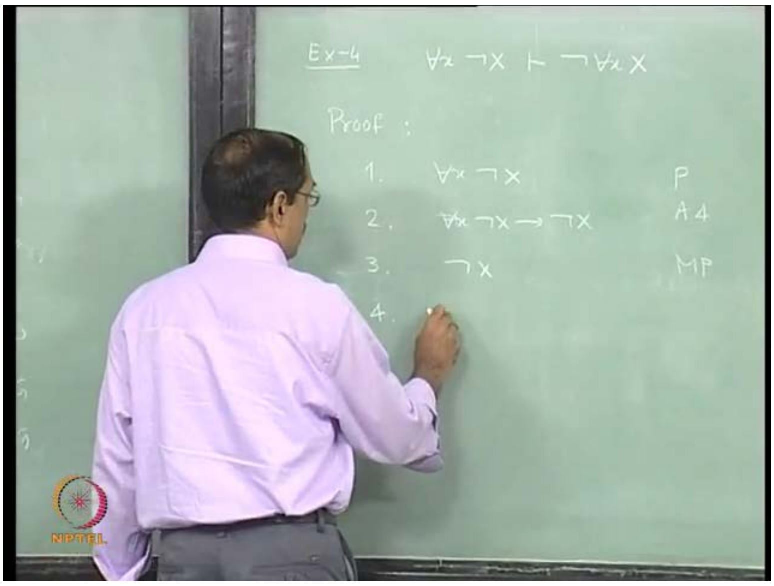

This is really A4 and MP; fine. Now, what is the relation between this and this? This implies this; this implies this; contraposition. You can always use PC theorems because it is an extension; right? If you do not want to use the theorem, you have to duplicate its proof; that is also fine. Let us use PC theorem; which is contraposition. Now, with contraposition, which one we want?

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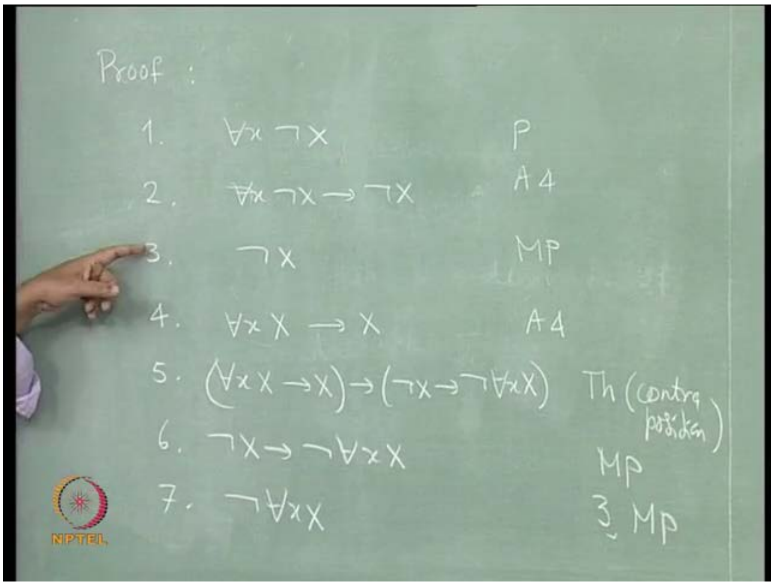

For each x X implies X; this an axiom. Next, we say for each x X implies X implies not X implies not for each x X. Ok? This is contraposition. We will just write as theorem; it is already proved in PC, right. You have already proved it as a theorem. If you want to find what is the name, you say contraposition, fine. Next by modus ponens we reach not X implies not for each x X. Next, use not X and modus ponens. I am not writing again 6; it is the previous line, so only remote line I am writing as 3. So, 3 6, modus ponens gives not for each x X.

Let us take one more example. We will take this, which looks very simple. But we do not have any axiom for it. We do not have an inference rule; symmetry of the equality relation. Anyway, we start with the premise; next?

Student: We use A7.

A7 in which form? Well, A7, if you look along with MP, see, it will be easier to think about the consequences. That is why you think always with MP instead of the axioms.

Suppose A7. A7, I can think of this as a consequence, this way: if I have already s equal to t, I have already X of s, then I can infer X of t. Intuitively, that is what A7 says. By application of MP twice. Now we want to find out that t equal to s is inferred. So, our X t should be t equal to s, right, so X should be equal to s equal to s.

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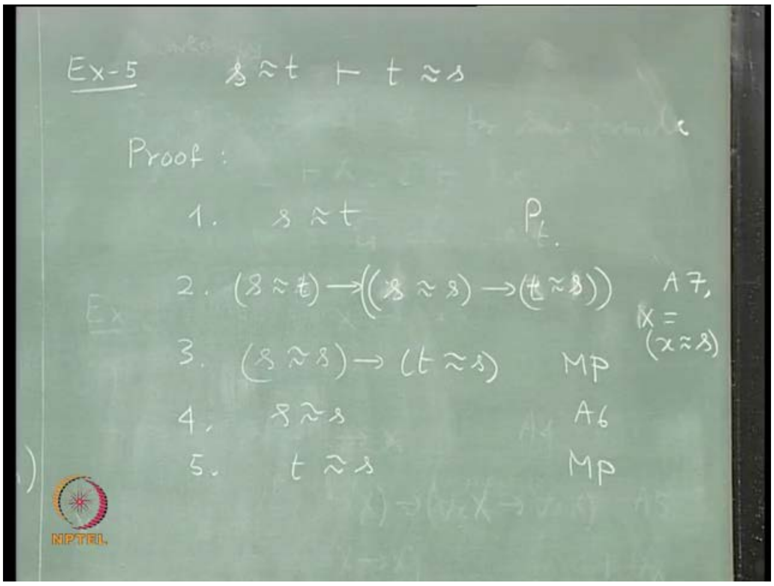

Right. But, if you say; see, there is a problem; writing in this form is giving the problem. If you write in the correct axiom form, it should not. Suppose I write this as X x by s and this is X x by t. Right. Now, you can think of X as x equal to s, fine? So, the axiom way, we are writing X x by s is not exactly X of s. Because you cannot substitute s by t directly; they have to be substituted by a variable and that gives also freedom. You can substitute partially, right? Is that so? We start with our X as or rather, axiom seven in the form: s equal to s, s equal to t implies x equal to s implies x equal to t, right. This is the axiom we want. This x will become now; this is s, this x will be substituted by t, that is what we want; is it right?

Student: Yes, s must be equal to substituted both in X x of x is.

This x you are going to substitute by s. So we start with this form of the axiom; is it clear? So, to give a comment, to make it readable, you may say A7: x equal to, x equal to s, that is all. That will make it readable. Next? We go for modus ponens. Next we introduce s equal to s, this is A6; and again by modus ponens t equal to s.

So, this is how we will be doing the proofs and derivations. Then immediately we should go for the meta-theorems, which should be helpful for us. That is what we saw in PC.

Let us formulate monotonicity. We start with two sets of formulas, one of them is a subset of the other, and then we have another formula. If sigma entails X then gamma entails X. Also we had another form.

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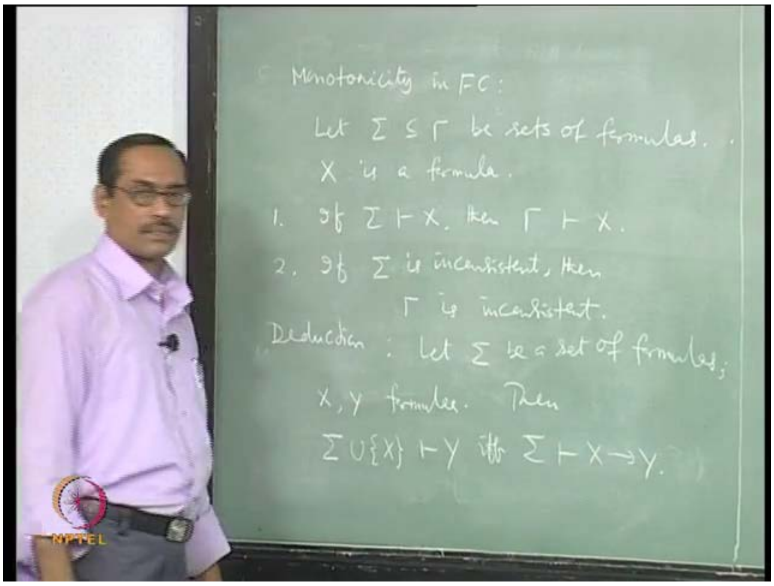

Well, we go the other way. We can write it as: if gamma is consistent, then sigma is consistent; fine? But, we will be proving this; we will be proving this; fine. How do we prove the first one? There is nothing to prove really. It is the same proof of PC, right, because there is a derivation. It is the definition of the theorem itself, definition of the consequence itself. Similarly, this one also. Once sigma is inconsistent, there is sigma entails X, sigma entails not X. By the first one, gamma is inconsistent. Second one implies first one. That will come only after reductio ad absurdum. First one implies second is easy.

Now, let us go to deduction theorem. This will say: let sigma be a set of formulas, then you need two formulas at least; X, Y formulas. Then sigma union X entails Y if and only if sigma entails X implies Y. One part of this should be easy; just an application of MP. Suppose sigma entails X implies Y. Then there is a proof of it. Take that proof. Call it P1. It starts as: 1 something and some m which gives X implies Y. All the premises used are in sigma only, fine. Next we introduce X which is a premise. Now, you have used premises from sigma union X. And then m plus 2 is Y, by modus ponens. That is all. This says sigma union X entails Y; just an application of MP.

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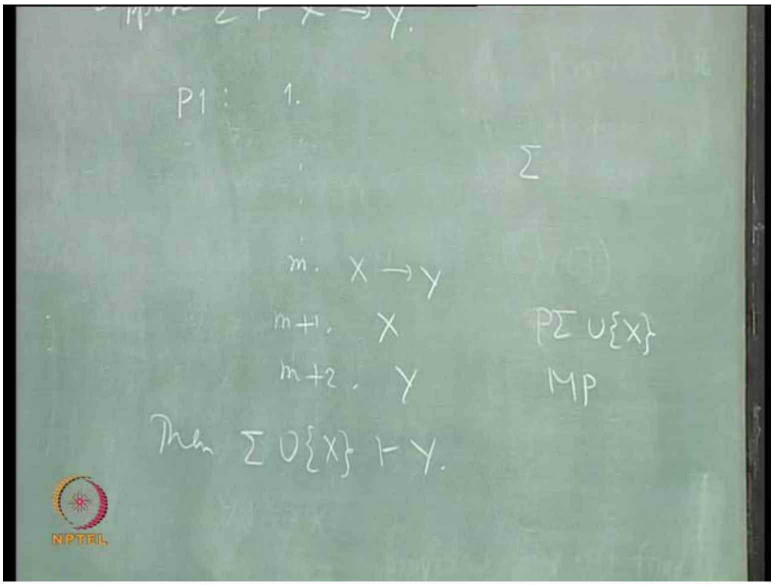

That means, sigma entails X implies Y gives sigma union X entails Y. The other part we have to show now. Assuming sigma union X entails Y, we show that sigma entails X implies Y. If you remember, we had done the proof by induction, right. The same method will be used here again, but then why repeat?

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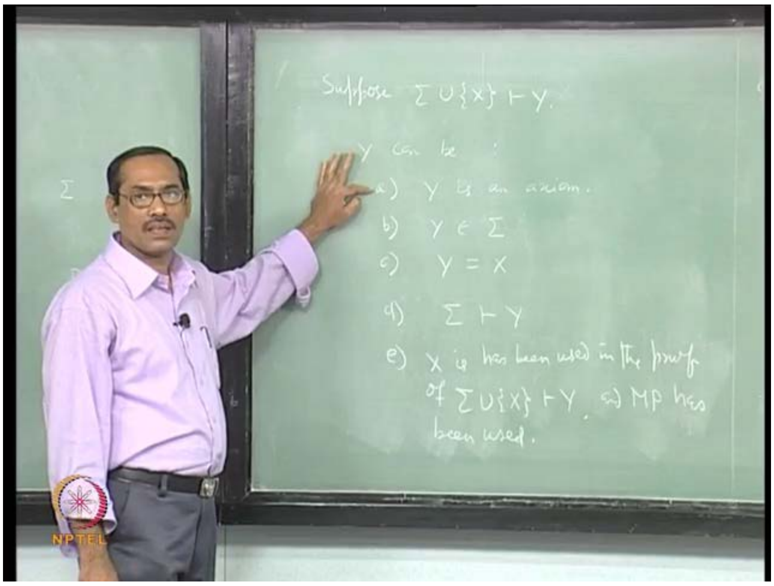

There is something extra here. Let us see that. We proceed with our assumption sigma union X entails Y. Our aim is to show that sigma entails X implies Y. Since sigma union X entails Y, what could be this Y? First, Y is an axiom. Second, is Y belongs to sigma. Third, is Y equal to X. So it belongs to sigma union X. Fine; anything else? Anything else? How can this come; that is what our problem. How can Y follow from sigma union X?

Student: Proof of Y.

So, there is a proof of Y; that is what it says, right. Let us take one simpler case; say, there is a proof of sigma entails Y, where X has never been used. First, let us take that X, some way, it has been derived; where X has never been used. Next case, we will take X has been used in the proof. So, X has been used in the proof, proof of sigma union X entails Y. In this case how that Y follows from sigma union X? Y can follow because of all these things; anyway we are not using that. It can follow by application of some inference rule, right? So, it can follow by an application of MP or it follows by an application of UG, right? Say, MP has been used. And the last case is X has been used in the proof of sigma union X entails Y and UG has been used. These are all the cases; you can follow, right? In each of these cases, we should show that sigma entails X implies Y.

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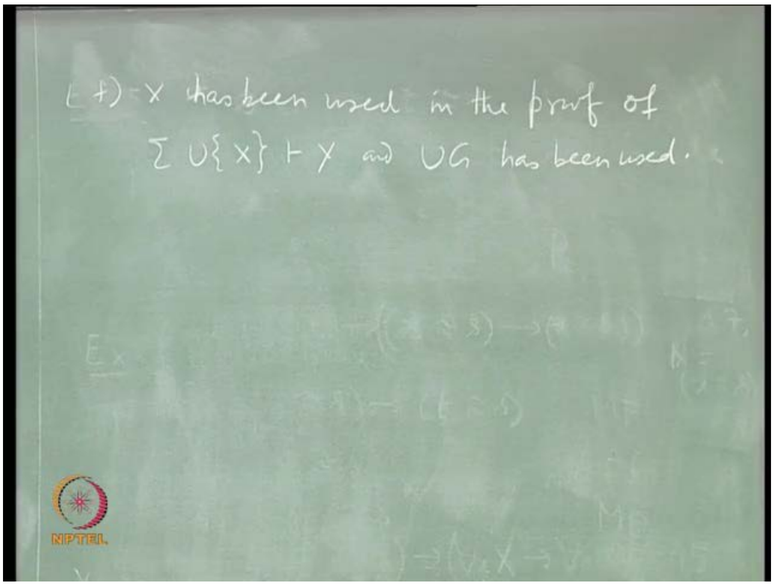

Let us see the first case. Y is an axiom. Then how do you say sigma entails X implies Y?
Student: Y has been taken. It is, let us take A7, axiom itself. Axiom one?

We want X implies Y not Y.

Student: Ah, Y is.

See, our aim is to show that sigma entails X implies Y. So, this is our aim; to show that sigma entails X implies Y, right? So, you need axiom one; you need axiom one. Right?

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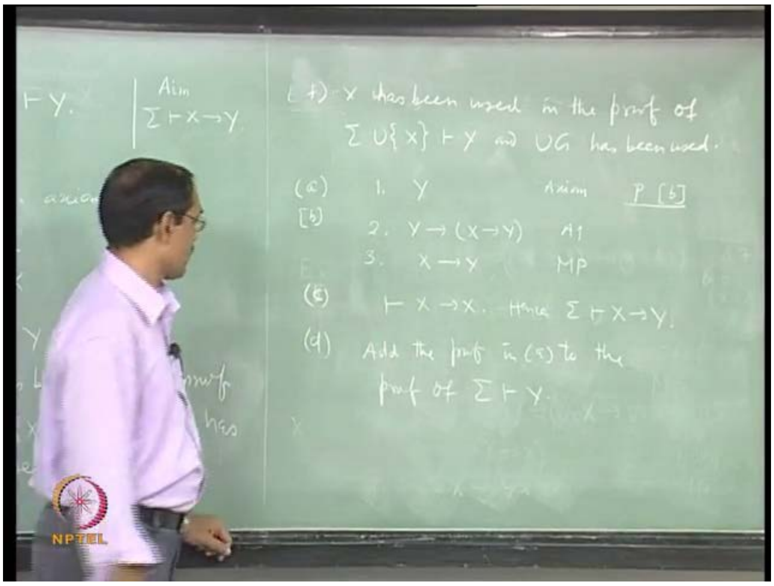

So, case a: we will give a proof of sigma entails X implies Y. We start with Y, axiom; some axiom, which one we do not know. Next, Y implies X implies Y; A1. Therefore, X implies Y, by modus ponens. This is a proof of sigma entails X implies Y. Even sigma has not been used. It does not matter; not that everything has to be used.

Next, case b: Y belongs to sigma. Same proof. Instead of axiom, hm, if it is b, you will be writing here premise; for b everything else remains as it is, right. For a you proceed like this; for b instead of axiom, you write P; it is a premise. Then also it says sigma entails X implies Y, clear.

Now, let us take case c: Y equal to x. X implies X is a theorem, right. Hence, by monotonicity, we do not need monotonicity, right, but, since we have proved, let us mention.

Next case d: sigma entails Y. What to do here? Just like this. Case b. Just like case b. Right? If sigma entails Y, then you have another proof of, up to this line, Y; where only premises from sigma are used. Then add this to that proof, right. Add the proof in a to the proof of sigma entails Y. You have sigma entails Y. Already, up to this, you have the proof, where is the last line of the proof of sigma entails Y. Then continue this way. If sigma entails X implies Y, that is all.

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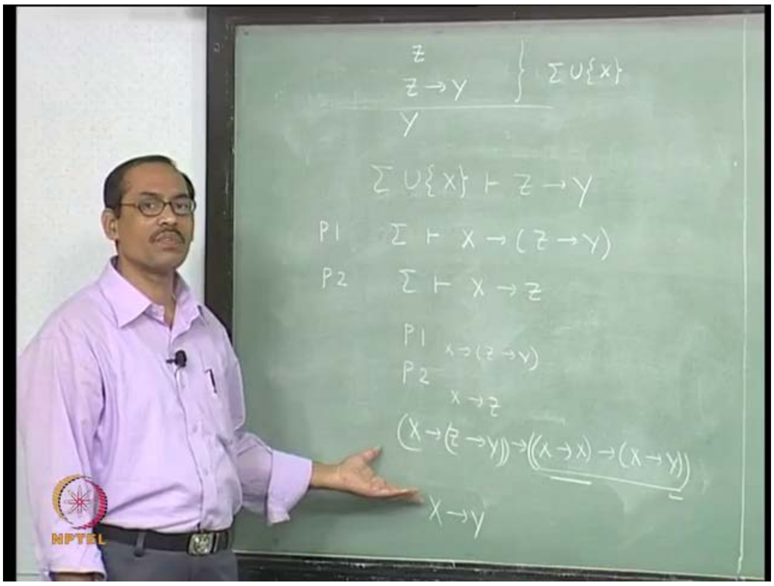

Next case. Now, we need induction. It has been obtained by MP. Suppose it has been obtained by MP. So, you have a proof where Y has been obtained, right. And you have sigma union X, where X has been used really, right, that is the case. Now, once Y has been obtained by MP, you have another formula Z implies Y before it. So that MP has been applied, fine, is that ok? Next?

Next is the induction hypothesis on this; next is the induction hypothesis; here proving it by induction on the proof of sigma union X entails Y, right. Now, this says sigma union X entails Z implies Y. That is what it shows. Then by induction hypothesis, you have sigma entails X implies Z implies Y; is that ok? But then, since we have applied MP you also have Z before it somewhere. So, apply again on that; you get sigma entails X implies Z. Now, add to this proof, add to the proof of this and proof of this. Suppose this has been proved by P1; this has been proved by P2; then what you do, take P1, next take P2; these proofs. Here, the last line is X implies Z implies Y. Here, the last line is X implies Z. Next, apply axiom two, implies X implies Z implies X implies Y. Apply modus ponens twice. Once you get this one; next time you use this, you get this one. You get X implies Y; that is all. So, this step is by induction on the length of the proofs of sigma entails, sigma union X entails Y; fine. It is just mimicking the same proof as in propositional calculus.

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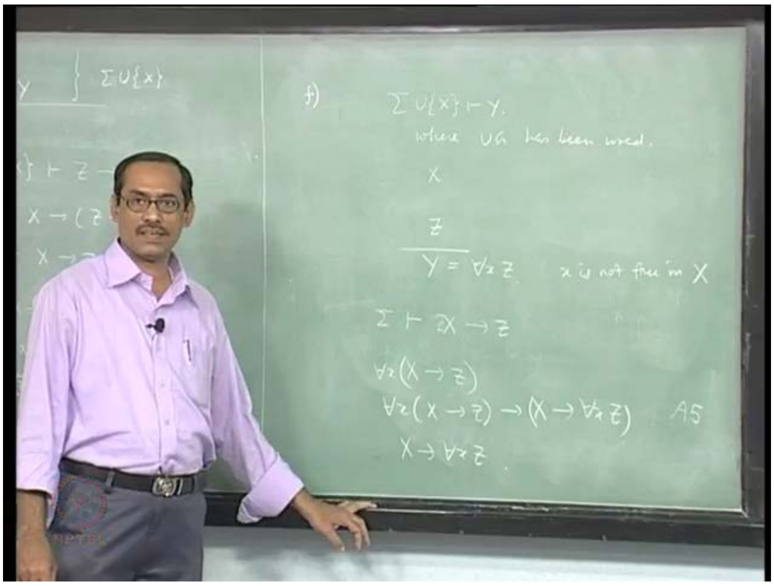

Now, we are in case f. You have a proof of sigma union X entails Y, where UG has been used. Again, the proof will by induction. You have a proof where you have obtained Y by universal generalization. That means, Y must be in the form for each x, some Z, some x. Because UG has been used. So then you have Z somewhere here on, UG, you have applied, and this x is not free in X, in all the premises. But, X has been used, right. So, it is not free in X. All these information are there. X is used somewhere and it is not free in X. Now, what do we do? Apply induction hypothesis. Then you get sigma entails Z, with X, right. We have sigma entails X implies Z. Now, sigma entails X implies Z. So, from this by UG, you get X implies Z, with, for each x. All that we need is x should not be free in the premises; and x is not free in the premises, it is not free in X; nowhere else.

So, for each x X implies Z. Then use the axiom for each x X implies Z implies X implies for each x Z; which axiom is it? A6 or A5? A5. And condition is, x should not be free in X, that is what it is, fine. Then use modus ponens. You get X implies for each x Z; that is what. And X has never been used in this proof, right? So, that is the end of the proof of deduction theorem. We will slowly go to other meta-theorems.