Mathematical Logic
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Lecture - 33
Skolem Form

So, now we know how to convert a formula to its prenex form, though prenex form is not unique. But all the quantifiers will be in the beginning and then there will be another formula without any quantifiers. And in order to do that we first start to rectify the formula that is not necessary really, but it helps; if you have done it, then up to equivalence, you can go for the next step easily. Our first concern was the prenex form.

Our main objective was how to get rid of the quantifiers, get only the matrix. But as it is, we cannot just omit the quantifiers; because if you omit you cannot reconstruct it back up to equivalence or up to satisfaction, or in some sense. So, you have to fix some sense and then try to see how to do away with the quantifiers so that that sense is preserved. Up to this, you are doing everything for equivalence. Now onwards we have to sacrifice on that. Up to equivalence if you proceed, you cannot drop the quantifiers as such.

There are two approaches here; you may take preserving satisfiabilty or preserving validity. We will see both the approaches; then later stick to one, because we see that they dual to each other.

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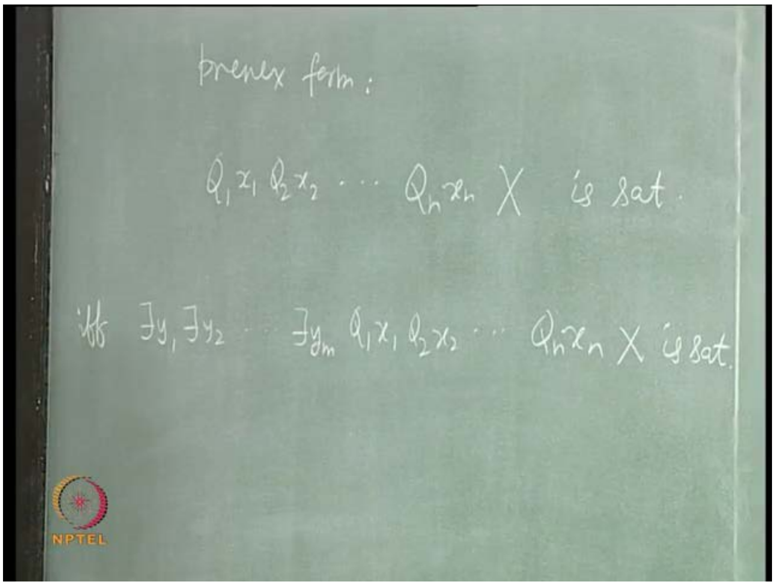

Let us consider say satisfiabilty, Suppose we have a formula in prenex form. It would look something like Q 1 some x 1, Q 2 x 2, some Q n x n, and then some matrix of the formula, which is quantifier free. Now, there can be some free variables in X which have never been quantified here. If the original formula is having those free variables, by rectification or by renaming those free variables, still remain as free variables. So, you preserve satisfiabilty; then our earlier theorem really helps here.

We can really convert to a sentence because we know that a formula is satisfiable if and only if its existential closure is satisfiable. Suppose we have free variables y 1 to y m in X. Then we can take there is y 1, there is y 2 and there is y m then Q 1 x 1, Q 2 x 2, Q n x and X. We say that this formula is satisfiable if and only if its universal closure is also satisfiable. Now, we have reached on sentence, which is satisfiable or not. Our aim is to get read of the prefix form the sentence somehow, still preserving satisfiabilty. May be one simple form of the formula will help.

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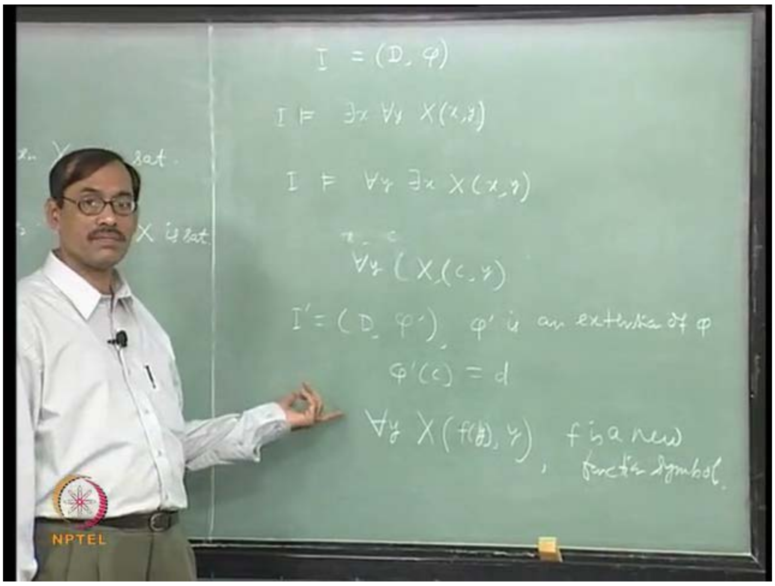

Let us take, say, it is in the form there is x for each y and X. We have just two variables x and y, x depends on only x and y. Let us write the other way; instead of the square bracket that might go for substitution. So, x comma y means there are only two variables here, it is a sentence now. Suppose you want to say that I l is a state, which is a state model of this. In that state how will you translate the sentence? You do not need the state now. It is enough for an interpretation, only because it is sentence.

Let us say I satisfy this. I is an interpretation whose domain is D and the mapping is phi. This phi should give us the meanings of predicates occurring in X x y. Somehow it will relate your relations; or if there are some function symbols, it will relate to functions with the domain D keeping arity. That is what it is. Then it will be a sentence; this sentence will be translated like there is some element in the domain D such that, whatever element you choose from the same D, the corresponding substitution when you think of relations instead of x. Let us say x prime where x will be written as d, y will be written as d prime; that will be satisfiable, that will be true; that is what it says. Now we see, that particular element d does not depend upon any other variables. There exists one d such that, then the other sentence becomes true on its own. Only thing is d has to replaced here; that is its meaning, when you come to the interpretation.

But if you consider another sentence, say I satisfies for each y there is x, X y. There is a difference, fine? It will say that if you have chosen that d prime here, an element, then the corresponding d might vary depending on this d prime in some particular way so that the formula X x y or X d d prime becomes true. The same d may not work; that is the intuition. So, if that is what it means, if there is a universal quantifier occurring prior to some existential quantifier then the existential quantifier uses the variable. That variable might depend on the variables used by the universal quantifiers; this is what the intuition says. So, if this depends or this does not depend, these two cases have to be clearly demarcated.

So, in this does not depend, for example, there is x for each y we might say that instead of x, I replace a new constant c. This new constant in the sense of new; when we say new, it means it is not occurring in this formula; that all we need is that. Suppose we consider for each for the first one, for each y X c y. Now, I has not interpreted this c till now; it is a new constant; it has not occurred at all. Now when you say that this is also satisfiable, how do determine this? I cannot determine because there is no interpretation of c till now. We may have to extend this I. That means, we will take I equal to same domain D, but another phi prime where, phi prime is an extension of phi, which takes care of c now. But now any extension will not satisfy. We want such extension which will satisfy the formula. That means earlier we have seen one element d under I was satisfying the corresponding sentence. Now then, in this case, we should put phi prime of that c to be equal to the same element d. That means under the interpretation I, if taking this x to d satisfies the corresponding formula we put that phi prime c equal to d, this is the extension we take.

All other values of phi will remain as it is in phi prime also. Then if this is satisfiable, this will also be satisfiable Under this domain, new extended interpretation, call it, say I prime. So, this is what happens. Then you go for the next step. In general, this is going to be happening, but one step we could prove it one existential quantifier, if there are many, proof will be by induction. This will be the crucial step in the induction process. Up to n you have, then n plus 1 existential quantifiers for the next one, you apply like this.

Now, what about converse? When you go for the converse, you do not have the extension business. That means the same interpretation should satisfy. Then how will it, c was new, so that was not new. Suppose I have one interpretation I prime, which has phi for some predicates, some way it has assigned and, phi prime of c equal to some element d. I do not know what d, this d was fixed because of this some d. Now, it is satisfied for each y X c y. Then in the same interpretation we just take the d, to mark here. There exists. One of them should satisfy, which one? That d. That means the same interpretation I prime will also satisfy the earlier formula: there exists x for each y X x y. So, it becomes if and only if statement, really. That means, when you delete there is x replacing all x by a new constant c, whatever formula you get that becomes satisfiable if and only if the original formula is satisfiable.

Now let us consider the second case, what happens here? For each y there is x X x y. This says, the x, which we will get from the domain corresponding to that, the element we get from the domain, might be depending on the values of y.

Student: how is it different from the extensional specification?

Question is how is it different from the extensional specification? In existential specification what you do? Suppose, there exists x P x and from this you want to have some entailment of another formula say Y. What you do is, you start with P c, then show that Y is entailed; y is not having that constant c; then you claim there is x P x entails this Y. All that we know, there is P c does not follow from there is x P x. Here also we are not telling that this formula follows from there is x for each y this. It does not follow. All that we say is, if this is satisfiable then this is also satisfiable. There is another interpretation. If it follows from, then the same interpretation should satisfy. Yes?

But that you are not concerned at the time of existential specification. All that we are concerned is from there is x from P c, if some formula follows without the occurrence of c, then the same formula follows from there is x P x. That is what basically existential specification is. But here we want to preserve satisfiabilty. New constant appears in both the cases, that is the commonality. Now let us take the second case. Suppose for each y there is x X x y is satisfied. That means in that interpretation we have interpreted y, say some element d, then there is correspondence of that element for x, which will satisfy, which will make the formula true. Now, if I vary this element y say y is going to d 1. Then the corresponding x may vary, may be constant; I do not know. But somehow the dependency is there. Now, you have to bring in this dependency, that is the main difference.

So, what we do here is, we consider another sentence for each y X f of x y f of y and y. Look at this form, and this form, x is replaced by f of y. You want to show that dependence. This f is a new function symbol; it is not occurring in x earlier. That is the main thing we have to care when you say new constant here. Similarly, we say new function symbol here. So, f is a new function symbol and that is the catch. If it is a new function symbol they can be re-interpreted. In fact you have to interpret it fresh, it was never interpreted earlier. So, suppose I satisfies for each y there is x X x y. Now we go for I prime same way where this phi prime of f, we have to define. We will define in what way? phi prime of f of y will be equal to that element d for the corresponding element y here. Whatever y, element you take, it will be that element corresponding to which this formula will become true. So that this new I prime will become a model of this.

Conversely if this is already interpreted to be true; so f of y has been interpreted to some d, take that d corresponding to this y. So, the same interpretation always satisfies for each y there is x X x y. That is the same as, the scenario is same as, for the earlier. So, this suggests that we have to find out when there is, there exists is occurring; before that what are the quantifiers occurring. So that all those variables used by those quantifiers can be the argument of a new function when you substitute that existentially used variable.

But there is another thing. When you say that it depends or it might depend, we have seen some cases where even if it occurs prior to it, it does not depend on that. It is because there may not be any atomic formula where both the things are occurring. We had one example of this form: for each x we took for each y. For each y there is x. Suppose it is in the form P x implies Q y. Here, when you translate, it will be looking like: for each element in the domain D some element are there corresponding to that element, say d prime such that, if d belongs to P prime. Then d prime belongs to Q prime. That is what it will be. But now you see P can be true or false, any way we like; Q can be true or false any way we like, for the d or d primes. So, it does not matter, the d prime really depends on d or not. It should be true; that is all. It should be true. The thing is, by making P d true or false, we are not going to change the truth or falsity of Q d prime.

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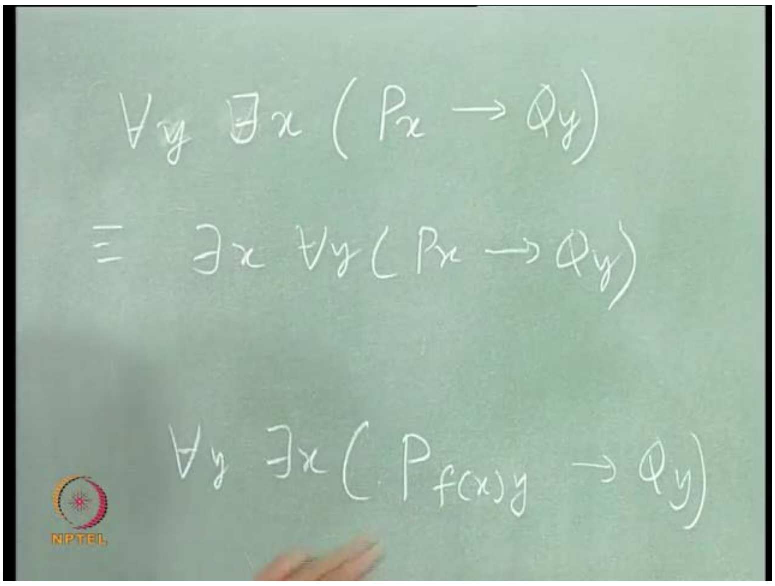

Student: Truth does not depend on d or d prime.

Yes. All that we need is Q d prime has to be true. It is for each y there is x. So, if there is at least one d prime for which Q becomes true, that we interpret here, that d prime we take it here. It is really does not depend on x. Is it clear? Even though for each y there is x is there, it will be equivalent to there is x for each y. This is what I asked you to prove, earlier. Intuition behind it is this: once P d becomes true it does not matter what d prime we use for d. All that matters is Q d prime should be true. So, even if there is something which makes it true, it is enough. Conversely, also same way. If this becomes false, it does not matter whatever P d or d prime is; automatically the implication becomes true. That is for the implication; it can be other connectives also, it does not matter. Still the intuition says that whatever be this elements d or d prime, substituted in Q or Q prime; it is a proposition. This proposition, truth of this proposition will not matter on dependence of d prime on d.

That is what we want. Dependence of d prime on d. That does not, because there is no relation which involves both d and d prime. So, P d can be chosen separately; d prime can be chosen separately, to make the whole proposition true. No relation which will connect both d and d prime together, there is no restriction on d and d prime in that sense. But had it been said, P x y then you have p d d prime. So, d prime has to be taken in such a way that given a d, P d d prime should be true. Is this clare?

So, the same thing we can extend to this place. We have to really check, even if there is one there is x and some quantifier, for each y is before it; We must check whether there is a predicate involving both x and y. If not, it does not depend; for the function, for this one. Function, anyway, is there. Whether it is a new constant or a function symbol, they are all functions. Thing is, whether you want to write f of y or not, f of y or just a constant?

Student: Whether f of x y or a constant?

That does not matter; that does not matter. So, once f of x y exists, it will be occurring in some predicate. See, suppose I have for each y there is x P f of x y implies Q y. So, there are predicates here, for f of x y is there, or even you write f x comma y Y. So, once this happen, we will say that both x and y occur in the predicate; not independently as x y, they are occurring with the same predicate, both the symbols are occurring. That why there is a possible dependence, that is what it says. If it does not occur, we know it does not depend, fine? So, you have to take care of this, and then try to eliminate the quantifiers for satisfiabilty. This says we can really eliminate, which one? For each y or there is x? X, existential quantifiers, because you are ending at for each y. So, existential quantifiers can be removed, can be eliminated. We are following this procedure.

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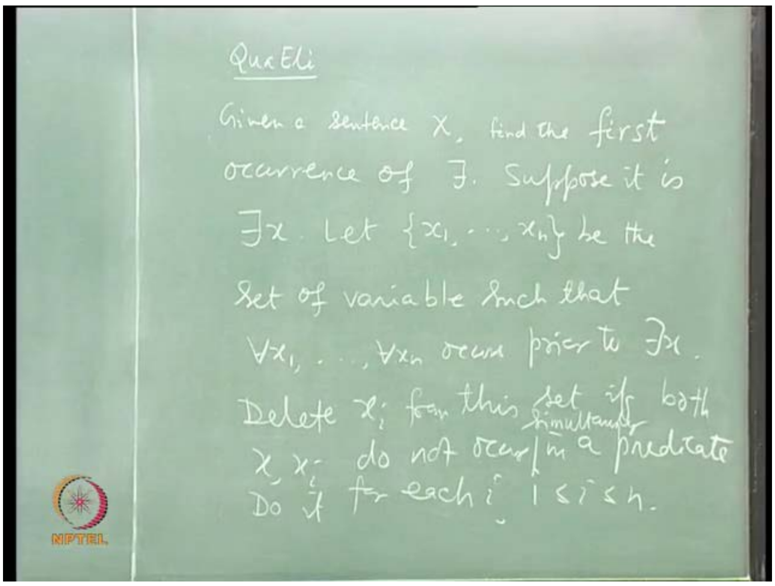

This procedure is called as quantifier elimination. Let us write Qua-Eli, we are just eliminating the quantifiers. The procedure is simple. What you have to do is, take any occurrence of there is, say, the first occurrence. Given a formula, or we can write a sentence, say, the other, because we have taken existential closure of the formula. Given a sentence X, find the first occurrence of their exists, suppose it is there exists x, it uses some variable call that variable as x. Let x 1 to x n be the set of variables such that for each x 1 for each x n occur prior to there exists x. Since it is first one, all of the others will be for all quantifiers, you just take blindly all of them that is your x 1 to x n. Now, what you do, delete x i from the set if both x, x i do not occur in a predicate; so it is simultaneous occurrence; do not occur simultaneously in a predicate. Do it for each i, so the updated set is the set of variables on which x depends. What you have to do is write the updated set of variables.

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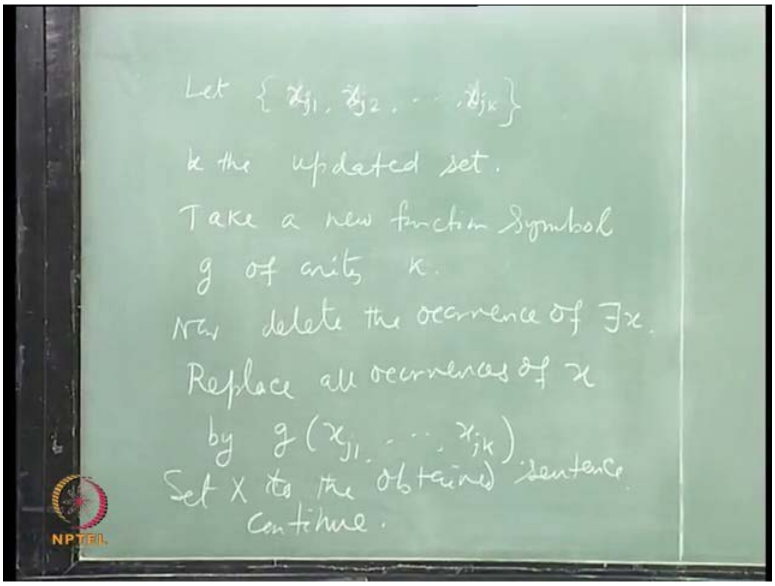

Let y, say, j 1, y j 2, y j k. In fact we can write x also, if you give subscripts x be the updated set. Take a new function symbol say, g of arity k. If there are k variables remaining, then take the k-ary function symbol g. Now, delete the occurrences of there is x; replace all occurrences of x by g of x j 1 to x j k. Then this is only one step.

The next step is: you take your new X as this new formula, what is obtained from this; and continue. Set X to this formula, to the obtained formula; it is again a sentence continue. That means, up to this you have eliminated the first quantifier, which is existentially quantified. Then again you apply; again look for the first in the remaining sentence, then continue.

So, all the existential quantifiers are eliminated now. This quantifier elimination is really quantifier elimination of their exists. Then at the end of it, the formula you get, there is no existential quantifier; only for alls are there. Now, drop the alls. Hence forward, we will make it a convention that in whatever you get, the free variables are universally quantified. You remember this and just drop the quantifiers. That is how quantifier elimination is over. Now, you get only some quantifier free formula, where all the quantifiers are; all the free variables are universally quantified.

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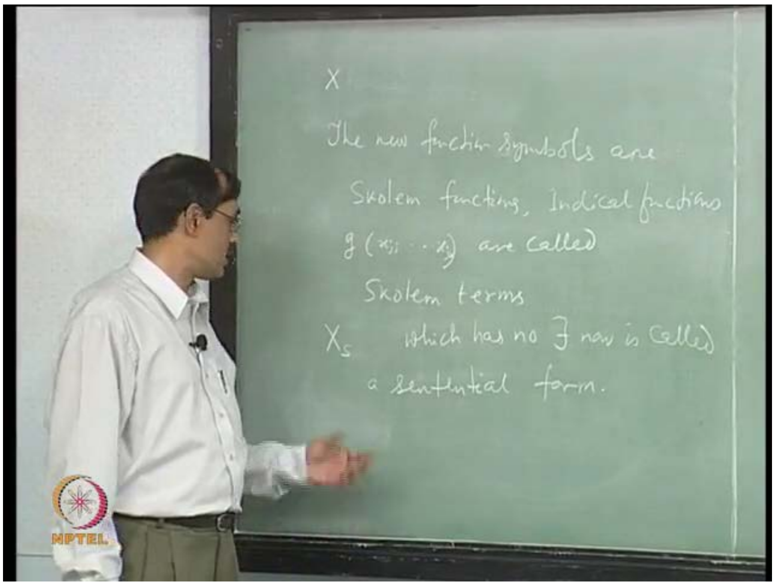

Suppose, we start with a formula X. Then from there you get one sentence, which is the existential closure of X. Then you are replacing their exists. So, if there is any free variable in X, all those will be replaced by new constants finally, because all those free variables will be existentially quantified in the beginning. Now, take any existential quantifier, before it is there is no for all, so, that we just replace by a constant. You can really modify quickly instead of taking the existential closure in the beginning. Just replace all the free variables by new constants that should be the first step; then continue, that also can be done. By this quantifier elimination, after the end of this step, you get no existential quantifier occurring in the formula. That formula is called the Skolem form formula and this process is called Skolemization. This functions we are introducing there called Skolem functions g's or Indical functions and the term you are using is called a Skolem term. There are some terminology here; we will just write those terminologies.

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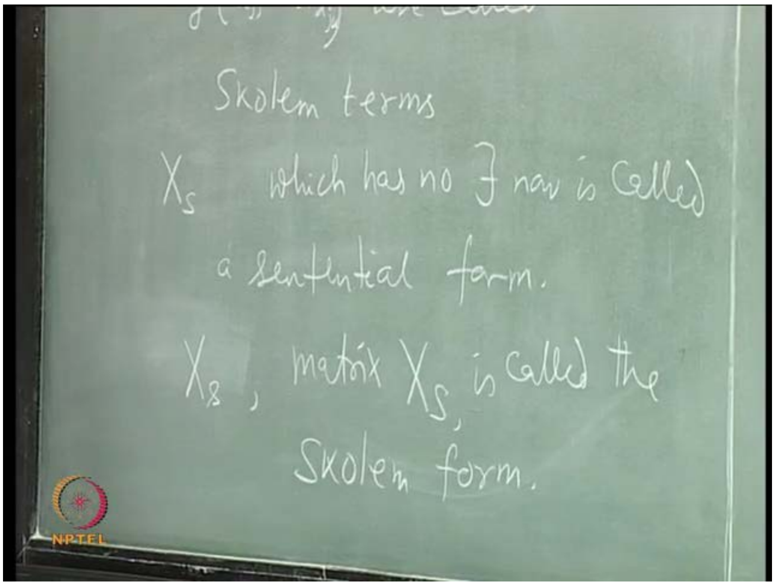

So, these g’s, the new functions or function symbols are called Skolem functions or even Indical functions; and then the g of, after you get, x 1 to x n and so on. x j 1 to x j k are called Skolem terms or sometimes Indical terms. So the sentence X s which has no there is, is called a sentential form; it is called sentential form of the formula X. We write as X capital, s and then prefix of this X s once omitted; that is, the prefix of X s are all universal quantifiers; because by this all existential quantifiers have gone. All that remains, quantifiers in the beginning are the for all's in X s. The prefixes are only universal quantifiers. Once you omit the prefix, that is the matrix of X s. There is also terminology for this X small s, which is matrix of X s, is called the Skolem form.

Then CNF equivalent of this Skolem form is called a Skolem CNF or SCNF. See, once you get X small s, in this Skolem form you have only the formula, there is no quantifier. There will be implication, since biconditionals, we have already eliminated at the time of rectification, at the time of converting it to prenex form. There can be implications, there can be OR, AND and negations. Use the propositional laws to convert it to CNF; the resulting formula is called SCNF. If it is converted to a DNF, we call it an SDNF. Let us give a notation this SCNF we will write as, say, X bar and this SDNF we will write as X tilde. We will take an example. Slowly, but first see how there we go. We have X then you take the existential closure, then apply quantifier elimination to get X s, then drop all for alls to get X small s. Then convert it to CNF to get X bar, which is SCNF. Convert same X s to DNF; call it X tilde, which is the SDNF. See, all these we have got by preserving satisfiabilty.

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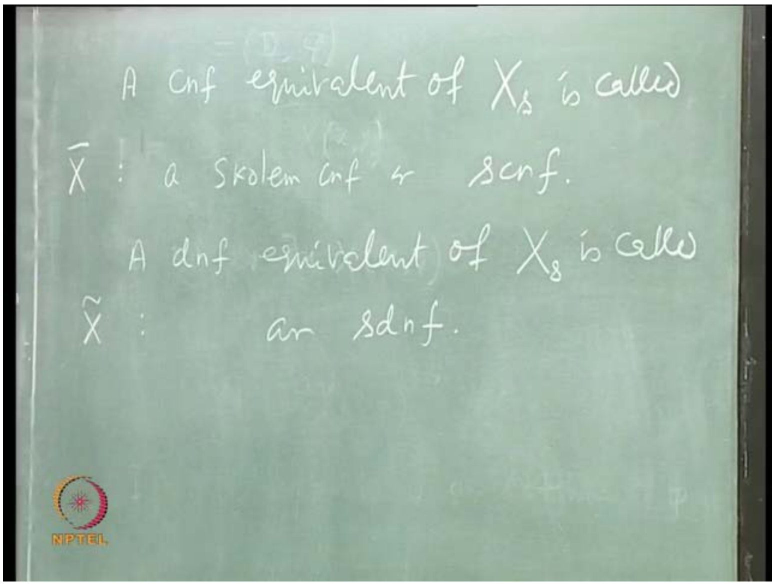

What this says is, X is satisfiable if and only if its existential closure is satisfiable if and only if X capital s is satisfiable, quantifier elimination, if and only if forgetting the for all, that is also satisfiable; if and only if X bar is satisfiable if and only if X tilde is satisfiable; throughout only satisfiabilty. The same way we can preserve also validity; but how to preserve validity? It is a dual concept. It is not difficult. You can check from this easily, you can also read the procedure.

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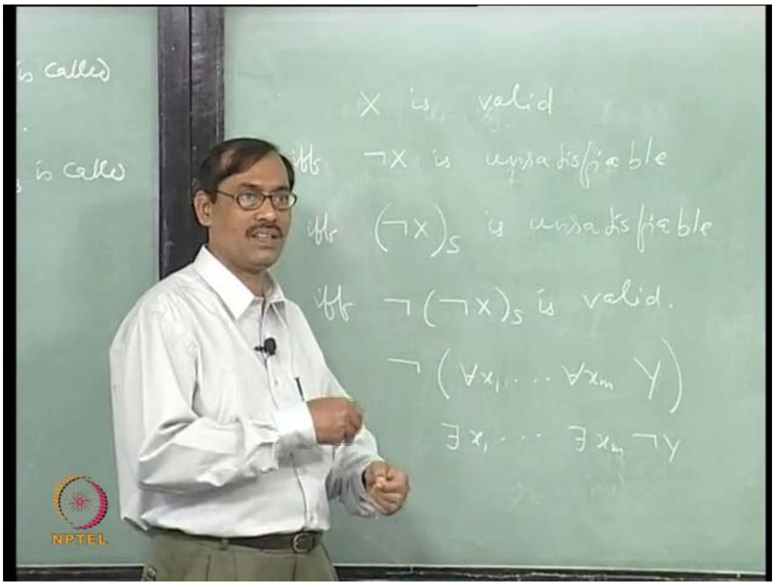

Suppose, we want to say that X is valid, how to preserve validity that is our question. Say, X is valid. This we may say, this happens if and only if not X is unsatisfiable. Now not X is unsatisfiable if and only if not X capital s, subscript, is unsatisfiable. This capital X subscript s says that you first take not X existentially, generalize over the free variables and then apply skolemaization; what you get is X capital s. But all those things you are doing with not X now; not with X, fine. So, if this happens, not X s is unsatisfiable. Now look at not X S; how will it look? In not X S, for that matter, take any formula Y and subscript capital S, that will be for all's and then some formula. So, in not X S also you will get similar way. It will be something like, for all, some variables are used, and then some formula. It will look something like this. Now, this is unsatisfiable if and only if not of this is valid, so we go back.

Now, how does this look? We have not, then for all, some formula Y; it looks like this. Once it looks like this, you take the negation inside; that will look as there is x, x m not Y. So, this Y has been obtained from not X by Skolemization. Here, what happens? not Y should be obtained from where? From Y, by Skolemization. You can use double negation there. So, from Y, if you apply Skolemization you should reach at this place, but there is some catch. In the Skolemization, after you finish, you end with for all; here you are ending with there is.

So, that means you have to switch there is and for all in this Skolemization process. You can simply modify your Qua-Eli. What you do is: look at the first occurrence of for all; then let this one be the variables prior to for all x used by their exist x, all existential quantifiers, then continue this Skolemization process. So, just treat there exists as for all, for all as there exists in this Skolemization process. Then you reach at the same place, which will preserve validity; that is what it says. Is that clear? So, let us call that procedure as Qua-Eli-for-all. You are not doing for that exists, that is, for all quantifiers are eliminated. Once you do that, but you finally get there is: let us call it X v; so which has no for all in it. It looks like there exists, there exists and so on and then some quantifier free formula; that is its validity form. We will call it the validity form of X. Next we will introduce X f. In X f you just take the matrix of X v just like the earlier. This is called the functional form of X. Just some different name, sometimes it is called also Skolem for all form, a Skolem validity form; and that is called Skolem satisfiabilty form.

Here, when you forget the quantifiers or you get some free variables, they are all existentially quantified in X f. In X, subscript small s, all free variables are universally quantified, in X f all free variables are existentially quantified. Just the dual things are happening.

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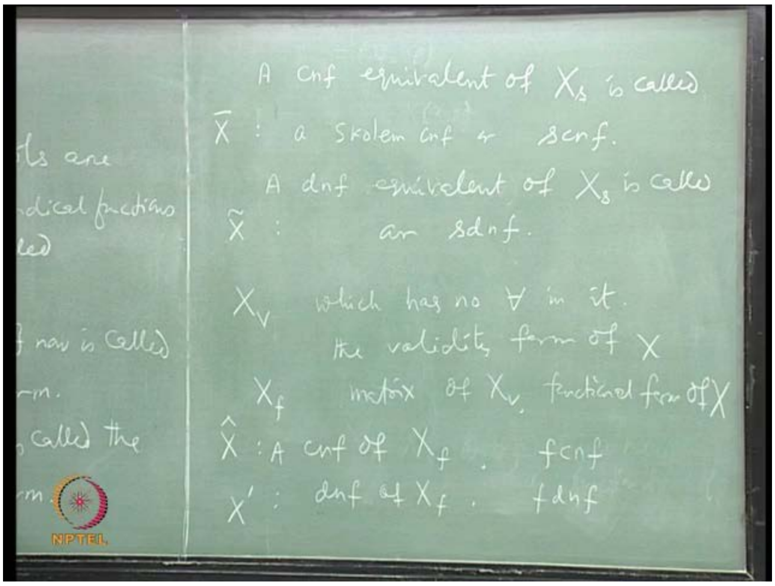

Then we write a CNF of X f; we will give a notation again. Let us call it say X hat, some notation we have to give. That is called the functional CNF, FCNF. And same way, we will write X prime as DNF equivalent of X f, which is also written as FDNF. Now, you can summarize what we have done for this, as a result.

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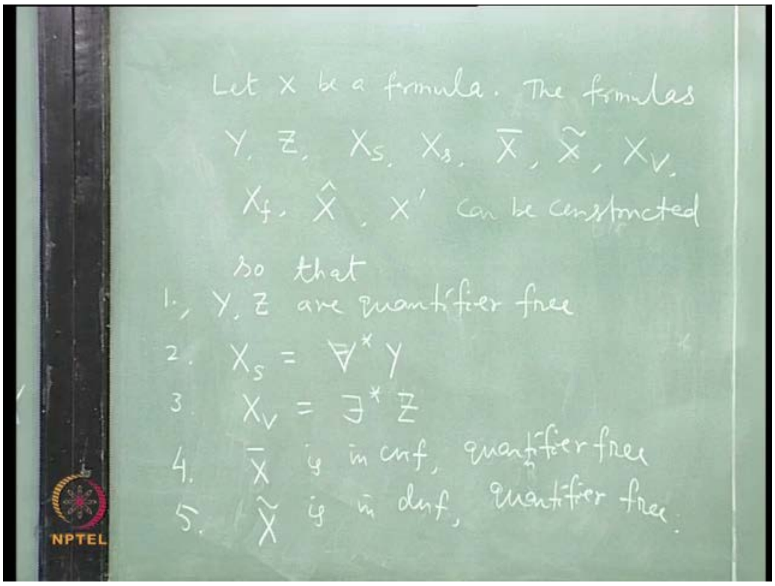

This conversion says the following. Let X be a formula. Then formulas Y, Z; and then X s, X s and X bar, X tilde, X v, X f, X hat, X prime can be constructed. So that what happen, Y, Z are quantifier free. X s is equal to for all star of Y, X v equal to there exists star of Z. Next, X tilde is in CNF and also quantifier free; X tilde is in DNF and also quantifier free. Then X f is, s, also we have to write X small s, which is really here Y. So then, we can forget this Y, Z; we write here X small s and here we can write X f.

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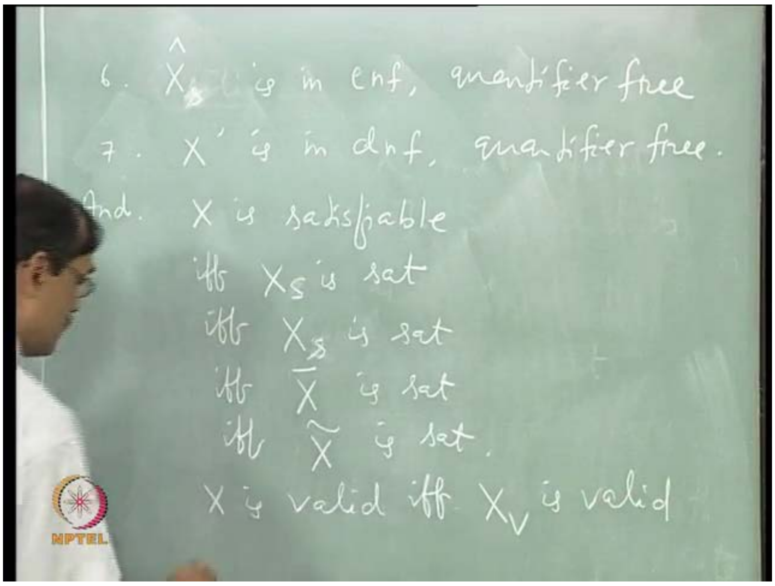

Then X hat is in CNF, quantifier free; X prime is in DNF and quantifier free. And what happens is, X is satisfiable if and only if X s is satisfiable if and only if X capital S is satisfiable if and only if X bar is satisfiable if and only if X tilde is satisfiable.

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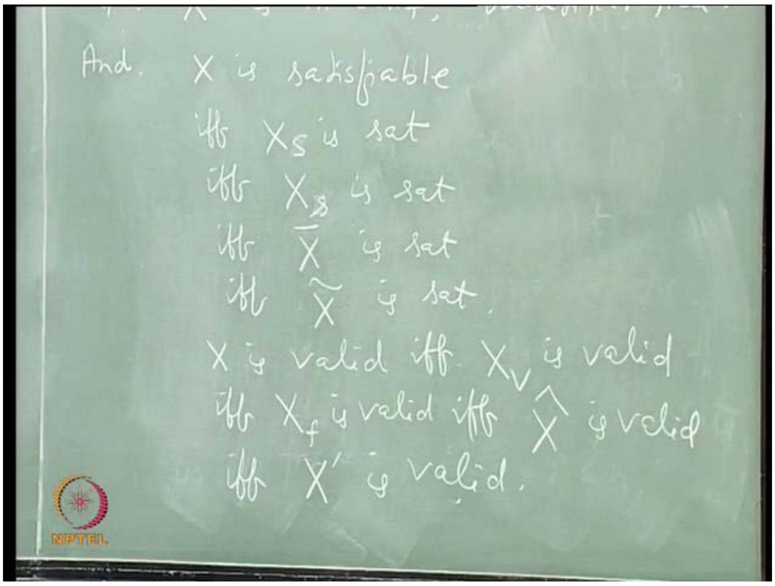

Similarly, X is valid if and only if X v is valid if and only if X f is valid if and only if X hat is valid if and only if X prime is valid. So, a big theorem we have proved. It takes lot of time in writing itself.

Let see an example; how it proceeds. You just took something arbitrary. Let us see. Now this formula is in prenex form; there, rectified and in prenex form. We will just see quantifier elimination; how it proceeds? Now, first Skolemization with their exists x. Let us preserve satisfiabilty first, then for validity also, we will see. First, we find there is x is here, there is no free variable, they have been already quantified; so it is a sentence.

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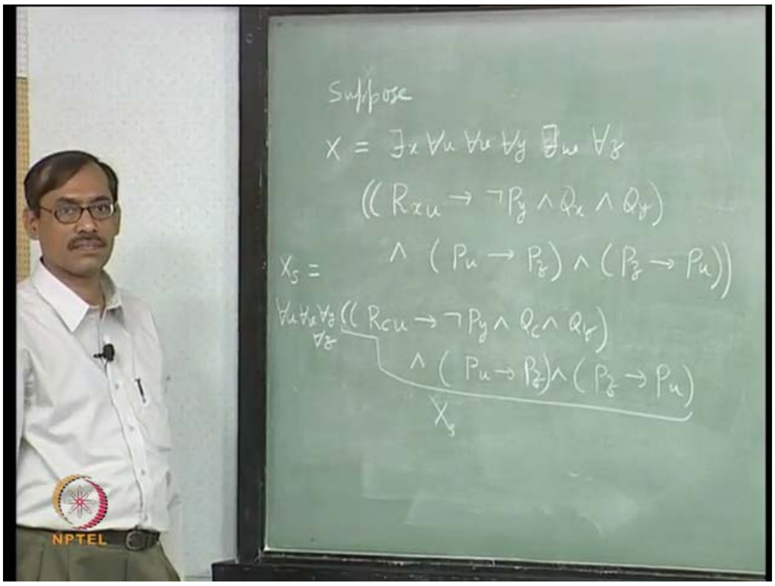

For this there is x, there is no for all before it; so if at all we have substitute, we have to substitute x with a constant c. So, this will give us R c u, not P y and Q c and Q i and P u implies P z and P z implies P u. In the first step, when forget this there is x, there will be for each u v y, there is w for each z, except this, all the others are in the prefix. Now, for each u, for each v, for each y, we do not have to do anything again; we concentrate on there is w. Here, what we see? There is w; before it occurring three for all's u, v and y. Now, next we find out where w and u occurs; nowhere; w v occurs nowhere, w y occurs nowhere, in fact w does not occur at all.

So, does not matter the algorithm does not find that. It tries to find whether with w, something occurs or not; nothing of them occurs. So, w is also replaced by a constant d; that is noted by the algorithm, there is no replacement, but that does not matter. So, no more replacements are here. Then for all z is also there. That means we obtain X s equal to for all, each u, each v, for each y, for each z. This is your X subscript s; then X small s will be just the matrix of this much; that is your X s, that is the Skolem form. This is what you obtain; it is the sentential form. We say that X is satisfiable if and only if this sentential form is satisfiable; for alls, and then this.

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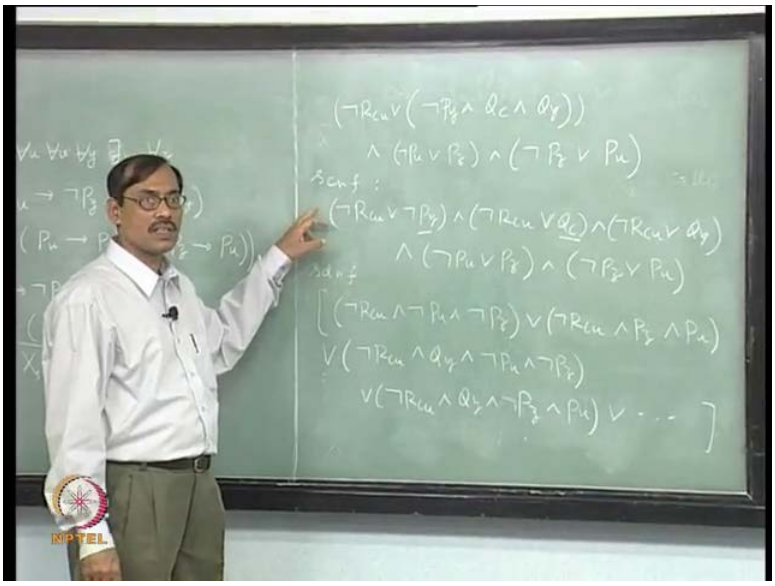

Let us go for the validity form. Now, suppose you want SCNF or SDNF; you can simply convert it. Let us see that also, before going to validity form. Here we have first one, which is not R c u or not P y and Q c and Q y, this and not P u or P z and not P z or P u then we convert it to say CNF. So, distribute it; not c u or not P y and not R c u or Q c and not R c u or Q y other two remain as it is, so this is its SCNF. Now see that in SCNF all the free variables are universally quantified. In SDNF also all the free variables are universally quantified, fine?

So, one SDNF will be, you can just distribute them, you take not R c u, not R c u, not R c u, not P u, not P z; this is one clause, or not R c u, all not R c u here and not P u and P u. So, that will never occur; it is equivalent to bottom; you forget it. Next, not R c u here, here, here, here. These two, we have already taken. So let us take P z and not P z will not come, P u will come, these two. Next, we take which one? This Q y and with Q y, then similarly with Q c with not P y, you have to take; and something else, that will be here SDNF. Now what you see, if you put this for all's, they are all universally quantified.

So, for all will be distributing over this and; it will be equivalent to for all star of this. And for all star of this, and for all star of this, and so on. But here, it will not distribute, for all will not distribute over or. You cannot take it to the clauses; some of them share this variables, but in SCNF they do not share; they can be independent. That is why it is natural to consider SCNF instead of SDNF because of this reason. Similarly, in FDNF there exists will go with all the clauses, but in SCNF there exists will not go. So, again it is natural to consider an FDNF. In that case the free variables are not shared by the clauses; they are independent. We stop here.