Mathematical Logic
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Lecture - 31
Examples of Informal Proofs and Calculation

We have just summarized all the equivalences and the laws. Then, the four quantifier laws that we propose for the first order logic. That, along with all the laws of propositional logic, you can use these quantifier laws and then conclude about equivalences, validity or consequences. We will take some examples today and see how does it go. It will also be coupled with symbolization. So that symbolization will tell you how much we have understood about the first order semantics, because semantics is already involved in the symbolization itself.

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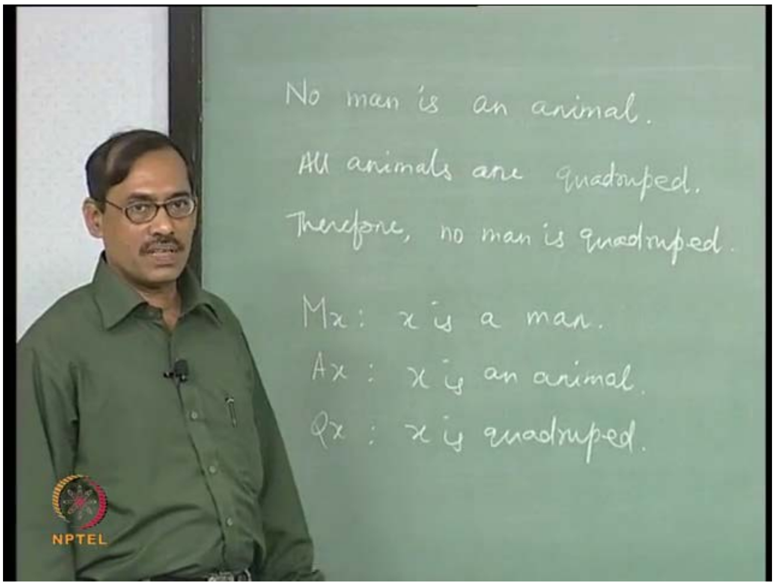

Let us try one. Let us start a symbolization of this argument. We are not concerned, so that the argument will be true or false, whether that is a valid consequence or not. Let us try to see how symbolization goes in. Suppose first one, you first identify the predicates and the connectives. There are no connectives used here. So, first one is, something is a man, something is an animal. So, you need two predicates there. Let us start. We will say Mx is x is a man and Ax is x is an animal. Then one more predicate will come here. Let us say Qx means x is quadruped. Now, coming to the premises and the conclusion. First one, no man is an animal, how do you symbolize it? First suggestion is, do not write as it is, a first order sentence or first order formula. First try to translate with using variables and maybe open formulas. Then try to see what quantifier you are going to use. Rewrite that in English, using the variables. First one, for example, no man is an animal. How do you write? For each x if something then something; or it is not; or there exists something for which something else happens. Someway you have to write it.

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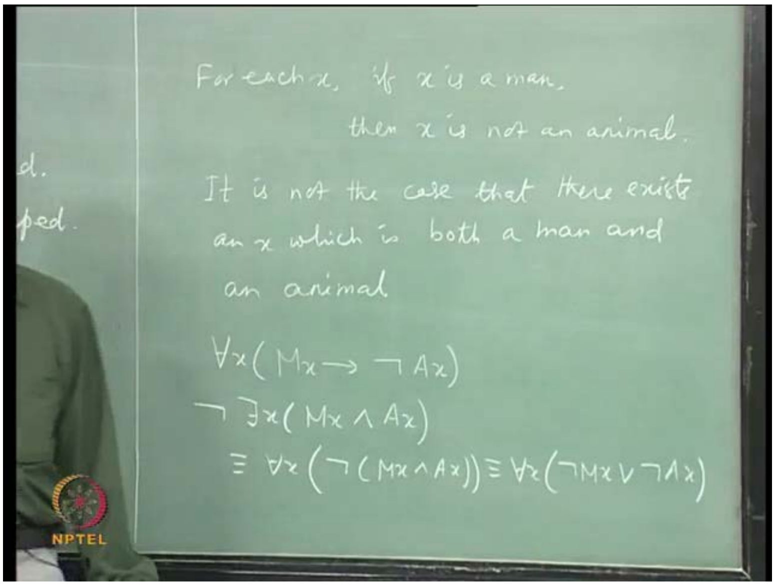

Let us say for each x if x is a man, then x is not an animal. You have translated the meaning like this. You do not know till now whether it is correct or not. Well, that can be, otherwise, like no man is an animal, when can you say it is false? If you can find one man who is an animal. That is not the case where you can find one person or one object which is a man as well as an animal. We might also write, it is not the case that there exists an x which is both a man and an animal.

Once you write this way, it will be easier for you to translate it to first order logic. Say, for first one, if you proceed this way, you would be translating as: for each x if x is a man then x is not an animal. In the second way, you would write as: it is not the case that there is one x which is both a man and an animal. Are they telling the same thing? Are they equivalent?

Well, you take this not inside, the De Morgan. That gives you: for each x not of Mx and Ax, not there is x is same thing as for each x not x, that is a De Morgan's law. So, this comes now, this will become: for each x not Mx or not Ax. Now, not p or q is same thing as p implies q. So, p as Mx here, Mx implies not Ax; it is the same thing. So, any one of the ways you can proceed, does not matter.

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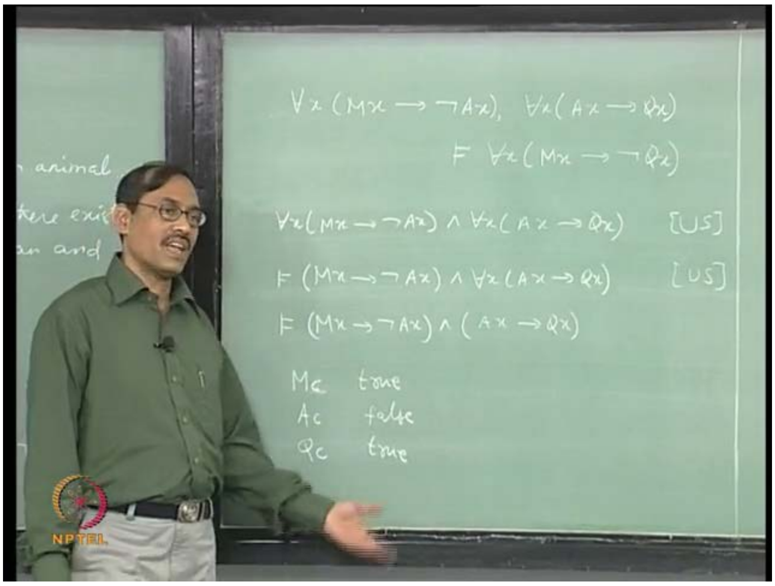

Let us take one of them, say first one. The first sentence we wrote as: for each x Mx implies not Ax. What about the second one? All animals are quadruped; that is easy. For each x if something is an animal then it is quadruped. So, Ax implies Qx. Therefore. What is the next? Last proposition is, last sentence: no man is quadruped. Here again, it is in the same path, in the first form. You can just write: for each x if it is a man, then it is not quadruped. Now, suppose you want to prove this. You do not know whether it is valid or not. Let us try proving it. Which one you will take? Informal proof or calculation?

Student: Calculation.

Calculation. So, how do you start? Well, there are only two. You can start with and of both the premises; it will not make much writing. Let us start with that, for each x Mx implies not Ax and for each x Ax implies Qx. As we apply quantifier law, say, universal specification and the first one, let us say. It is like universal specification and the first one. If you want to specify, what x you are taking? You may write x by c or x by x itself, does not matter; okey?

Let us take x by x. That gives Mx implies not Ax, and for each x Ax implies Qx. Now, again we can apply US with the same x. So, Mx implies not Ax, and Ax implies Qx. Now, how to use both of them? Our aim is to get something with Mx and not Qx, or Mx, Qx. Ax should be eliminated. It does not look, I can eliminate Ax. Had it been not Ax, I could have used hypothetical syllogism, or it had been not Qx. I could have also applied same way because contradiction. But nothing is there. So, I am stuck; I cannot proceed. It looks, it may not be valid. Well, I am not able to do; that is why I am thinking it can be invalid; that is not a proof.

Now, to prove the invalidity, what we will do? Find one state which is not satisfying it. That means you have to get a state, which is a state model of both the premises, but it is not a state model of the conclusion. This is what we want. Now, how to proceed from there? Say, I have some a here in the domain. I have M of a. I know that is also not A of a. Then I have Aa implies Qa.

Suppose Ma is true. Then I have to get not A of a. So, that A of a is false. This will be vacuously satisfied; because not A a is true. So, A a is false. Once A a is false, Qa will be true or false; it does not matter. Your sentence will be correct; it is true. Then in that case, I end with Ma is true, and not A is true. These are the two things. Q can be anything. Now, conclusion is? Conclusion is: Ma implies not Qa. So, when Ma is true, Qa should be false. Now, to give one counter example, I take Qa to be true, instead of false. What it says is that I should take Ma as true, Aa as false, Qa as true. If I try with that, then probably it will succeed, is that okey? Now, how to declare all those things?

This requires Ma to be true. Let us write c instead of a; it will be easy to speak now. So, Ac will be false and Qc should be true; this is what I want, at least one a. So, that is there, for which it should happen. You can take only single element domain say c and make a relation where Mc can be there, Ac is not there. On a single element, how do we say Ac is false? I take the relation as an empty relation. Possible. I can take empty or if you take a two element set, you take the other one to be true and this is false. So, let us try that.

That will be easier to see. Say, let I be equal to D phi where D is a b and we want M, we want A, we want Q; all these are required. So, one interpretation is enough; because all these are sentences. I do not have to go to states; there is no free variable in either the premises or the conclusion. I just give one interpretation which will falsify it; that is all. With these, we take M prime we call M prime at c should be true, we have c here and b, let us say. I should take c only; it is a unary relation; it is the subset of domain D. I take only this subset c, and then I take A prime, A prime at c should be false. So, c should not belong to A prime. I take only b. Now, Qc should be true. So, I will take only Q prime equal to c again, is that ok?

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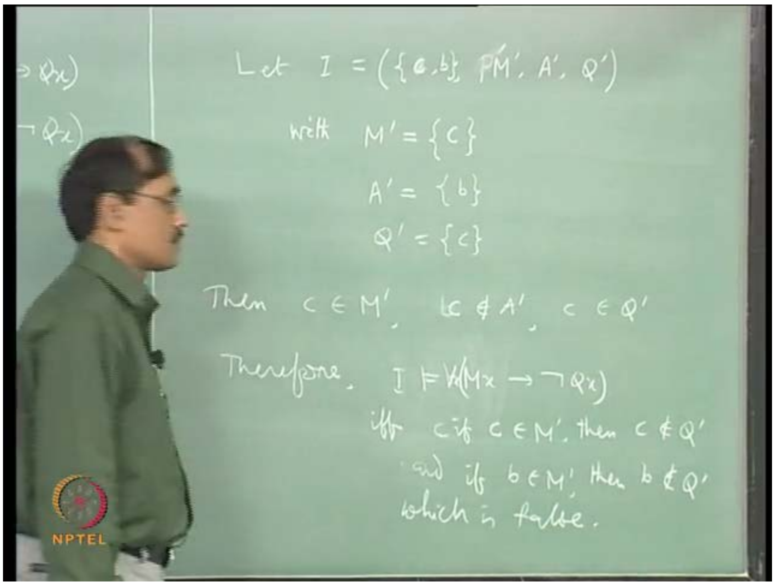

Then what we see, then c belongs to M prime, and c does not belong to A prime, c belongs to Q prime; therefore, what happens, I satisfies Mx, not Mx. You should, Mc only, right? But there is no c. Yyou have to see first sentence: for each x Mx implies not Qx. Is it satisfied? Directly you can see or not? I, c satisfies for each x Mx implies not Qx if and only if, I have only two elements in my domain; for each element I have to see if and only if. So, first element is c, if c belongs to M prime, then c does not belong to Q prime and if b belongs to M prime then b does not belong to Q prime. You are going every step of the formal semantics. See, for each x Mx implies not Qx. Now, this says x can be either c or b. For c, we will translate as Mc implies not Qc; it is prime.

That will be translated as c belongs to M prime, then c does not belong to Q prime; for c. For b similarly, if b belongs to M prime, then b does not belong to Q prime. Now, with this data what we see is, c belongs to M prime, c belong to Q prime also; c does not belong to Q prime; so that first sentence itself is false. So, the whole thing becomes false. Because it is all, which is false, right? But then what we wanted, that is the conclusion for each x Mx implies not Qx; it is false. But to show that the consequence is invalid we have to see also premises to be true. This is not over yet. You have to see all those things again.

Let us try that. Now, the first sentence will be I satisfies for each x Mx implies not Ax. This holds if and only if, again with c and b will take; it c it says, if c belongs to M prime, then c does not belong to A prime; and if b belongs to M prime, then b does not belong to A prime.

Now, you have to use the data, c belongs to M prime. Yes, what about c does not belong to A prime; that is also true. The first assertion holds. Next one is, b belongs to M prime. No, it does not belong to M prime. Therefore, vacuously, that sentence also wont, then the sentence holds. Therefore, this is true.

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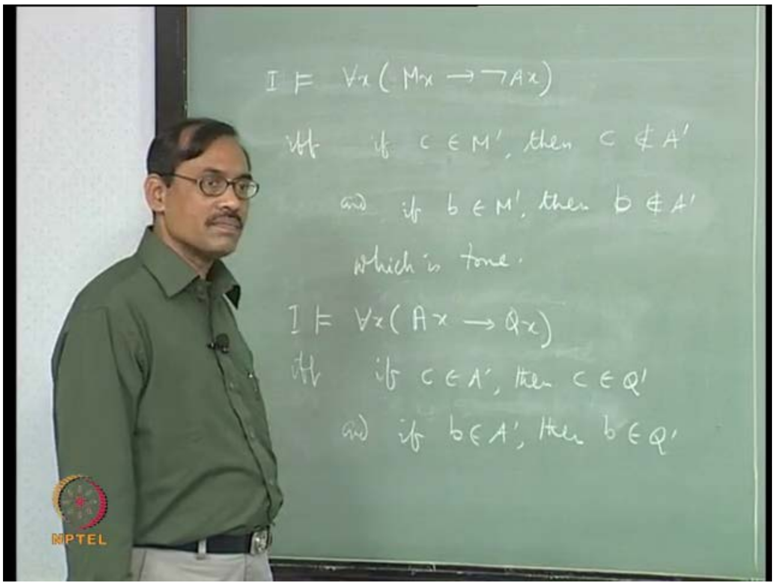

Now, you go for the second premises. That says, I satisfies for each x Ax implies Qx. This holds when if, with c and with b. So, c belongs to A prime, then c belongs to Q prime; and if b belongs to A prime, then b belongs to Q prime, okey. Now, use our data; c does not belong to A prime. So, first sentence is vacuously true,right? So, second one is b, b belongs to A prime or not? b belongs to A prime, yes. And does b belong to Q prime? No; but then it will be false, right?

No? We want to make this true. So, this will be true when for both the things it will be true, fine. So, c belongs to A prime; c must belong to Q prime; that is vacuously true, because c does not belong to A prime. Now, what about b? b belongs to A prime iff b belongs to A prime. Then b must also belong to Q prime, which is not the case. So, our model, what we have constructed, the interpretation does not satisfy these premises. You may have to readjust. So, we have to readjust. We include b here. Then you have to verify everything else also. Well, let us try that.

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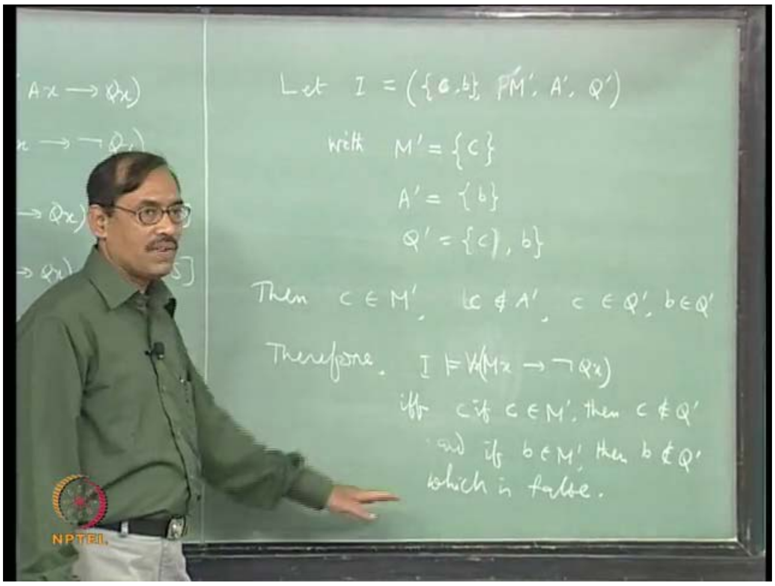

Suppose this is b. Here, c belongs to Q prime and b also belongs to Q prime. With this let us verify again. This, wherever is Qb, there you have to check this anyway. You do not bother one of them only not true. So, that is, that passes. What about the second one? Second one, there is no Q. So, it holds as earlier. Now, the third one. If b belongs to A prime, then b belongs to Q prime. That, now holds. It is all; interpretation is correct; it solves our purpose. So, you have shown that these interpretation satisfies both the premises, but not the conclusion; is that clear?

Now, look at this. The way you have constructed, you can make it better, without going through all these detailed points. What we have done? M prime is c, A prime is, it is compliment in some sense; c is not there, and Q prime, you could have taken the old one, right. Suppose I interpret in the set of natural numbers. I will take M prime to be set of all prime numbers, and A prime to be set of all composites. Let us say. And Q prime is everything of M. Now, is it okey?

First one, let us try to read that. This says, if x is a prime number, then x is not a composite number. And this says, if x is a composite number then x is a natural number, yes? This says if x is A prime number, then it is not a natural number. Now, that is creating problem. That is what we wanted. This should be true and this should be false. Now, you see this is simplified. So, we start with that interpretation; that might be easier right? We just forget all these things. But this is what formal semantics says. You can always construct a model or show that it is invalid by semantics itself, fine. Even if you are misguided, you will be getting the result somewhere, but it could have been easier by that, without common sense, is that clear? That is how you have to proceed; to show whether it is valid or invalid. For showing validity your laws might be helpful. Showing invalidity, semantics will be helpful. Because laws can't say whether it is invalid or not.

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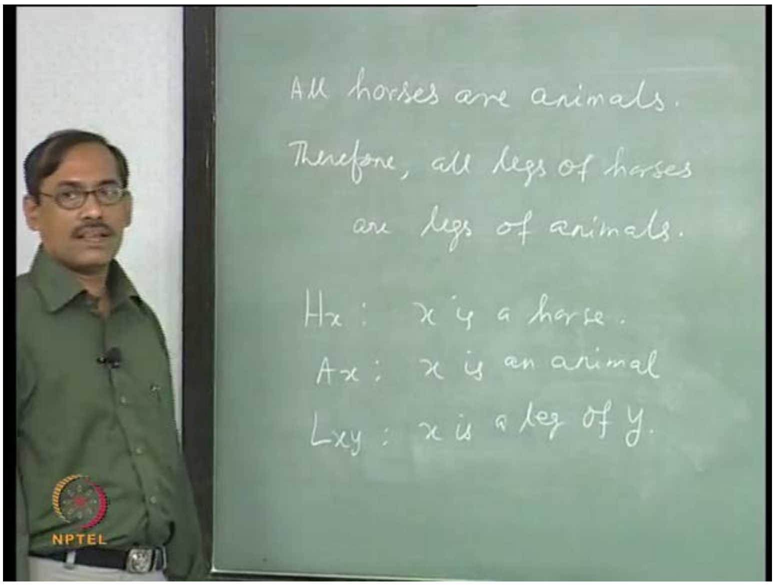

Let us take De Morgan’s example. See what happens. All horses are animals therefore, all legs of horses are legs of animals. This seems to be valid argument. Now, first thing is how to symbolize. It should be easy, yes. First one says, I have to write one predicate with Hx, x is a horse. Then, Ax, x is an animal. Then the next one. I need a predicate, legs of horses, leg of what? Something; can be a leg of something, here it is a binary predicates. So, let me take Lxy, x is a leg of y. So, first sentence is easy to symbolize. The premise which says for each x, Hx implies Ax. Therefore now, all legs of horses are legs of animals, how do you write it? First with x, y and semi-formal English, yes. Is it? What if x is leg of all horses, then x is leg of all animals? Because all horses are some horses; that we have to see and some leg or all legs that also we have to see. Anything is possible. There, it is a binary predicate. Is it that the sentence is telling: you take x which is a leg of all horses, then that is also a leg of all animals? There is no leg of all horses, because if two different horses I am taking, their legs are also different. So, I can't take that, there is one x which is leg of all the horses. Is that okey?. So, what should I write? If x is a leg of some horse, then x is a leg of some animal; and this x should be there, for each x, whatever x is there, it does not matter. Is that okey?

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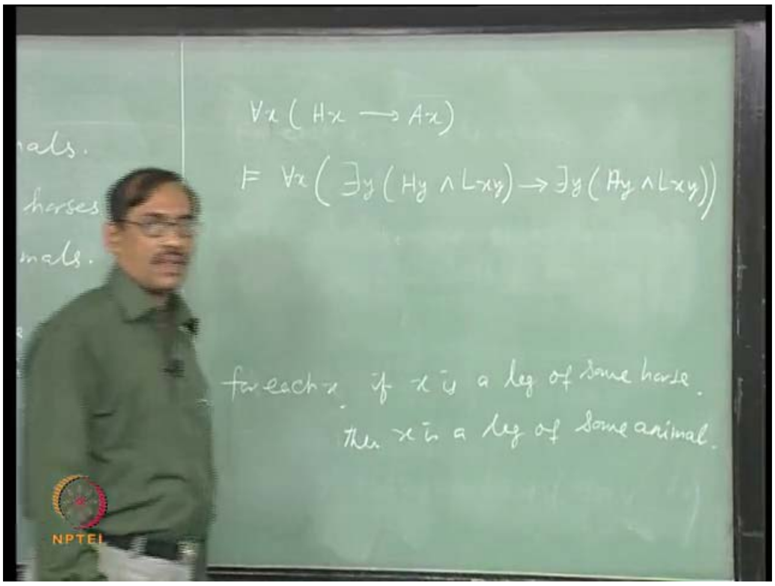

The semi-formal English says, for each x, if x is a leg of some horse, then x is a leg of some animal. Now, x is a leg means? This A is what? All legs. Well, all legs of any horse you take, they are all legs of animals, some animal, but it does not matter now. You can take also some. Now, how to write? For each x will come in the beginning, anyway. Now you forget that. If x is a, so after if, before then. This looks like one formula; x is a leg of some horse; however you are going to symbolize it.

Student: All or some?

Some horse. So, there exists a horse of which x is a leg, fine. That is what it is. There exists a horse of which x is a leg. Fine? So, there exists y such that Hy and Lx, Is that right? So, it will be, for each x there exists y, Hy and Lxy. Since in the predicate we have taken x is a leg of y, we do not have to bother about this A. Its meaning is 'some' now, whatever you take. So, A we are not bothered, it comes there on there exists y where y is a horse. Then what will happen? Similarly, the other one. Instead of horse you have animal. You just use the formula: there is y such that Ay on Lxy. Is that okey? If you want to use another variable you can use there exists z such that Hz and Hxz; they are same thing; is this clear? Now, it looks to be valid; how do we give a proof of validity?

Problem is, here is a premise, here is a conclusion. The conclusion is having only Lxy. In the premise there is no Lxy. So, in the informal proof or in the calculation, you have to introduce that Lxy somewhere. And you do not know where to introduce, how to introduce it. So, the best method will be to proceed by reductio ad absurdum. You have all the information; your target is to getting the bottom; that can be easy. This is a strategy, which helps many times because you do not know where to introduce; you know what only to introduce Lxy. If I have all those information, probably they will cancel, give me bottom, right?

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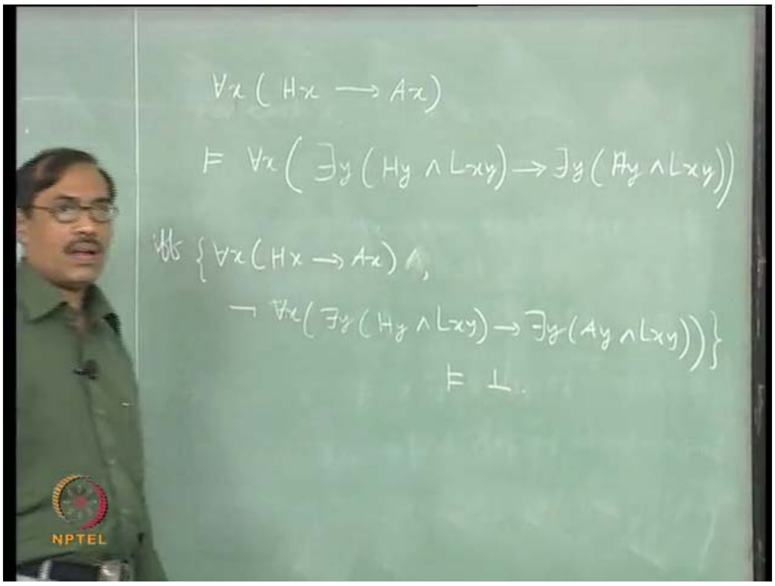

Let us start with reductio ad absurdum. So, this happens if and only for each x Hx implies Ax and it is not that for each x there is y, Hy and Lxy implies there is y Ay and Lxy. This entails bottom; it is unsatisfiable, which says that it should entail bottom. So, you try a proof of this.

Now, where should I start? This one is simpler. I can use, Hx implies Ax anywhere. So, let me start with this one. It looks complex, fine. Second one? Let me try. First premise is this. In fact, you can write also in the form of a premise and conclusion, this one.

Now, you have two premises and one conclusion which is bottom. Let us start with the premise: not for each x there is y, Hy and Lxy implies there is y Ay and Lxy; this is my first premises. I will give a comment, it is a premise. Second, you should manipulate this; try to get some conclusion from this. What conclusion we will get? Yes? Starting with a not, I am not able to use anything. You take that not, first inside, using De Morgan. That says it will change to there is x and not of the whole; so, not of one implication. Here it is, not of there is x; it is in the form not of p implies q. That is p and p not q.

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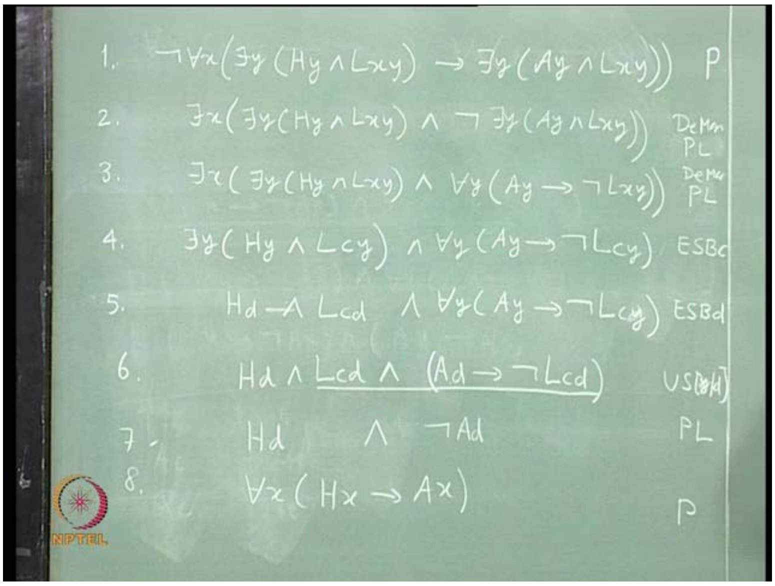

So, I put directly there is y Hy and Lxy and not q, not there is, y Ay and Lxy. I have to give a comment here, what I have used? I have used De Morgan and also implication: not implies. You can write this way. You have to write it fully. Then you will write not of p implies q is equivalent to p and not q, or in first order logic. We assume that you have become matured. You know all the propositional laws. So, you just write PL, some proportional tautology. we get it.

Next again, there is a not here. You can take it inside. Let us try that. There is x there is y Hy and Lxy. For each y, not will go inside, not of Ay and Lxy. That will be not Ay or not Lxy or we can write Ay implies not Lxy. Again PL, and also De Morgan, both are used. Is it correct or not? Verify it. So, this will be not of Ay or not Lxy; this is not Ay or not Lxy. Next, there is y is in the beginning. You will be using existential specification. But you cannot just write after this. From there is x Px, we cannot write Pc, right? But we can take it as one extra hypothesis. Once you take the extra hypothesis, you have to note it somewhere, that you are taking an extra hypothesis; just like in your deduction theorem. And your existential specification begins there with that constant c, fine. Suppose I instantiate x to c, a constant. I would get there is y; I would like to introduce this as an extra hypothesis. This and for each y

Ay implies not Lcy. Now, I must write existential specification begins with this new constant c. So, after this block is over, if there is no c there, whatever is concluded I will accept it; that is the procedure. Within the block whatever is there, it depends on c, so I cannot conclude anything. That will be dependent on this new premise, with this new constant; that is the understanding here.

Now, let us see how far we can go. You have to use our premise somewhere, Hx implies Ax, but that also requires Hy, that H of something is needed; then I can go to A of something. You can another existential specification we may have to do. Before that, for each y. But for each y, I can deffer, for longer. I do not know which constant I will take. So, if I use this, I might unnecessarily take one specification, which will not be used. Let us not use it now; keep it deferring. Now, with this we have, say, Hd and Lcd. For this there exists, say, I am only taking other things; I keep it as it is. Again, existential specification begins with the new constant d, fine. Now you see, Lcd is there, here not Lcy. It suggests that I should take the universal specification with y as d, right? Let us take that and existential, sorry, universal specification follows. That is, if for each x Px, then you get Pd. It is entailment; it is not an extra hypothesis. So, we can just introduce it, as it is. This give Hd and Lcd, and Ad implies not of Lcd. Here, what we have used?

Universal specification with y as d. You can write y as d, y is substituted by d. Not in here without the for all y, there only you substitute. Next, you have Lcd and Ad implies not Lcd; that is modus tolens. Otherwise, you have to write one equivalence; it will give you Lcd implies not Ad, by contraposition; then use modus ponens; that is equivalent to taking modus tolens. That gives Hd and Lcd; let us keep it, does not matter; and from this, I get not Ad, propositional logic, PL. If you know the law, you can write modus tolens. If you want to say from where I have got it, is from this, underline it; you have to show what it is, right?

From these two you get this. You can keep Lcd. But it looks, you do not need any Lcd anymore; you can even forget it. If you forget, it will look like this, fine. Now then, I have not used the other premise till now. And I cannot conclude anything from here, because d has been flagged; we say it is a flagged constant. New constant, so block is not over; block will be over only when d does not appear; but d appears, still now. I can close the block. So, I will introduce the other premise, which is for each x Hx implies Ax; this is a premise. Then it is clear what to do after this step.

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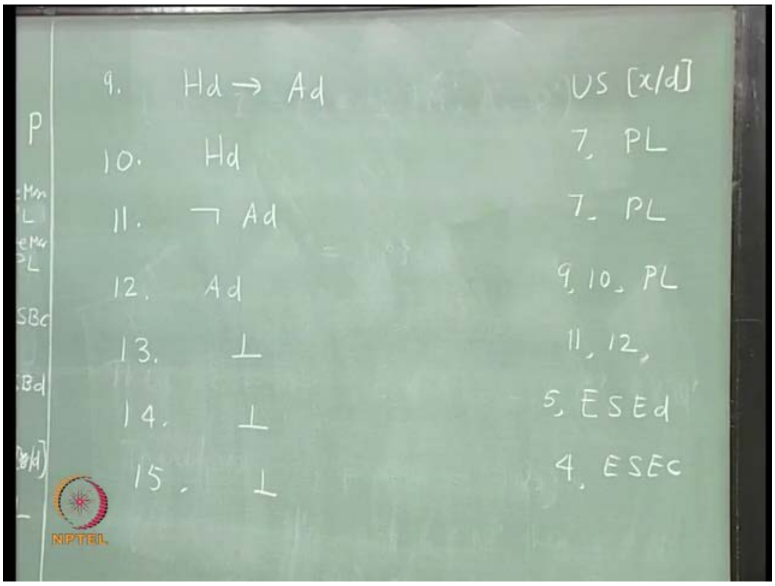

You take x as d, universal specification, right? That gives Hd implies Ad. It is universal specification, where we had taken x as d. Here, implicitly followed another convention. Here, if you are getting a formula from just the last line, you are not writing the line number. If you are using the remote one, then at least you write it; mention it, because otherwise we cannot read it: where from it is coming. All these are coming from the last line.

You have not written the line numbers in the justification column. When you write, here for example, P. Next line, I should have written 8, US, x by d; from 8 it follows. Next, I have 7 where x, d is there. Right, but Hd and not Ad, here Hd implies Ad. That is a contradiction, right? But I have to make it, which way it gives the contradiction. So, first I go for Hd, it comes from line 7, by? So, that is p and q entails p, that is why. Again, not Ad. Similarly, p and q entails q. Next, from these two I get Ad, 9, 10, modus ponens. Now, 13, you have 11, 12, gets bottom, okey?

Once I get bottom, I see that it does not have d. So, I can close the existential specification. You can write, you have to go for the next line, ESEd. Nesting is there; do not write c here; both are closing there. But you have to go only on the nesting, of the loops, nesting of the blocks. The last one was introduced as ESEd, in line number 5. Let us write 5 here. You say where it is closing. It closes with line number 5, ESEd. Once it closes, it does not have bottom. So, I could infer bottom; 5and the previous line; it closes there.

Next, again I have to write bottom; because I have to close the other line; there it ends. Is that okey? So, it is valid. This could not be shown by Aristotelean logic because in Aristotelean logic, it is like all four sentences, types of sentences; they are having only monadic predicates, unary predicates. Px implies Mx or Px and Mx, with for all, there is; and nots, used somewhere; but this is having a binary predicate Lxy. That is why the famous example of De Morgan. It could not be done by Aristotelean logic. We need first order logic; is it clear?

Now, you can write one calculation also, if you want. This premise, you can introduce later also. In calculations, you can go on writing one premise whenever you need to you use that, right? But there is another thing. Since these blocks ESBd, ESBc; you have to take care of these existential blocks in the calculation. This does not follow from the last line. When you do calculation, you have to write another symbol, that implies symbol. We will be writing with indentation, which says that another block is starting; it is a lemma inside the theorem, inside a proof of a theorem, right?

After that indentation is over, you bring it to left side. That closes that indentation also, because it says, it does not follow from the last line. It follows from, after the block is closed, it follows from all the premises whatever had been used in the proof. That is why the block, that is why the indentation; is that clear what you are going to do?

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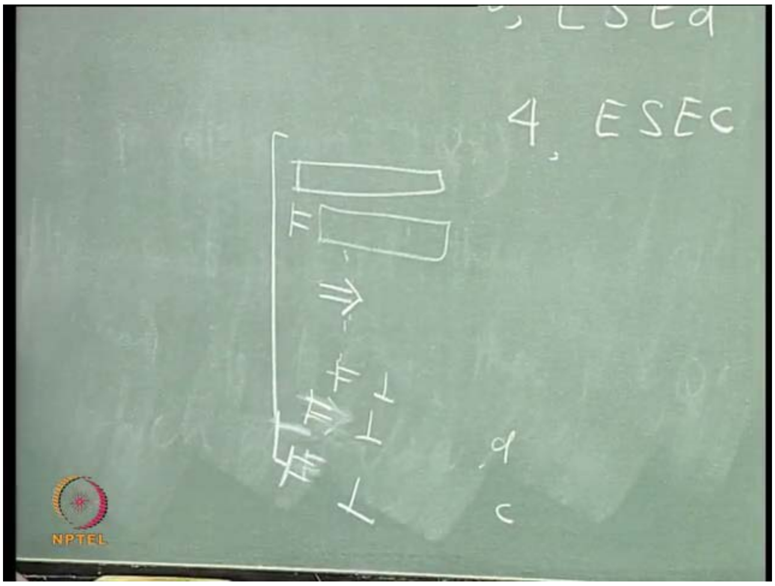

Here, the structure will look like: in a calculation, it will start something like, all the premises; then you have entails, proceeds, somewhere you have one implies symbol will come with indentation; it is over you, get some entails bottom; then next line you have to close the indentation with bottom. Either implies or even entailment, does not matter now. Once more, this is for the d, d block then for this c; again you have to go for entails bottom, once more. There will be two nested loops with c here; that is how it will look. So, one indentation here, one indentation here, which should have been, somewhere it is introduced. This will be for c, another will be for d similarly, somewhere. That is how the structure will look.

Now, just like you are earlier things in propositional logic, suppose you go for a normal form; how do you proceed? In propositional logic, we had a normal form which are helpful for proving the completeness of the calculations. We proved that, completeness of the calculations by converting to normal form, fine. In first order logic how do you introduce a normal form?

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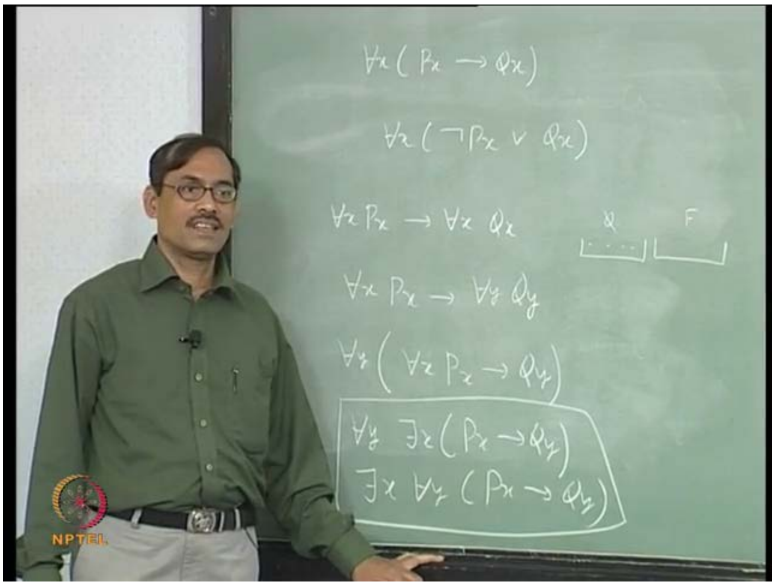

Like, you wanted to be in the form of conjunctive conjunction of disjunctive clauses or disjunctive of conjunctive clauses. So, what will be a clause? Any formula? Where are the quantifiers?

For example, let us take this: for each x Px implies Qx. One, you may think: for each x, I have not Px or Qx. In another case, I say for each x Px implies for each x Qx. Here, if I apply some propositional tautology, I would get not for each x Px or Qx. Now, there is some problem in this form and in this form. It is not clear whether we should accept mixed form like this or not. There is one nice way to go about it. The nice way is, you take any formula in propositional, in first order logic, you can always bring it to a form, where all the quantifiers will be in the beginning. It will be something like this. Can you bring it, for example, this one, it is already there. Now, what about this? Yes? Now, how to bring all the quantifiers to the beginning?

Well, first thing is, you may need to be introducing different variables; using renaming. You have used renaming. After renaming? For all y, I will bring here; it is not done. Because what we want is there should be one place where all the quantifiers will stay, and then from there onwards, one formula without quantifiers. It is not in that form still, and there is some symbol which is before a quantifier; there should not be any other symbol, only quantifiers using their variables, that we accept. Then what should we do?

Yeah? This is a formula where x does not occur. So, you can use the distributive law, fine. It will be: for each y there is x, Px implies Qy. This becomes so, because you look at this in the, this form; it will look, not for each x Px or Qy. Not for each x Px is same thing as there is x not Px or Qy. So, that is there is x goes out now. Not Px or Qy, which is Px implies Qy. But you do not have to do all those things if you remember the correct distribution law. This is how it will be coming. But also you could have done some other way.

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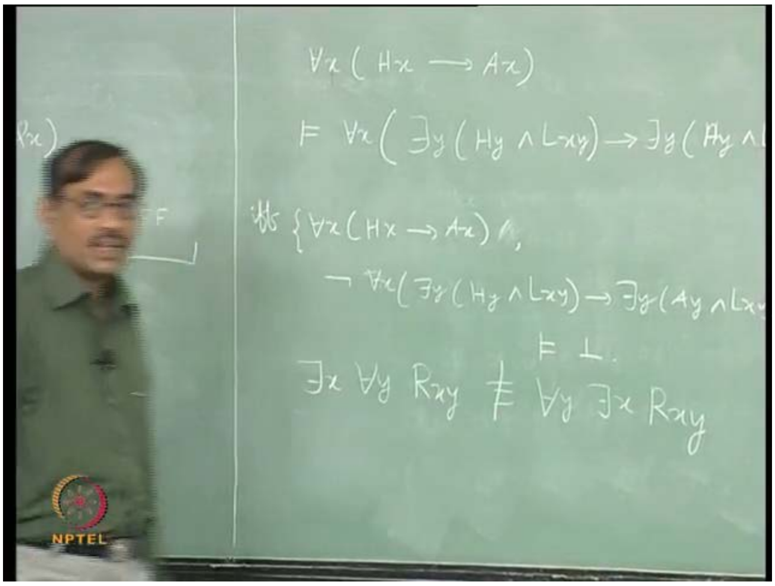

That, you can bring to there is x for each y, Px implies Qy also; to this, because I do not take for all y now; I keep it there, first I pull this. That gives there is x Px implies for each y Qy. Now, I pull for all y. So, it will be something like this. Are they same or not? Well, this is an exercise: find out they are equivalent or not.

Student: May not be equivalent.

But you are getting through equivalences only.

Student: So, each of them is individually not,

But each of them is equivalent to for each x Px implies for each x Qx. So, they should be equivalent. It is true that this is not correct. This is what you are telling. If you take there is x for each x Rxy, it is not equivalent to for each y there is Rx. But it is not of the form Rxy. So, think about it.