Mathematical Logic
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Lecture - 32
Prenex Form Conversion

So, you are considering how to convert first order formulas into some normal forms like your DNF and CNF. Then we came across one hurdle, which was to bring the quantifiers first to the beginning. And that again adds some problem like there can be some free variables, some bound variables with the same name. The same variable can be both free and bound. So you need to define some terminology, then you can be easily talking about them.

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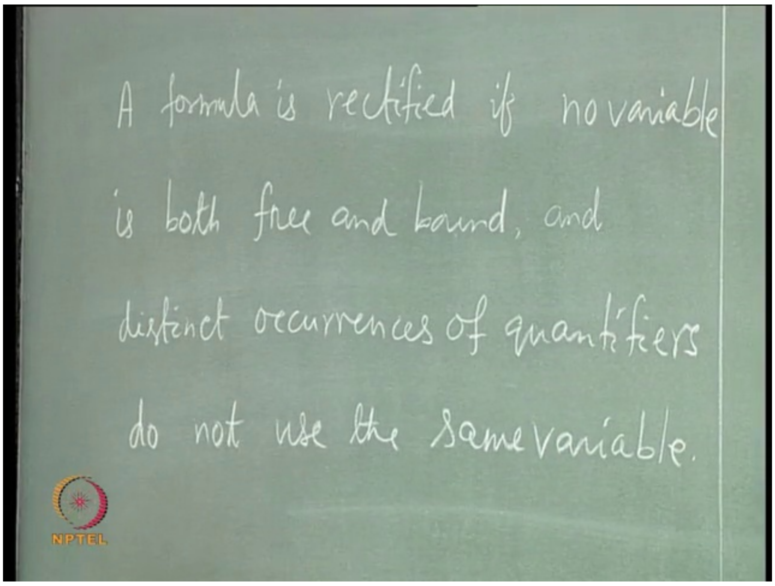

We call that a formula is rectified if no variable is both free and bound. And also we need another, that the different quantifiers do not use the same variable, and distinct occurrences of quantifiers do not use the same variables. So, first we want to convert each formula to its rectified form. Rectified form may not be unique, because you may have to use renaming of a variable; depending on what name you are using, the formula will also differ. But then when you implement in a mahine, what it does is, just renames all the variables from the beginning itself. If x is there, it takes x1, it does not know what x is. It rewrites all the variable first; then when some variable is not in the correct form; so, it rectifies the old formula by renaming to the next index. Suppose x1 to x10 have been used in the formula and there is a sub-formula where it needs renaming. It uses x11. In a certain way it will be unique. Let us see how to rectify.

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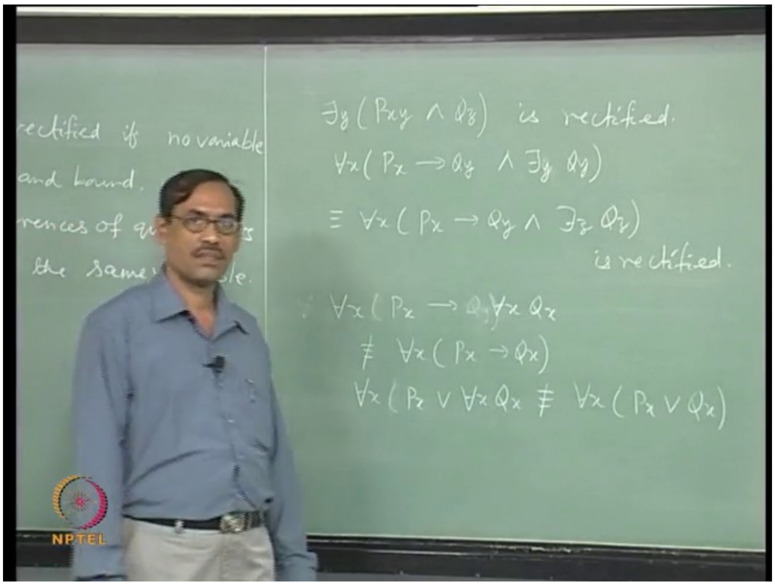

Let us take the example: there is y Pxy and Qz. Now, is it rectified? There are three variables occurring in this formula, x, y and z. And now you see x is a free variable and x is not bound. There is only one occurrence of the quantifier which uses y, that y is bound here; it is nowhere also free; z is a free variable. And this is already rectified, fine. We say that this is already rectified.

Suppose we take another, say, for each x Px implies Qy and there is y Qy. Now again, there are three variables occurring here x, y and z; x is a bound variable, it is not free; y is a free variable here, but y is also a bound variable. So, it needs renaming. Let us take this sub-formula where it is bound, which we can rename. Free variables cannot be renamed; it will not preserve equivalence. We write that as: for each x Px implies Qy and there is z Qz. There are really two variables occurring here. There are three variables occurring here. x is bound, y is free, z is also bound, and there are two occurrences of the quantifier; they use different variables; so this is rectified. Rectification is a simple process.

Now, our next step is to bring all the quantifiers to the beginning. How to bring them? You have to use certain laws you need equivalence there. Now, once it is rectified you can see that. Suppose you take some formula, say this. I can use my distribution laws there. For distribution laws, you need variables to be differently named; otherwise the distribution may not occur. Like, you have seen, for example, for each x Px implies Qy; say, for each y here, or let us say, here, for each x Qx. Now, this not equivalent to for each x Px implies Qx; it is not equivalent to that. If we take the distribution in AND and OR, the same thing happens for each x Px or for each x Qx is not equivalent to for each x Px or Qx, and it will be true, for all, it will not be. And if you have existential quantifier, or, it will be true, and, it is not true; duality is prevalent. Similarly, for implies also it will happen. For implies both the thing will happen because quantifier will become different, once you use the distribution.

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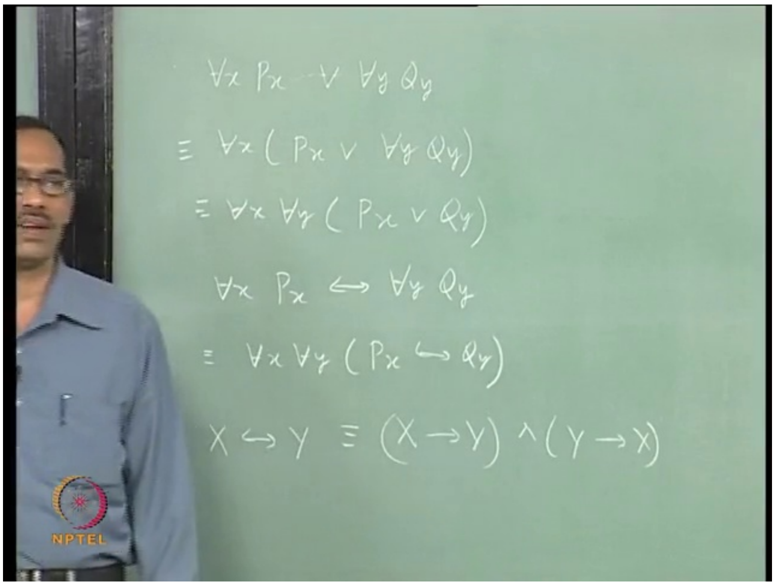

But once it is rectified, the thing can be different. For example, here suppose it is for each x Px, say, we take the OR case first, or for each y Qy. Now, what happens is, in this sub-formula x does not occur, this is OR. So, here x does not occur. Since x does not occur at all, I can bring the quantifiers outside. I say it is equivalent to for each x Px or for each y Qy, because x does not occur there. But for what formula you put in there, it does not matter if x does not occur. Next what I do, say, for each x each y Px or Qy. This is the advantage of rectification. Now, you can bring the quantifiers to the beginning, but you can never bring it to the form for each x Px or Qx; that is fine. The quantifiers can be bought to the beginning; may be with many more quantifiers, and so on. They will become equivalent now. This is the thing we want to happen now. In general what to do? There can be if and only if also bi-conditional occurring. Biconditional; you do not have a distribution law directly.

For example, if you take for each x Px, for each y Qy; this will not be equivalent to for each x for each y Px if and only if Qy, as happened for for all. For implies, you have something to do, not for biconditional. So, first we need to convert the biconditional to implies AND, or any other form. Let us take to one. Say, it will be changing all these biconditional: X biconditional Y is equivalent to X implies Y and Y implies X.

But there is one more problem; there can be quantifiers here. Suppose quantifiers are there in the beginning or somewhere else. These biconditional, quantifiers cannot be pulled as it is, or that might be because the formula what results need not be rectified. It might require again another renaming, that is possible. So, what do we do is, before rectification eliminate this, and then rectify; that is possible. Once that is realized, we can go for the procedure. How to proceed? This, we will be writing as: starting from elimination of the biconditional.

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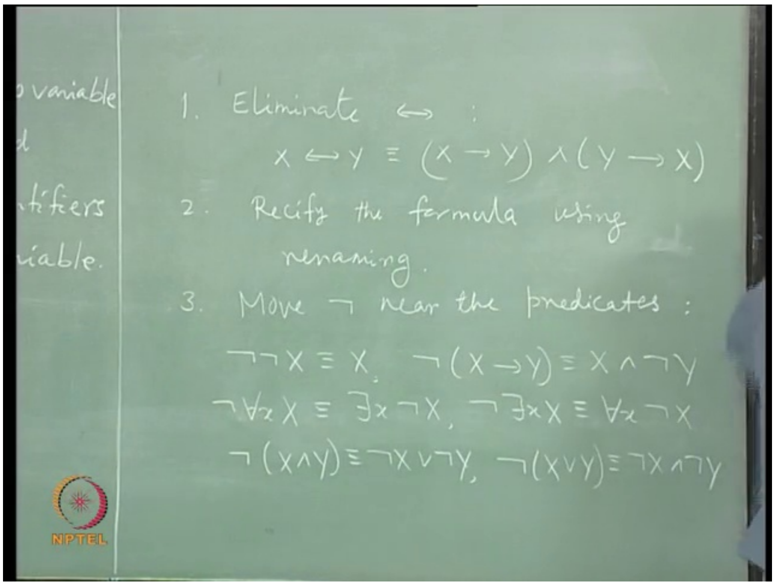

First we would like to eliminate biconditionals by using the equivalence: X biconditional Y is equivalent to X implies Y and Y implies X. Second, we want to rectify the formula using renaming, that is clear. See, our aim is to go for one form where quantifiers will be in the beginning. Now suppose this is a not symbol, for example; it will start with not for each y and everything else. For each x then not for each y. Now, this not, you cannot bring it, you cannot take it to inside the formula; because our aim is to bring all the quantifiers in the beginning. But there is a not; so that has to be removed now. That means we have to move this not beyond this block of quantifiers. Let us make it. That not should be going to predicates directly; that can be done.

That will be stronger; but that is fine. That can be done by using double negation and De Morgan. Let us do that first; near the predicates. In fact, we do not need to move it near the predicates; we can keep it outside the block of the quantifiers that is enough for us. For example, that is not of X implies Y; we do not need to put X and not Y; it is enough. If it is not inside the quantifiers, but better to do it any way, we will be going for the DNF and CNF. So, this step is over-burdened. But we will take it; we will go along with it. So, this can be done by using double negation, then implication and then De Morgan. Here, if you want them to move near, the not signs near the predicates, you have to use other two also with AND and OR. It could need it; this much is enough; you can have all the quantifiers outside. Then, what awe use is: this not for each x X is equivalent to there is x not X; and also not there is x X is equivalent to for each x not X, fine. Here, you may say, if you need to make it better, you may write not X and Y as not X or not Y; and also use the other one for not of X or Y; this is not X and not Y.

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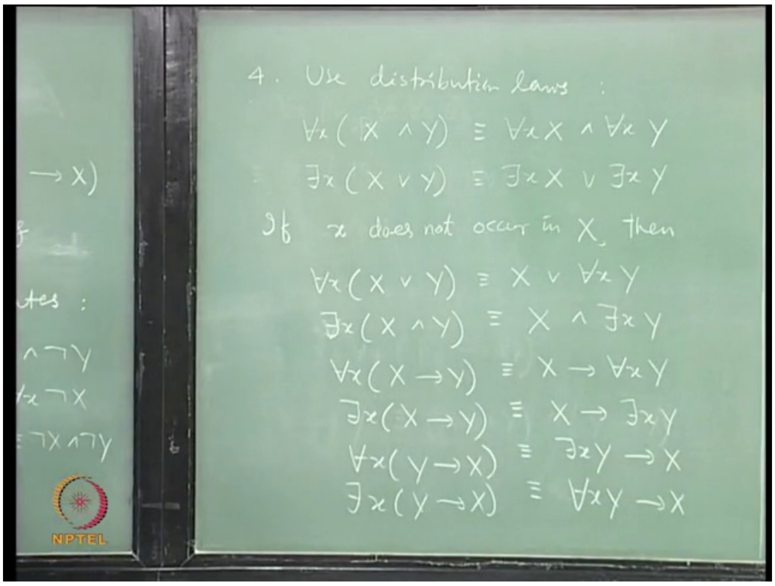

Then comes the distribution part, to do similar things. Next, we use distribution laws. These distribution laws will be for the quantifiers; we need for the quantifiers. Let us write: for each x X and Y is equivalent to each x and for each x Y and holds for every. Similarly, there is x for OR. With OR only one side only holds, but if it does not occur, then that is also alright.

You may say, if x does not occur in X, then some more we can have. Like: for each x or Y is equivalent to X OR for each x Y, because this variable does not occur in capital X; there is no need to keep any quantifier, that will be equivalent. Similarly now, for implies, for each x X implies Y will be equivalent to X implies for each x Y. And there is x, similar; then for each x Y implies X, that is, there is x Y implies X; it will change the quantifier, because of this implies. And because this is really not Y or X, it will come to, for each x not Y, which is not there is x Y OR X, that is, there is x Y implies X. Similarly, there is x Y implies X gives for each x Y implies X. This should be sufficient. Then whatever form you get of the formula that is called a prenex form or the prenex normal form. We will use this prenex form only. We will see how to bring to the normal forms.

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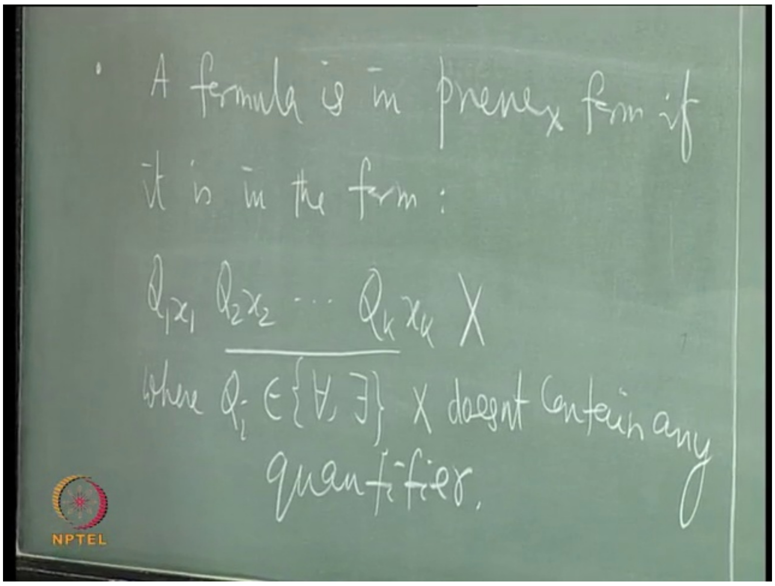

We say that a formula is in prenex form if it is in a form, where each Qi is either for all or there is, and X does not contain any quantifier. So, in the prenex form the block of quantifiers in the beginning is called the prefix of the prenex form; and then the formula having no quantifiers is called the matrix. They are some names; it will be the prefix, then the matrix. The prefix has all the quantifiers; matrix does not have any quantifier. This procedure is called prenex form conversion.

You take the first one: there is x Pxy implies not for each y Qy and Ryz and Qx implies for each x Sx. You must start with this one. Now, what our procedure says is, if there is any biconditional, first eliminate it. There is no bionditional; so go back to that, pass that step. Next, what you have to do? Rectify it.

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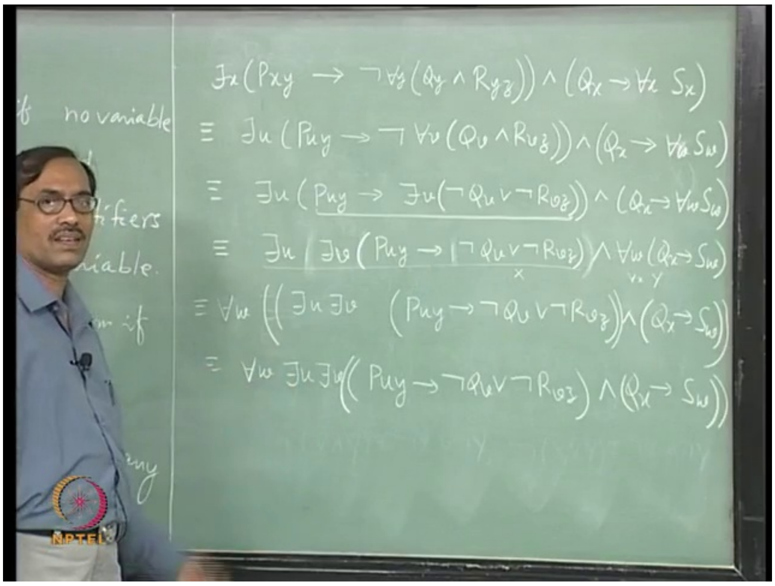

First, let us check. Your x is bound, x is free here. So, all the bound occurrences of x are to be renamed first; and then what about y? y is here free, y is also bound. So, bound occurrence of y, and, also has to be renamed; and z is only free, it is not bound. So you can leave it as it is. Now, for x, you have a bound occurrence here, you have another sub-formula where also it is bound. You will be using two different variables to rename it; and y also should be renamed. We need three renamings. First, x let us try, there is u Puy implies, now y also is to renamed, but not this free, only bound occurrence is to renamed. So, this gives not and then this y is to be renamed; let us write as v, Qv and Rvz; and this is a free occurrence, it has to be kept at it is. Then, this bound renamed differently. Let us write each x Sw now, is it rectified? u is only bound, no where it is free; v is only bound; and then w is only bound, z is free, x is free.

Next what you have to do is take the not signs inside. So, this is equivalent to there is u Puy implies; now, this not will go inside, so, that gives there is v not of this, and that is, not of Qv or not of Rvz and Qx implies for each w Su; that is done. Next is the distribution law to pull the quantifiers to the beginning. Usually it will be easier if you start from the inner most things. You have to start from somewhere. If you can identify the inner most, it will be easier. But on a machine when it is implemented, it will go from the left side. It just comes from the left, whatever it finds, tries to pull it. Let us follows that, see what happens. So, this becomes there is u, already in the beginning, I do not have to do anything. This is the formula. This, there is v can be pulled directly. Which law you are using? X implies there is, something like this, something implies there is x something else. That gives you: there is x will come to the beginning. That gives there is u Puy implies not Qv or not Rvz. I do not need to have a bracket here because of precedence rule; that is all. So, I also do not need this one. This one, I wrote, not this one, that is enough. There can be more brackets, but does not matter; you can omit them later. Then what about here? It is in the form X implies for each like this; that will be for each w Qx implies Sw. Next step will be pulling this. In fact, when you use it, this for all w will be in the beginning, where? This y x OR for each x y. So, X and for each x Y, it will be here; then, X and for each x Y. So, for each x, X and Y, that is here, because that does not occur there; this is for Sx.

Student: This for the first?

Yeah, it is only for the first one. First step will be what? If you apply that blindly, it will come to this: there is u there is v like this and this; it will come to this. Again you have to pull it, so it will come to for each w there is u, there is v. You can also do it someway so that this for each w will be there, in the beginning, there is u, there is v can be outside. That is also possible, because first you pull this one then go for this, and combine them differently; that is also possible.

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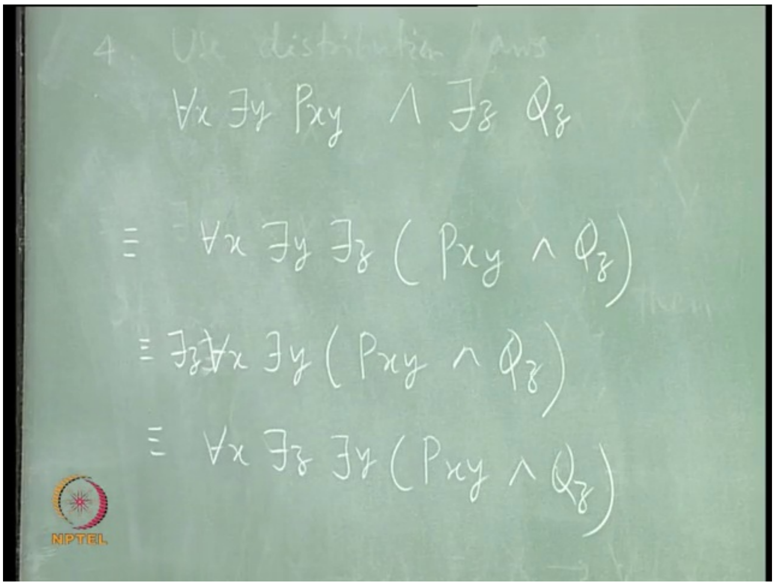

And what I wanted yesterday, gave you for the exercise is: show that they are equivalent. It does not really matter. Here, for each w there is u there is u for each w does not matter. We will see the reason; why it is happening?

Well, let us consider this formula. Suppose, you consider this formula; or even you may say there is z, it does not matter, it will be easier to see this. Suppose, I interpret this sentence in the set of all human beings. Now, P will be something like, say, y is father of x; x has a father, it says. It says x has a father; P is for parent, x has some parents. And suppose Qz means z has 12 fingers; 11 fingers, let us say. You have one model for it. Now what you do? This sentence can be read as each person has parents and there is a person who has 11 fingers.

Now, when you bring it to this form, how does it look like? x, y do not occur here; z does not occur here; you can just bring out, in any way you like. It may look something like: for each x there is y there is z Pxy and Qz. And it can also be seen as there is, for each x, there is y, there is z can come here Pxy and Qz. You can also bring it to the form for each x there is z, there is y Pxy and Qz. But for each x there is y will be together in that order, not necessarily together. There, z can come in between; that ordering is important. You cannot write there is y for each x, that will never happen. First, for each x will come, next there is y will come, in same order; but it does not matter for z. Now, can you read this? What does it say? There is z has come to the beginning; what does it say? Again the same thing it says. Because this z does not matter what this x y are. For, that person has fingers, 11 fingers are not, it does not matter for anyone to having parents and not having parents.

So, this z does not depend on x and y. And the reason is? This is happening because there is no predicate in the whole sentence or the whole formula, which having both x and z together. There is no one which is having y and z together. In a sense, they are independent; there is no dependence between the variable z and the variables x and y. But there is a dependence between x and y. In order that this is true, sentence is true, there is dependence; this y will be a something like a function of x, not exactly a function, it is like a function; it depends on this x in a particular way.

When you say y is a parent of x, that fixes x in some sense. It cannot be anything arbitrary. But the other z can be very arbitrary, it does not depend on x and y either. The formal semantics, whatever we have defined earlier takes care of this. Though I have not fixed it anywhere, we have just defined it by induction; anything you can have. Some similar change will be like, not sign you have to take care. But here it does not matter because z is simply independent. There is no predicate having z and any of x and y. It can just be pulled to the outside, yeah? But you have to show that, what I told yesterday, by formal semantics. It can be done; intuition is this. Okey.

Now let us see. This prenex form is not necessarily unique. There can be different ordering of the quantifiers, but they will be equivalent in this sense. If there is no predicate having those variables combinedly then those quantifiers can be in different order; that is what it says, it is allowed in the prenex form. Now in this prenex form this is the prefix and this is the matrix.

Let us take another example; just start with this. The first thing you have to do is, eliminate biconditional. This can create problem in the rectification. So, first we have to do that. There is x Py implies not there is y Py implies Qx implies Qy and for each x Px implies for each y Qy and for each y Qy implies Px. Next thing is, we rename the bound variables if necessary. So here it is bound, x is free anywhere? x is free here. So, you have to rename the bound variables, but x is not free here, it is not within this scope, it is within this scope, so not free here.

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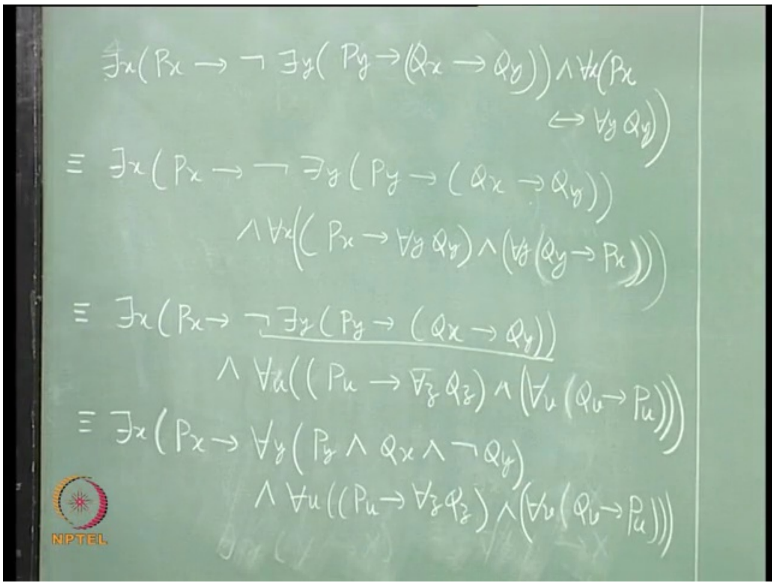

Here? Well, here it is not free, but within the same x so there are two quantifiers using the same variable x; this has to be renamed. Next here is one quantifier there is y, there is also for this y, there is also for this y; they have to be renamed. Let us rename them: there is x Px implies not there is y Py implies Qx implies Qy. Now, this x is to be renamed; say, write for each x say, Pu; now x will be u, and this y is to be renamed. Let us write for each z Qz, this y also to be renamed; for each v Qv implies, this x within this scope, that is u. So, next thing is, take the not to the inside. There is x Px implies, this not has to go inside; so this becomes for each y not of this whole thing.

So, that is Py and not of this; that is again Qx and not of this, it has come up to this. And now for each u, there is nothing to be done here. We had biconditional here, for each y Qy then, it gives Px implies for all y Qy and implies Px; there is no bracket here. The bracket is here, bracket should be here. So, bracket is here, then? You have bracket here also, that is the modification.

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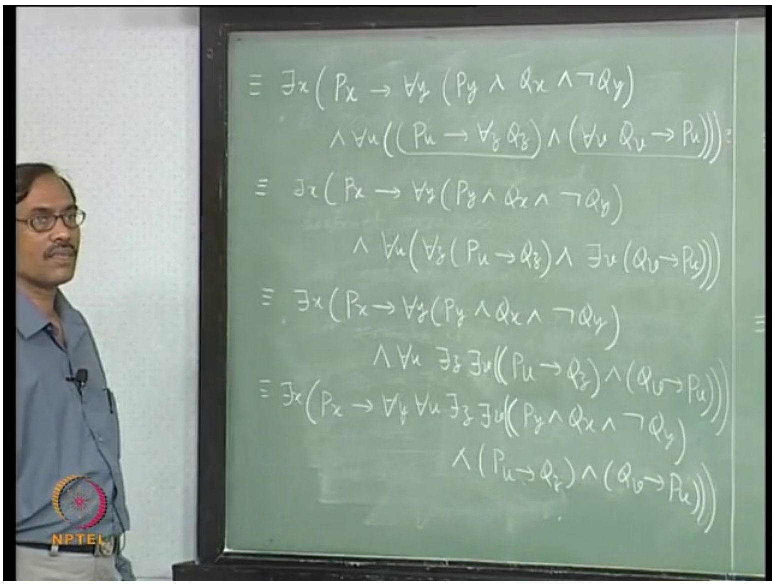

Now, what should we do here? Start from this. Let us start from this one. We are doing one only. Write there is x Px implies for each y Py and Qx and not Qy and for all u. What about this? For all z comes out. We write for all z Pu implies Qz. And here also, what will happen to this? It becomes there exits, so there exits v Qv implies Pu. Let us keep that now and here, what should we do? There is z Pu implies Qz and there is v Qv implies Pu. z does not occur here, v does not occur here. So, you can just write for each u there is z, there is v Pu implies Qz and Qv implies Pu. Next, after implies, this and, this whole formula. There you have to pull the quantifier similarly. You have u z v; u z v do not occur here, and it is and; so all those things can be coming together. I may write there is x Px implies all the quantifiers together in any order you like now. This or this; that only you can play with, you cannot play with this order now. So then, let us write this way: Py and Qx and not Qy and Pu implies Qz and Qv implies Pu. Next you have to pull this block out. It is like there is x Px implies X or implies Y, where x does not occur in Y. Now, x does not occur in this formula, it occurs, x occurs here. First, you have to think of this Px implies this quantifier, quantified formula; forget about the each x. Now, Px implies this, x is free here, forgetting this, x is free and right side also there is x, x is not in one of the quantifiers. So, that can be pulled to the beginning, and since it is on the right side of the application, nothing will change; it will be kept as it is. So, it will look like there is x for each y, for each u, there is z, there is v, Px implies the rest.

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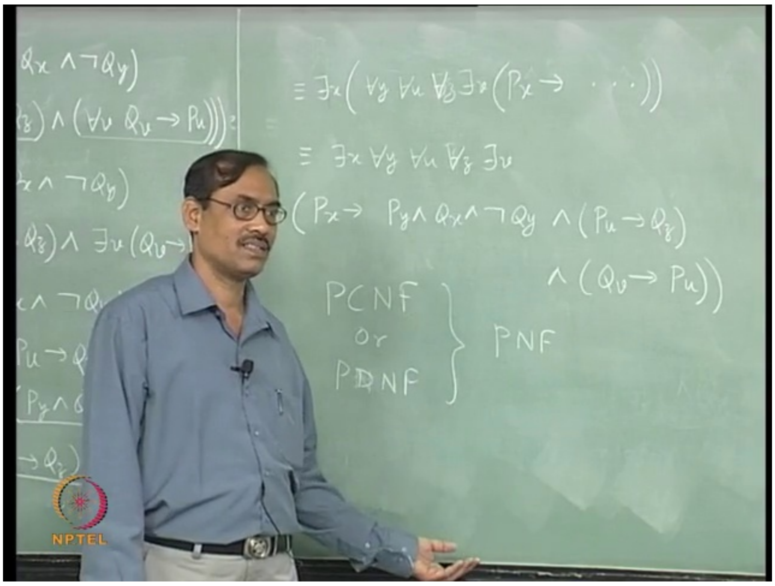

Student: The last part?

Which one, this? So, what is the change? For all y is here, for all y, in this part is there; now only this part we are concerned. For all u, there is z, this should be for all z, that is right. Otherwise it will carry over, for, this for all will not change here; it will be kept as it is. Now, let us look at this. This gives Px implies the rest; the rest means this one. Then next, just forget the bracket; this was for each z; that is how it will be proceeding. Here, you have really entered formal manipulations now.

All that we see here is what? Any formula can be brought to one, which is in prenex form. If you want to convert it to some DNF or CNF you have to redefine what is a literal, what is a disjunctive clause and so on; that is easy. You just say that any atomic formula is a literal, any negation of an atomic formula is also a literal; that comprise your literal; either an atomic formula or negation of an atomic formula. Then ANDs of all those atomic formulas are, literals, in general, they will be conjunctive clause. Then, ORs of literals will be disjunctive clause. Then CNF is conjunction of disjunctive clauses; DNF is disjunction of conjunctive clauses. That is how we will get DNF or CNF.

Now, all those definitions will be useful for converting the matrix of a formula. Once it is in prenex form, you consider its matrix; forget the prefix now. This matrix can be bought to CNF or DNF equivalently, using that definition. Once you brought that form, you call that formula as Prenex conjunctive normal form or Prenex disjunctive normal form. So, you just tell PCNF or PDNF; both are combinedly called PNF, Prenex normal form.

This is the Prenex from. Once you covert it to either CNF or DNF, it is called prenex normal form. If it is in CNF, the matrix, you said it is PCNF. If the matrix is in DNF, you say it is PDNF. We will not use these normal forms very frequently because we have something better; after this, we will be doing, where we will be using these normal forms again. But after doing something, we will be converting to normal forms.