Mathematical Logic

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Lecture - 35
Herbrand's Theorem

So, last time we had introduced Herbrand expansions; we proposed some statement, it might be true or false we do not know. But we saw from the examples that something is happening. Given a formula you find its Herbrand expansions; then say that that formula is satisfiable if and only if the Herbrand expansion is satisfiable; that was our proposal. For that we need the Herbrand domain or the Herbrand universe, then that generates the Herbrand interpretations, then the Herbrand expansions. We started with a formula in a c n f. So, let us keep that.

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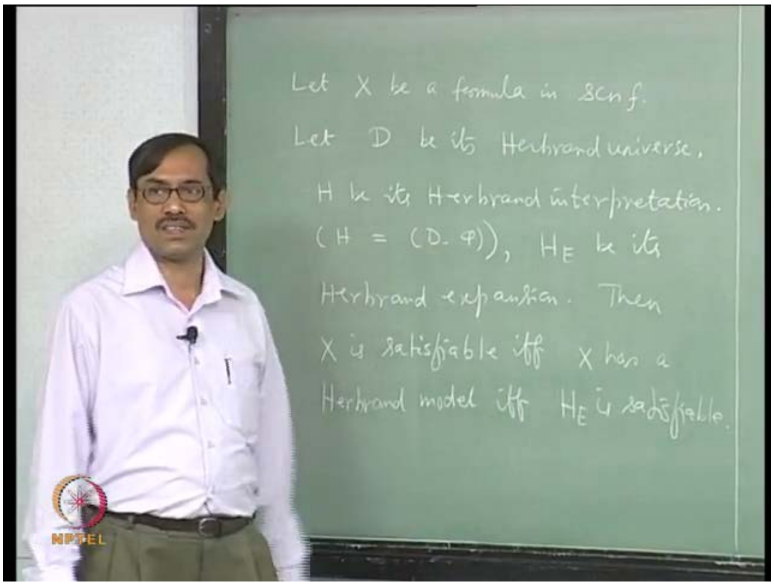

Let X be a formula in c n f. Of course, any formula can be converted to a c n f preserving satisfiablility. So, there is no harm in starting from the c n f formula itself. If satisfiablility is all that we are concerned with, then you say you find its Herbrand universe. Let us write it as D. Then we have automatically the Herbrand interpretation. What we had seen is that if you take its Herbrand interpretation, the map along with this, which we are calling as the Herbrand map, should specify the predicates, how they are interpreted, should specify the function; how they are interpreted. Then, here we do not need these functions to be interpreted, because the domain of the universe itself are having the terms. So, you just interpret as f and any predicate P is also interpreted as P. So we do not need Herbrand map to specify those things. But we need it to specify whether any atomic formula is true or false. That is the specific aim of the Herbrand map.

So, here this H is equal to an ordered pair D and phi, where phi specifies how these atomic formulas are interpreted; either to 1 or to 0. They are either declared true or declared false. Some of them can be true; some of them can be false. Then we need also the Herbrand expansion. Let H E be its Herbrand expansion. You just get the Herbrand expansion from the c n f, X itself, by substituting the variables with the elements from D. These are just terms generated from the constants appearing in the formula and function symbols appearing in the formula. If no constants is there, we start with eta; some symbol we have started with; we start with eta; and if no function symbol is there, then eta is the only element in our domain D. If at least one function symbol is there, then the domain D will become infinite; it is denumerable. So then we want to show the satisfiability. Then, X is satisfiable if and only if X has a Herbrand model if and only if its Herbrand expansion is satisfiable. This was our proposal.

Then let us see which ones can be dispensed with easily. We can start from the beginning; say, X is satisfiable. We want to show that X has a Herbrand model. That is really the main part of the proof; all others will be simpler. Let us see the main part first.

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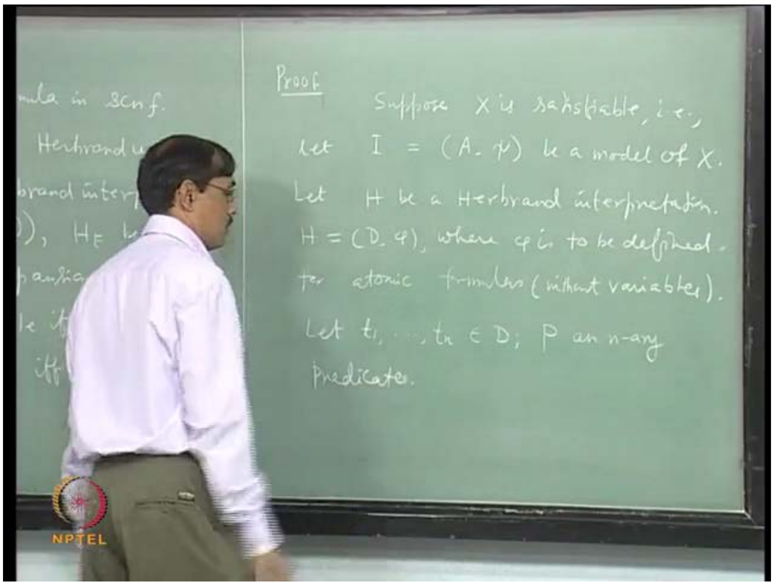

So, suppose X is satisfiable. Now, we can take X to be a sentence, because all the free variables are assumed to be universally quantified there. It is a sentence. Then we need not go to states; we can be at the place of interpretations itself. Once X is satisfiable, it will have one model. That is, let I be equal to, you cannot use that D; let us write A phi be a model of X. So, A is some nonempty set, phi is that map which interprets the predicates, function symbols, and constants in it; and they will map always to A; A is our domain. Now, you have to show that this has a Herbrand model. So, where from we get the Herbrand model? In the Herbrand model, we have the Herbrand interpretation. First that should satisfy. So, Herbrand interpretation is already determined. D is fixed. Once X is given, D is fixed; phi is also fixed, in the way it is. How it is giving P and all the predicates, the function symbols? phi is not fixed. Only in one sense that which propositional or which atomic formulas it declares to be true, which one it declares to be false, that sense only. We have some freedom there. Otherwise everything is fixed. So, we start with A. Let H be a Herbrand interpretation. So, H is equal to, let us write, D phi, where phi is to be defined.

They will be defined for atomic formulas only; for the atomic formulas. And where ever in those atomic formulas, there is a variable, also only closed terms can appear there; because that is how this phi will be over this D; D is having only closed terms. Once you have some P, P of t 1 to t n, and so on, so this t 1 to t n are closed terms, there is no variable in it. So, this phi to be determined only for those formulas. So we may write atomic formulas without variables; this is to be defined. How do we define? Well, suppose you take any predicate P. Let us say, P of s t; a binary predicate, where s is a closed term, t is also a closed term. The terms belong to D because all those terms are here; nothing else is there. Now, how do you define P s t to be 1 or 0? Well, you will define the same way as it is being interpreted here. Because we want this to be implied. This, we have the freedom of defining it; we define it in such a way that it becomes a model. It should comply with whatever this I interprets. Is that right?

So, let t 1 to t n be terms in D; P be a predicate; P an n-ary predicate. Define phi of P t 1 to t n; let us forget the commas; is equal to 1 if and only if what happens, phi of? You want to connect with our original model; phi of t 1 is defined; phi of t n is also defined, because they are occurring. All those functions are also occurring in the formula X. So, if this belongs to phi of P. If there is a complicated formula built up from this kind of predicates; you have connectives and all other things. So, the connectives are taken care in the same way as in H also. There is no quantifier. There is nothing to prove. But then there can be free variables in X. Okey? There only, connectives are not there. For connectives, there is nothing to be proved; only for the free variables we are worried now.

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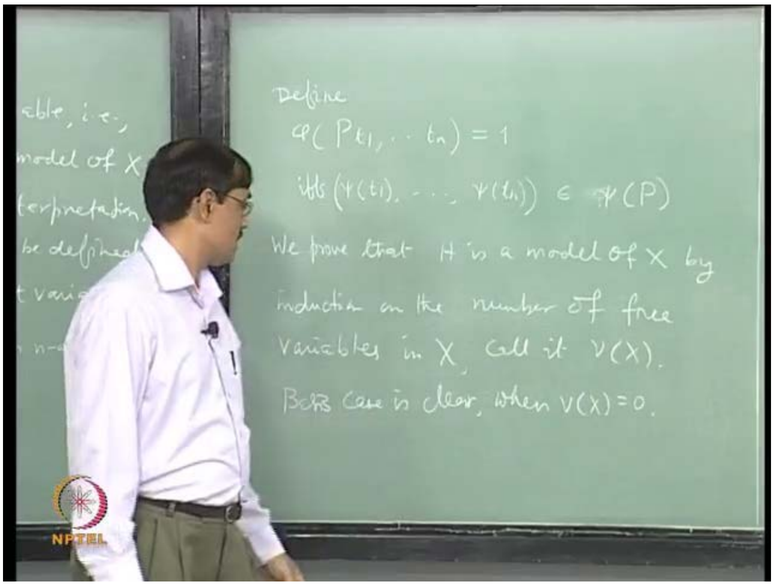

In X, suppose there is one small x, a free variable. That means, it is universally quantified. Then you may have to do something. How it goes over. Because, in that case, it might quite well happen that D is denumerable but A is not denumerable; it is finite. That is also possible. Then, in that case, many of these d’s are collapsing into that A, because of this. So, structure can be complicated. A proof is required only for this case; whenever there are free variables in X. If there are no free variables, there is nothing; it is just propositional. Fine? So, what we do is, but that is also not difficult; you can see for one step. For example, there is one: for each x X; x is a free variable there; so it is interpreted as: for each x X, due to our convention. Now, you go for: I satisfies this; that is given. Now, we want to show that X satisfies it. Then, you go for every element in D, this is what happens. That for every element in D it happens, then every element t, it also happens. Because all this t’s are mapped into that D, somehow, because phi’s are coming here, they are implicitly being mapped to D.

So, the proof should be going that way.But a formal proof will not finish with that. We have to start by induction, on the number of free variables in X. Let us try that directly. We prove that H is a model of X, by induction on the number of free variables in X; call it say, nu X. Now, if nu X is equal to 0; it is propositional; there is nothing to prove. Basis case is clear. It is clear when nu X is 0, so X is simply a proposition, only connectives are there. And these are 1, so connectives will take care; in both the things connectives are same.

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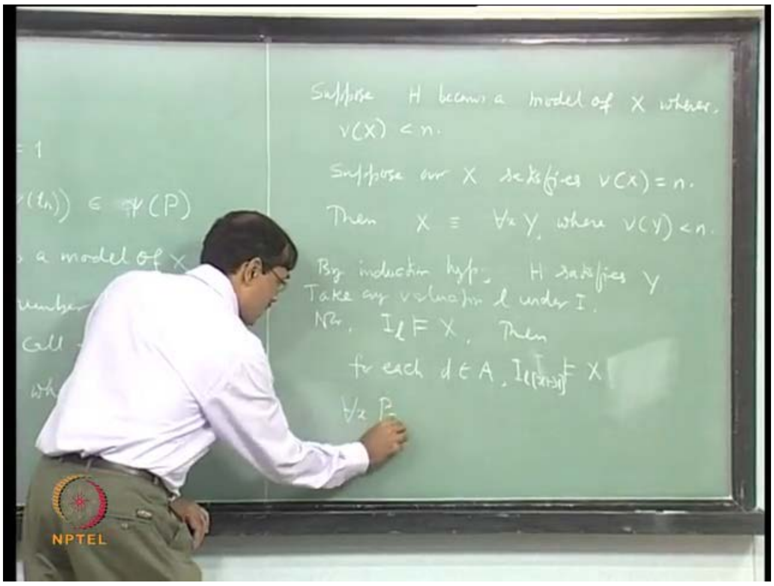

In the induction step, suppose H becomes a model of X whenever nu X is less than n. Suppose our X satisfies, nu X is equal to n. Then X is same as for each x Y, where nu Y is less than n. There is one more free variable in X, so call it small x, then X is really for each x 1 for each x 2 for each x n, then some quantifier free formula. So, first for each x, after that whatever remains, we will call it Y; so Y has number of free variables less than n; this way you are writing this. Because each variable in X is universally quantified, so we can write X is same as for each x Y, for some Y with nu X less than, nu Y less than n.

Now, by induction hypothesis you get H satisfies Y. H is a model of Y; that much we know. Now, let us see what to do. For H now, I is a model of X; that means, for each d in A, we have I satisfies X x by d; it is not really formally correct. What do we have to do is find one valuation there l, then we have to take l x by d that satisfies this. That is how it looks, fine. But informally, this is what happens. When you substitute that x by d, whatever you get, that is satisfied by I; that formula is satisfied by I; that is happening. If you write it formally, it will, say, you have to introduce one l, then put l x by d. Right? Is that okey? If you want that, then you may say: take any valuation l under I; now you say: I l satisfies X, because X is satisfiable, so I l satisfies X; then we will be writing: for each d in D, I l x fixed to d satisfies X. This is how.

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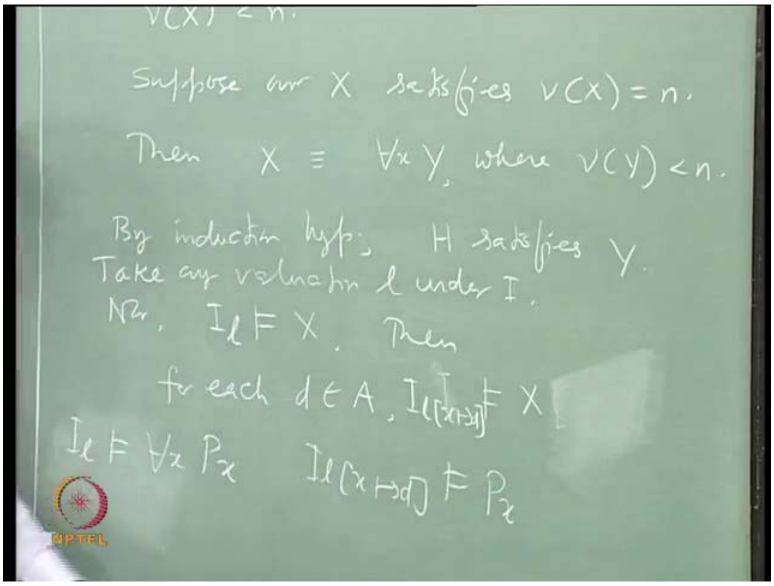

You have for each x P x. Now you say I l satisfies this. This happens when for each d in A, I l x to d satisfies what? P x. Not for each x P x, right? It should satisfy P x. So, for that what we do, instead of free variables, you say universally quantified variables, that will clarify it. Then, continue induction on that, then there is no problem. That is true because of the terminology, it becomes the opposite. So, now here what you do, suppose I l satisfies for each x P x; which means I l x to d satisfies P x for each d in A. Here, that means it will not be X; it will be Y; it should be Y, how do you proceed? For each d in A, you have this. Now, here I l x to d satisfies Y. This d is an element in A. So, we can think of some term which is mapped to this d; which term? It is that x, that x has been substituted in our Herbrand expansion by something, some term. So, take that term, whichever is mapped. You have a d there; you are really substituting that term in Y instead of x by d. That is how we can write informally, as x by d itself. This is the reason. Is it clear? We may say: for each, if we write that way, it will become clearer; d in A, I satisfies X x by d. It is really x by t, and phi of t equal to d. Once this is realized, there is nothing to prove. After this, induction hypothesis applies. The H satisfies this, then you say for each d in D, H satisfies this; over. Quantifier is taken care. Okey? So, after this there is nothing, in this sense, for each d in D, you will be writing that H satisfies x by d, x by some term t, really. You may say, some term t. Now, you can say t belongs to D. Now, it is formal. Once this happens; that means H satisfies X. This was the crucial thing in that theorem. All others will be easier, after this. Clear?

See, what you are doing here is, you have A and you have D now. The whole of D is mapped to A. So, for each d in A, this is happening. Then, for each t in D also, the same thing is happening. All these terms are given some values in D, by I, interpretation I, right? Now, for each d something is happening. That means for each functional value of this t's, something is happening. For each functional values of these t's, this thing is happening. Therefore, for t also; for all t now. The crucial idea is, you have to see that everything of the Herbrand universe is mapped to that domain of the interpretation; that is all. There is another way of working around it. You just define one function mu from the Herbrand universe to this domain D. Then, take mu everywhere, work with that mu; psi is doing that job. Once you realize this, there is no need of getting another mu. Right, is it clear? Psi is precisely doing that. It is taking all the terms in the Herbrand universe to the domain D. We are simply, simply of thinking of psi inverse d is here; all the psi inverse d, may not be singleton; they can be many, but then, for each element also that satisfies.

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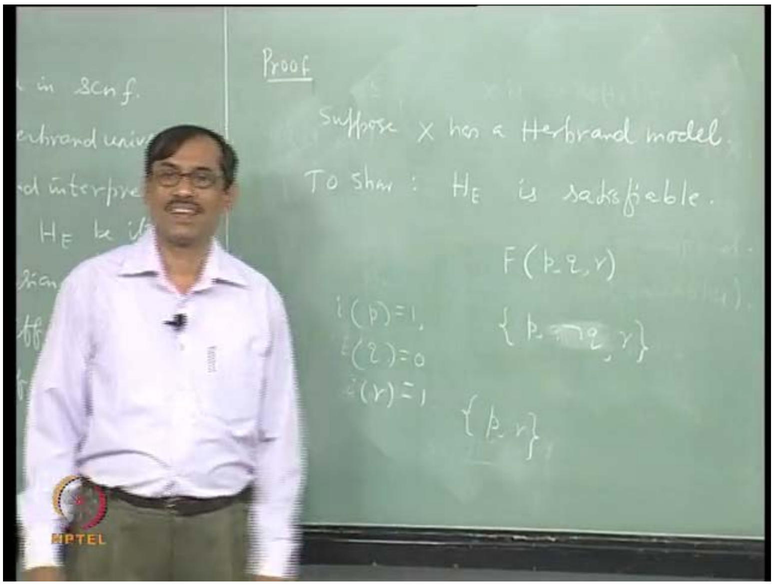

So, let us see the statement. You have already proved one thing. X is satisfiable, then x has a Herbrand model. Now if X has a Herbrand model, then H E is satisfiable, that we will see. How do you show?

Suppose, X has a Herbrand model. Now we have to show that. To show its Herbrand expansion is satisfiable. As the meaning of X has the Herbrand model, you have the Herbrand universe of the terms, of the closed terms, are there. Now, those closed terms, at the predicates, when you substitute at the predicates all those closed terms, they are declared to be 1 or 0; something is declared, that is what happening in the Herbrand interpretation itself. After knowing that, you evaluate X. You see that X is satisfied; that is what happening; that is the meaning of Herbrand model.

Now, in H E what happens? Instead of evaluating there, you are writing it out. In Herbrand expansion, they are precisely to be 1; what else it can be? If you can see it, then there is nothing to prove. But there is still another way of expressing it. In the propositional logic itself we could have discussed that.

Suppose i is one interpretation in PL. I have i of p equal to 1, i of q equal to 0, i of r equal to 1. Now, i of some formula involving p, q and r. I want to have a model of this. Assume that this is a model of this. I can write this model in a different way. Instead of writing i of p equal to 1, i of q equal to 0, i of r equal to 0, I will write it as a set p, not q, r. I can just write this set. I will say that this is my model. They are equivalent. This is what we have done while constructing the normal forms also. So, what we do, I will even forget this q, I will say only p, r nothing; else will be there. It means if you look at the formula, look at the propositional variables there, if that propositional variable is absent, it is negated. I can put that convention always. So, I may say that this model can be written as p r nothing else.

Now in the Herbrand model, this is what happening. You are just declaring some of them to be 1; forget the others, they are 0's. This is the way you are writing the Herbrand model; and Herbrand expansion is something like this formula. Now you say, this is a Herbrand model of this; that means you go back to this and that satisfies this. When you say Herbrand model is satisfiable, you just find out from this, one of, kind of this. Is that clear why it is happening? So, is there anything to prove there? It is just another way of writing the Herbrand model; that is what it says. But you have to give the argument.

The argument will go like this. Suppose, X has a Herbrand model. Now you want to show H E is satisfiable. Once it has Herbrand model, then the formula is satisfied; it is universally quantified. It is satisfied, for, now all the free variables are substituted by the terms from the Herbrand universe; so there, substitution is equivalent to telling that all the formulas in H E are satisfied; that is all.

Next, if H E is satisfiable, then X is satisfiable; is it clear? Well, this statement is if and only if. From H E also you can go back. Now then, we have to prove only, if X has a Herbrand model then X is satisfiable. But there is nothing to prove. If it has a Herbrand model, it has a model. So, it is satisfiable. So, that was only the crucial step; first one implies the second. Okey.

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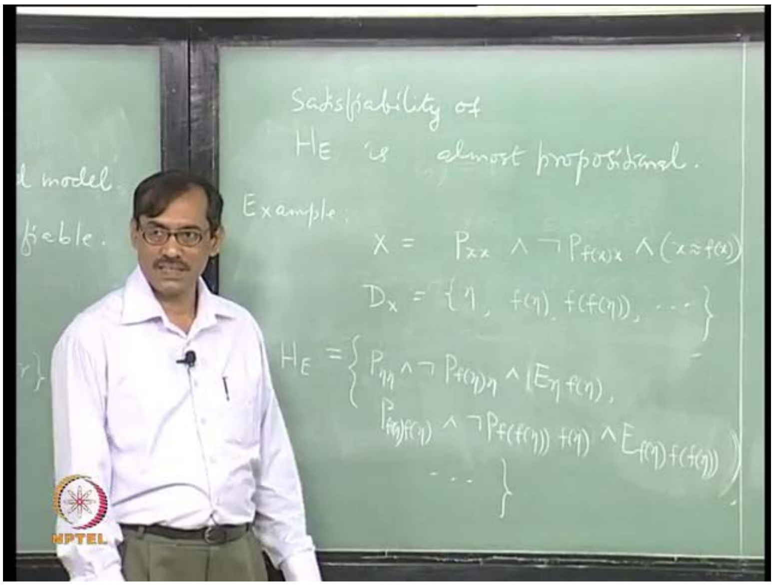

There is one hitch in this. H E is almost propositional, satisfiablility of H E is almost propositional. Let us write, why I am telling almost. Satisfiability of H E is almost propositional. We will see the reason, why this almost. Let us take one example. Say, X is equal to P x x and not P f of x x and x is equal to f of x. Suppose this is my X. Now, let us go for the Herbrand expansion. There is no constraint occurring in X. So, D X will be equal to, starting from eta; but there is a function symbol f; which is a unary function symbol; so, it will generate all the terms from eta, taking its composition with f; so I would get f of eta, f of f of eta, and so on. All these are there in the Herbrand domain.

Then Herbrand expansion will look like P eta eta and not P f eta eta and eta is equal to f of eta, then P eta f eta, and not P f of, x has to be f eta f eta, P of f of f eta and f of eta and f of eta is equal to f of f of eta. And it continues. It is a denumerable set. Writing all this x as eta, and then f of eta, and then f of f of eta, and so on. Fine? Now, is this satisfiable or not, H E? H E is satisfiable or not?

Why it is not satisfiable? Because of some property of E equal to f eta. When you are writing in H E, you will not also write eta equal to f eta, because equal to is not permissible in the expansion; it has to be as E. So, we have to write really as E eta f of eta. Similarly here also we have to write as E f of eta f of f of eta. Now is it visible? It is not satisfiable. We have to go for the properties of E. You know that if E s t is there, then anywhere s is there, you can substitute by t. If it is a predicate, truth will be invariant. If it is a function, it is a term. So, then those terms will be taken as equal, that is what E means. Then here, what we do is, eta f of eta can be substituted for each other. I can write here: f of eta, f of eta. I can write this eta as f of eta. I get P f of eta f of eta and not P f of eta f of eta; that leads to unsatisfiability. So, H E is unsatisfiable. It should be, because from the formula itself we can see that it is unsatisfiable; doing the same thing here, x substituted by f of x, I get P f of x f of x and not P f of x f of x; which contradict each other. This is the meaning of this almost.

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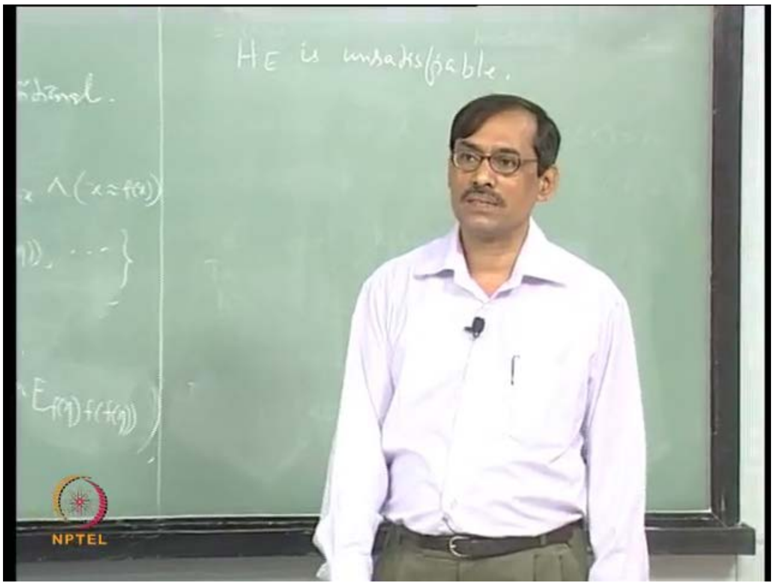

Except this equality predicate, everywhere it is propositional. but then how to make it propositional? It can be done. There is a way. Because E is an equivalence relation; so on the domain itself you find its equivalence classes. Instead of domain as D, you consider the equivalence classes inside D; take one representative from each equivalence class; that is your D. That then induces one equivalence relation on H E. So, H E will become different. That H E is the Herbrand expansion if you have the equality predicate. In that H E, it becomes propositional.

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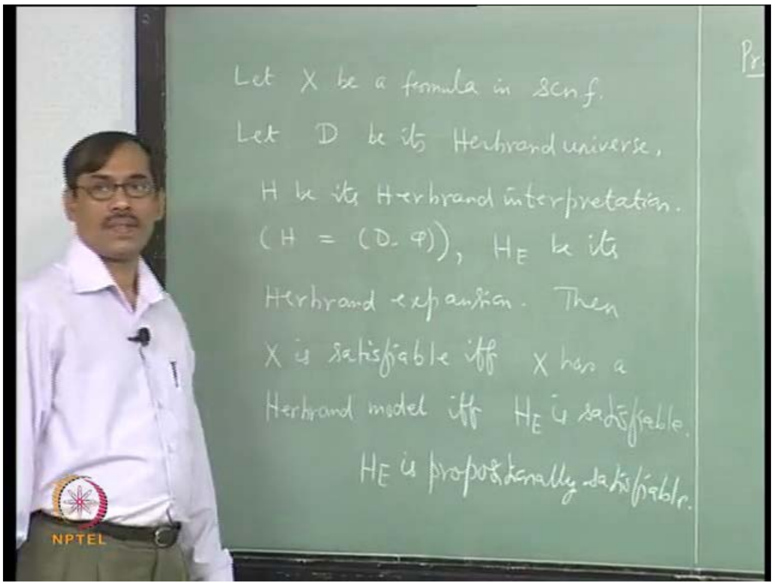

Sometimes in this theorem it is written as H E is propositionally satisfiable; that is the sense. Once you do this equivalence relation or equivalence classes business, then you see that H E becomes propositionally satisfiable. It amounts to doing something in the proof theory. In the proofs, suppose you have to prove something to be unsatisfiable. So what you do? First apply all the quantifier rules. That is what we have done to find out H E. Apply all the quantifier rules. Finishing it you end up somewhere, where you do not need any quantifier rules to be applied further; everything after that will be propositional. That is what it says. So, in any proof system there will be a way to handle it, if quantifier rules are separately given. If not given, then you have to comply. So it says you can have a proof system where quantifier rules are different, and you can go on applying quantifier rules; then, forget the quantifier rules; you apply only propositional rules then onwards to show unsatisfiabilty or satisfiability. Right? Or even proof for validity. Because validity and unsatisfiability will be dual to each other, right? So, It will go there.

But there is something else. It gives like, you have H E is propositionally satisfiable; now, use the compactness theorem for propositional logic; it says H E is unsatisfiable if and only if it has a finite subset which is unsatisfiable. Now, H E is, suppose it is unsatisfiable. You get a finite subset which is unsatisfiable. That finite subset would have come from some finite number of formulas in X itself. If you have one formula, it is fine. If you have many formulas, for that you get the H E, by their conjunctions; then you also reach at some finite number of formulas only, which will be unsatisfiable. And that is compactness theorem for first order logic.

So, it is very powerful in that sense. It can give much more. Because, you are converting satisfiability of first order logic to propositional satisfiability. Many of the theorems can be lifted, now to first order logic. But we will do compactness latter, in a better way, also from the proof theory.

Then, there is something else. It says something about validity also. If unsatisfiability, then validity, similar way. That validity can be propositionally done, first applying the quantifier rules, then propositional validity; that is really Herbrand's theorem. We will have some other applications.

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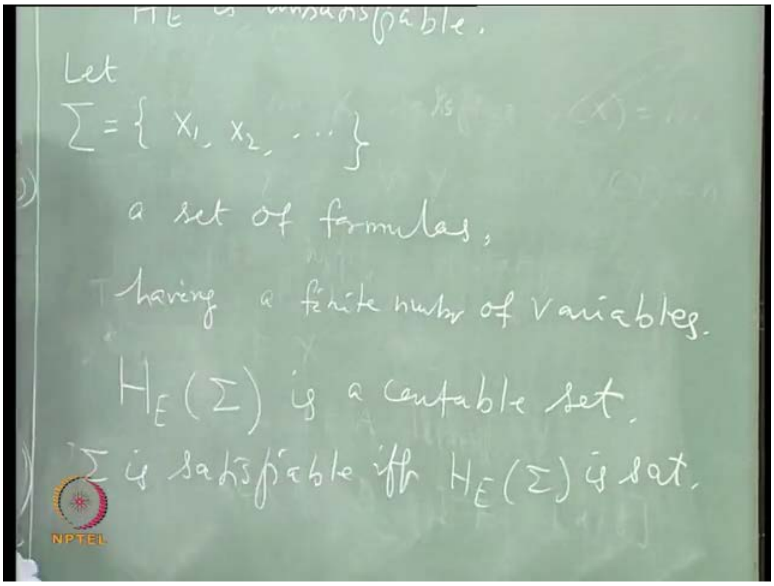

Suppose you have a set of formulas. Let us say, Sigma be equal to X 1, X 2, and so on, a set of formulas. Of course, we need one restriction here; formulas having a finite number of variables, having a finite number of variables. This means, suppose X 1 uses, say, some variables, X 2 uses some other variables; even with quantified and non quantified, whatever it is, X n uses some variables. If it is infinite; it can be potentially infinite, number of variables. Now, you are putting a restriction on it; that does not happen. Finally if you go up to some finite stage, you get all the variables; during that only; those variables repeated. That is what happens here. That is, only finite number of variables used in this. Then you can think of a Herbrand expansion of Sigma, because all that you need is a finite number of variables in the Skolemization process. If there are infinite number of variables you can not Skolemize, finish Skolemization; so the algorithm will not terminate; though Skolemization can be done theoretically, but no algorithm will finish this Skolemization process.

Now, if there are number of finite variables, this number of variables is finite, then you can have a Skolemization process on it; then you can think of H E of sigma, Herbrand expansion of sigma. Similar to each formula, you go on doing it. The number of indicial functions introduced to be finite, because number of variables is finite, right. So the process terminates there, for this Skolemization. Then this one is a countable set, potentially infinite. It can be denumerable. If this is denumerable, this is also denumerable.

Now, what happens, we say that Sigma is satisfiable if and only if H E is satisfiable, because for each formula you have a countable set. It is a countable union of countable sets. As a particular case, suppose you take a finite set Sigma. Then automatically you will get a countable set, maximum. It is finite, then automatically you will get.

Let us say group theory. Group theory can be axiomatized with four axioms; forget the Abelian groups; only four. So, closedness that operator; some group operator is there; then you have associativity, then existence of identity element, and existence of inverse element; these are the four axioms. So four sentences can be written. Now, take those four sentences. Any model of those four sentences is a group; that is what a group means; anything that satisfies those four axioms, right, is a group. So, every model of those four axioms is a group. Fine? Now what you do? Apply this Herbrand's theorem on and of all those four formulas. You get one Herbrand expansion. Now you have a model which is countable, of those four axioms. So, there is a countable model for those four axioms. It says that there is a countable group. You do not have to produce; you do not have to really produce a countable group. It simply says there is a countable group.

And it is an infinite, there is a denumerable group. Because, there is at least one function symbol there, used in a group, definition of a group. So, this becomes denumerable, right, like natural numbers. So there is a denumerable group; that is what it says.

Now let us think about something else, say, real number system. It has, similarly, axioms, some eleven axioms, field axioms; then completeness axiom. It is a complete ordered field. Now, all those axioms, you add them together; you get one big formula X. It has many function symbols. Now, go for its Herbrand expansion. You get a countable model for it. There is a countable model for the real numbers also. But, Cantor's theorem says R is uncountable. So, what is R there? Anything that satisfies those axioms. Then, you have a model which is countable. How is it happening? This is called Skolem’s paradox. There is a problem here. There are ways of explaining it, but not very satisfactory ones. One, best one till now. The explanation is that yes, it is indeed countable, but the function which makes it countable, there is a function which makes it countable in the sense that there is a map from that place to natural numbers. So, that map cannot be constructed within the axioms, within the system; that map has to come from outside, somewhere.

So, hat means the cardinality concept itself is system dependent. If you have different system, cardinality concept will be different. So, there are something like you have empty set, so empty set can be different. It will be because cardinality itself is different. So, empty set is having zero elements, can be different in different contexts. You say number of people in this room having eleven fingers is an empty set. But outside, it is not empty. There are people having eleven fingers in their hands; so there is some relativization of the concept of cardinality. It gives rise to that. It does not give a paradox; it does not give a contradiction; but relativization of the concept of cardinality itself. So, there are many non-trivial results out of this Herbrand’s theorem, like this. Let us stop here.