Mathematical Logic
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Lecture - 30
Quantifier Laws and Consequences

We were discussing the four quantifier laws. Out of that, we have really completed two of them, the simple one was universal specification, another was existential generalization. Then we have to discuss the other two. One of them, the universal generalization, we have discussed. But proof was informal; we have not proved it completely. Next, our job is to formulate existential specification itself. Formulation for universal generalization was, let us rewrite it again.

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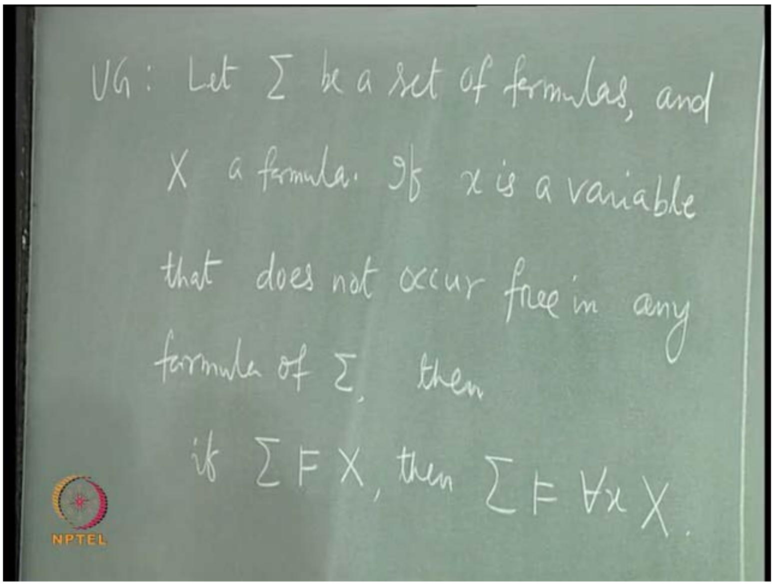

We start with a set of formulas and X a formula. Then from X we want to go for every x X, that was the point. Our formulation was if x is a variable that does not occur free in any formula of sigma, then, we had from sigma entails X, we can go for sigma entails for each x X; that was our universal generalization. All that we need is if x is not free in sigma in the premises, if x is not free in the premises, then we can go for generalizing over that particular variable x, that is the crucial point. Then we wanted to formulate existential specification.

Here again, our motivation was in arguments when you have one premise, for example, there is x Px, you use Pc, where c is one ambiguous name. We do not know exactly to which element in your domain it refers, but it can refer to one of the elements, that much we know. So, it is an ambiguous name. And then you just write Pc. Now, from Pc and along with other premises, we conclude that something, say Qc. Then from Qc generalize or even if there is no c at all occurring in the conclusion. Then, whatever you had concluded that follows from the original. That was our point. Somehow this ambiguous name is eliminated in the proof. That means, suppose you have sigma as the set of premises. Then you have sigma along with there is x Px. From there you conclude some Y, then you say that from sigma and there is x Px, also follows Y, is it clear? The whole proof methodology goes like this: you have there is x Px, but you cannot say that there is x Px entails Pc. You start with sigma along with Pc, infer Y. From this you conclude that sigma along with there is x Px also gives you Y.

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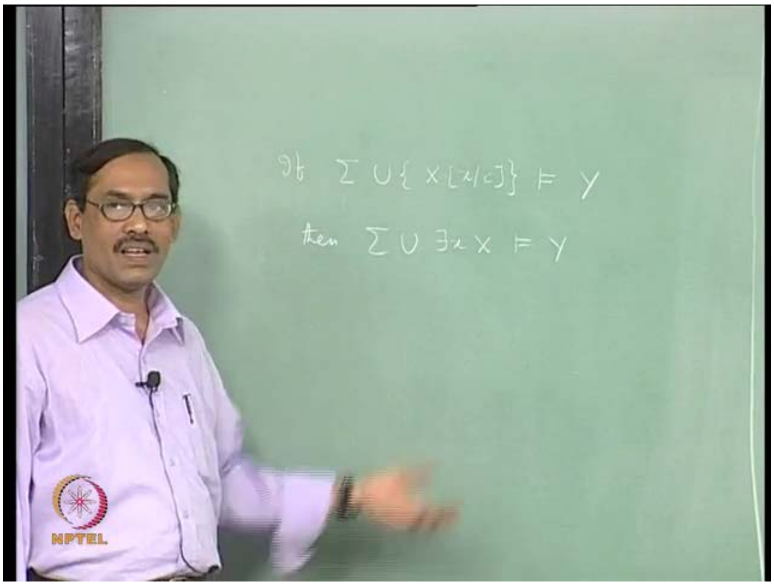

That means, it would look something, the conclusion of existential specification would look something like this, sigma union some formula entails Y then sigma union there is x X entails Y. Along with the premises, you are really starting from this, you are starting from this in your proof, but you would know what there is x X will give, the element, you do not know exactly. So, you just take one ambiguous name c and instantiate this. Now from this, along with the premises, there follows Y. Therefore, you conclude from this, follows Y. It is a bit complex in this formulation. But while using, this is what we do. So, let us formulate it. Here, what are the constants we need? That, this constant c should not occur in sigma. Suppose, you have already told Socrates is something, now you say there exists a man with this property; can you tell Socrates is also that having that property? We do not know, unless anything specific is told about Socrates. So, this c should not have occurred in sigma and then our proof procedure says that the c has been eliminated in the proof somewhere, so that we get Y. That means c also does not occur in Y. There is also one more constraint. Suppose, in X itself originally, there is c. For example, Pxc, there is x Pxc. We cannot say there is x Pxc will give you Pcc. That c can be different now.

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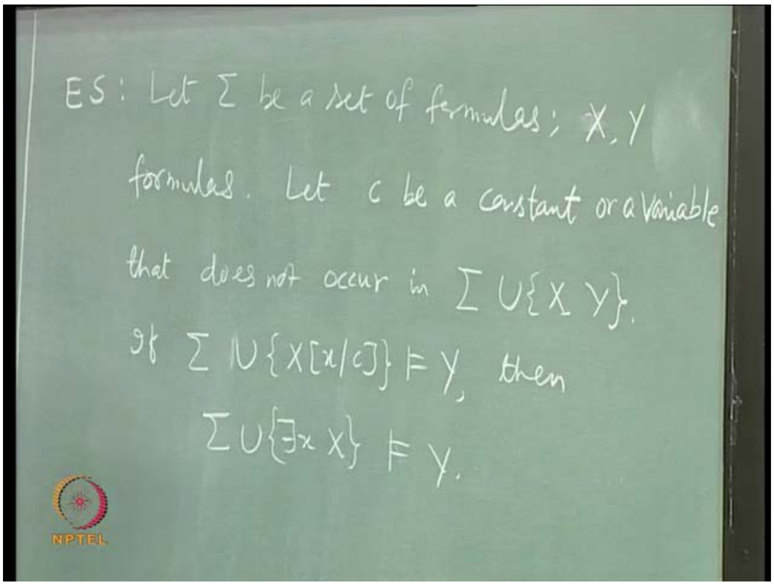

So, that means, this constant is completely new; it neither occurs in any formula of sigma nor in X, nor in Y. Now, you can formulate by writing these conditions. Let sigma be a set of formulas; X, Y formulas. Let c be a constant that does not occur in any of sigma or even in X or Y. Now, if sigma union X, X by c entails Y, then sigma union there is x X entails Y.

This constant, anyway, is being eliminated. Instead of a constant we can also take a variable, is that? It is just a symbol, that symbol should not have occurred; that is the main point. You may have also a variable. Once it is a variable, and you say the variable does not occur; it means the variable does not occur free, due to renaming. If that variable occurs bound you can also rename it. Equivalently. It will amount to a variable that does not occur free; both will be acceptable. Let us now prove them. This proof of universal generalization, we have already discussed.

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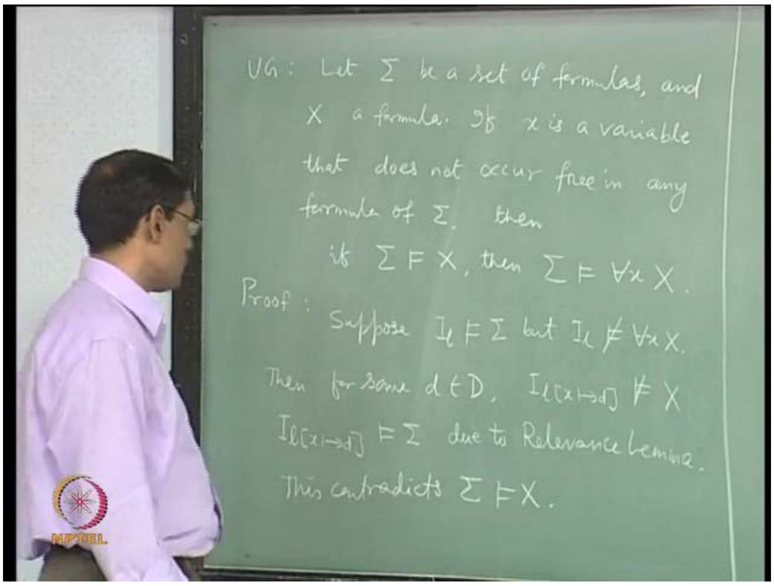

Let us give a formal proof now. So, how do you proceed? Let us give it by contradiction. It may be quicker. Let us. We want to prove sigma entails for each x X. So, we will start with one state, which is a state model of sigma, and try to show that, that becomes a state model of for each x X; by contradiction. If you want we take one state model of this which is not a state model of this.

So, suppose I l is a state which satisfies sigma, but I l falsifies for each x X. Once I l falsifies, it means there exists one element in the domain of the interpretation under which I l is a state such that, such that I l x fixed to d does not satisfy X. You want to, falsifies; that means then, for some element d; I am not writing it completely; this I l, assumption is I equal to D, phi; I l is one valuation under that. For some element d in D, I l x fixed to d falsifies X, because when you come to I l satisfies for each x X, that will mean for each element d in D, I l x fixed to d satisfies x. That is not, that does not hold. So, there is at least one element, for some element, I l x fixed to d does not satisfy X. Just from our formal semantics, where we have introduced the semantics, see, we have to use this sigma entail X somewhere. Once you say I l satisfies sigma, is the same thing as telling I l x fixed to d satisfies sigma. Because x does not occur, does not occur free in any formula of sigma, x itself does not occur. That is your relevance lemma, right? If x does not occur, then whatever value fixed to it, it does not matter; whether it satisfies or not will be fixed with the original valuation. So, you see now that I l x fixed to d satisfies sigma due to relevance Lemma.

Otherwise, you can see what is its effect. l and l x fixed to d work the same way, and sigma that is what you want to see. x does not occur and l x fixed to d agree on every variable except to x, x does not occur. So, this satisfies or I l satisfies are the same. That is exactly relevance lemma. Now, you see I l x to d is a model of sigma, state model of sigma, but it falsifies it; this contradicts the assumption. This contradicts sigma entails X. That is it. In short, proof comes because of proof by contradiction.

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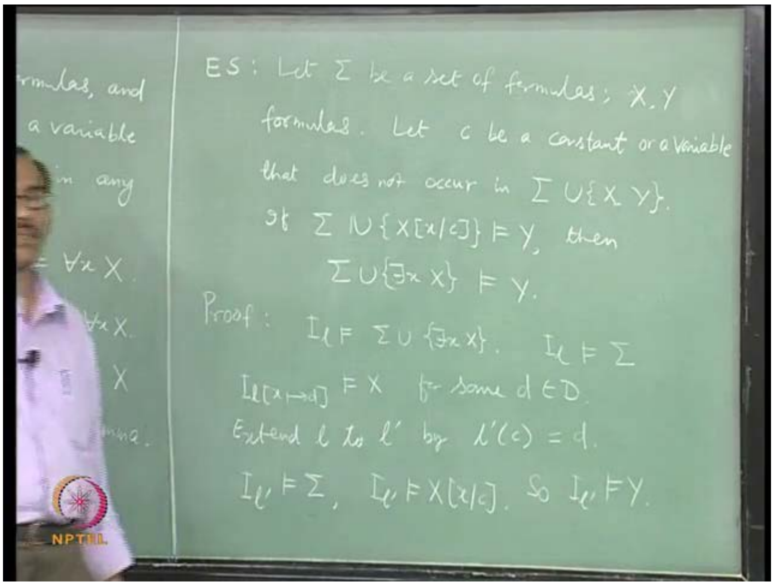

Proof for existential specification is also similar. And here, all that we have done is x actually occurs in capital X; otherwise x to d will be vacuous. If x does not occur, then X or for each x X are equivalent by empty quantification. There is nothing to prove. So, we leave that uninteresting case. Similarly here, if your x does not occur in X, so that also becomes vacuous; there is nothing to do. This and this are the same; both of them are X only equivalent to X.

So, how do you proceed? Here again, we want to show sigma union there is x X entails Y. We start with one state, which is a state model of sigma union there is x X. This means I l is a state model of sigma and I l is also a state model of there is x X. That means I l x fixed to d is a model of x for some d in D. This is the meaning of the existential quantifier there is x X, is that? We want to show this entails Y. To show that entails Y, we have to use this, because we already know this entails Y. So, sigma is, I l is there, but x X be c, that is to be same. Now, this c is new to whole of sigma, X, Y. c never occurred. Now, l never interprets c as it is. So what do we do? The relevance lemma will take the minimum things only. l is exactly appropriate to whatever is there, because c is not there. So, we can extend this l. It has to be interpreted. What do we do?

Extend l to l prime. Once you say, extend, it means whatever l assigns to variables and constants, all the terms, they have been kept as it is. We are giving some extra, fine, and calling it l prime. So, l prime, by taking l prime of c equal to this d, because you want this to satisfy. This formula should be satisfied. We have already something like that. So, we take l of c, l prime of c equal to d itself. Is that? Now, what do we see? I l satisfies sigma, it is something as telling I l prime satisfies sigma; l and l prime agree on everything except c, possibly, and c does not occur in sigma. That is again relevance lemma. So, what we see is, I l prime satisfies sigma. Now, I l x fixed to d satisfies X. Suppose it is like Px. I l x fixed to d satisfies Px. That means, when you substitute x, in terms of d, it says d belongs to phi of P. That relation d belongs to phi of P. Look at this: l prime satisfies this, means what? In Px, you have substituted x as c; so it is Pc. Now, l prime of c equal to d. So, this also says d belongs to phi of P; is that? Because l prime of c is the same thing as l of x fixed to d, there evaluated the same way. So what we see is, I, l prime satisfies X x by c.

Now, since sigma union x X by c entails Y, here is one state model of sigma union x X by c. That state model should also be a state model of Y; that should be a state model of Y due to this. So, you conclude that I l prime satisfies Y. Now, I l prime satisfies Y; l prime and l agree on all the variables, constants, except possibly c; and c does not occur in Y. So, again due to relevance lemma, I l satisfies Y. That is what we wanted to show.

So, you start with one valuation l; you get the state I l; then I l satisfies there is x X. Therefore, you have one element d such that I l x fixed to d satisfies x. Now, if you take, define, your l prime such that l prime of c equal to d. Then, it is equivalent to telling I l prime satisfies x X by c. Then use sigma union x X by c entails Y to conclude I l prime satisfies Y. Since c does not occur in Y, I l prime or I l satisfy or do not satisfy, they agree. So, it is equivalent to telling I l satisfies Y; that is what we wanted.

Now that all the four quantifier laws are proved, you need not even go through all the other laws. You can prove them by using these quantifier laws. That should be easier. But there is a hitch. It is this: existential specification is not in its full generality; it is not very general. Why? For example, you say for each x there is y Pxy.

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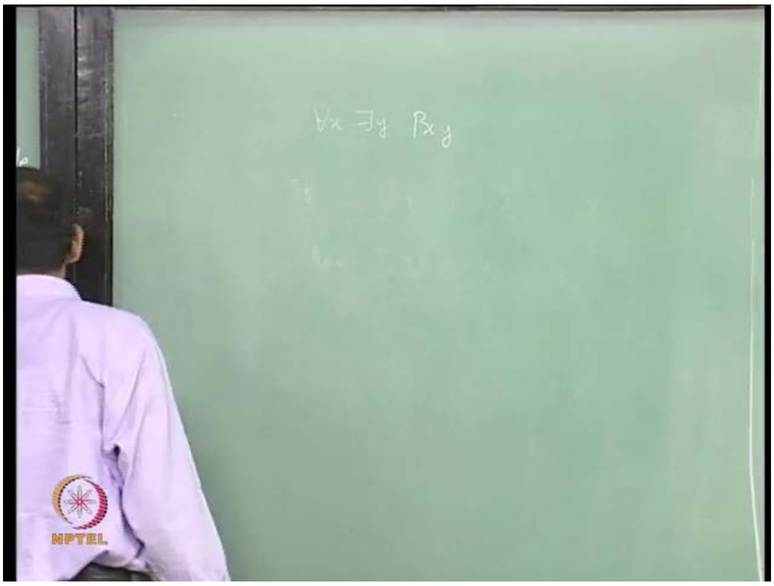

Here, to capture this existential generalization, existential specification, you need really y to be depending on x. You chose one x, for that x you get some y such that Pxy holds. You chose another x, it need not be the same y, for the new x. So, y is really depending on x. But this form of the existential specification says that we will be using the same c for everything, though it is not same. It is ambiguous; it can vary. But it does not show up this dependence; it should be shown somehow. We will do that later. We will wait for some time, to come to this generality. Now, for our purpose this is enough.

Let us see some examples. How to tackle the consequences by using these quantifier laws? Let us take one easy example. Say, show that this sentence is valid. That should be easy. How do you propose to go? Say, look at the inside of this quantifier for each x; think of not P implies Q implies not Q implies P; that is a propositional tautology, contradiction and double negation. From a tautology, we should get it. But you get, what you get? Only not Px implies Qx implies not Qx implies Px. That is all. Then universally generalize, right, to reach x, for each x. So, you start, say, this way. You say, top. It is equivalent to not Px implies Qx implies not Qx implies Px.

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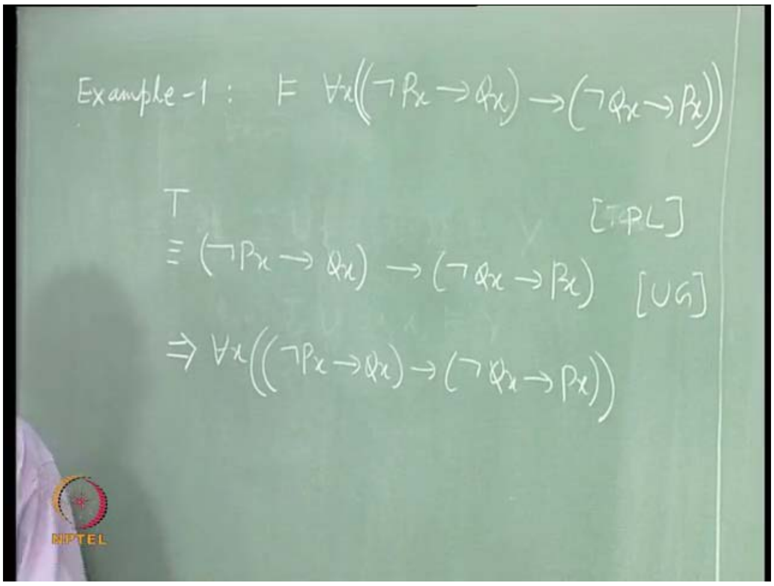

So, justification is tautology. Or, we will write just PL, for propositional logic. Then what do you want? To say, from this will follow not Px implies Qx implies not Qx implies Px. And you want to use universal generalization. What will you write here? In a calculation, it is not equivalent; this side entails this, but from this does not entail Px, does not entail for each x Px. Then, what you want to use? Universal generalization. Which says if x is a variable that does not occur free in any formula of sigma, in the premises, then you can conclude.

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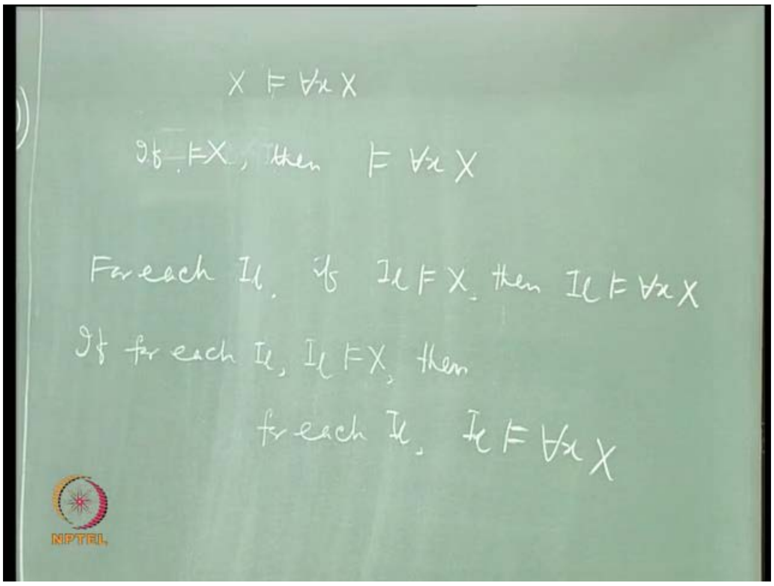

Now you see, this x on which you are quantifying universally, quantifying, that does not occur in the premises; there is only one premise here. So, you can use universal generalization. But then, what to write there? What symbol we will write? We will devise one symbol. We will write here implies, because this is not entailment. So, what is that we are really writing? It, here, it is a meta-sentence: if from all these things X follows, then from all those things for each x X follows. That is what we are writing here. It is not X entails for each x X.

What is the difference between them? You have X entails for each x X; another is sigma union X, or let us forget sigma, take it that way. Say, if X is valid, then for each x X is valid. If you have sigma, you will write sigma here, sigma here, sigma here, sigma here. Let us look at these two sentences. This is a meta-sentence, this is also a meta-sentence, but it is of different level, still a higher level, it involves talking about the entailment itself. This talks about validity, this talks about entailment itself. It is still a meta-sentence. It talks about entailment relation itself. Now, what does it say?

This says X entails for each x X; means, whatever state you take, if that is a state model of X, then that is a state model of for each x X. Or, you may say, for each I l if I l satisfies X, then I l satisfies. What does this say? Now, if then and this for each are interchanged. This says: if for each I l, I l satisfies X, then, for each I l, I l satisfies for each x X.

There is some difference. This can be vacuously true if there is one state which falsifies it. For example, I say if Px is valid. Then X is valid. That is true because Px is not valid. It can be vacuous. But this one is stronger. If this holds, then this will hold; not conversely. Now, in the calculation we want to write this sentence: if sigma entails X, then sigma entails for each x X. It is not entailment; it is weaker than entailment; it is stronger. That is why you write some other symbol. Let us call it implies. Write it as implies in mathematics. This is the symbol we always use. It means exactly this. If from all those premises something follows, then from all those premises, then the other thing follows. It is not really entailment; implies in mathematics.

Now, what happens is, it says, it is only connecting the previous line as per other symbols in the calculation. In a calculation we can use equivalence, we can use entailment. These two symbols connect to the previous line always, right? But this symbol implies does not connect only to the previous line; it connects the whole proof before it; whole calculation that has gone before it. If all those things hold then the next one also holds; that is the difference. Therefore, you can write this with an indentation. If it continues up to the last line, it does not matter. It is unlikely. But if it is not the last line, you need another sub-calculation here. For example, while applying existential specification something else you are assuming here; which is not given, you require to prove this.

Now, in this context X, X by c becomes an extra premise, right? Of course, it has connection with this. But it is an extra; it does not appear in the premises. Now, this one, you assume, continue to find Y, and then you conclude this. That proof for sub-calculation itself will slightly be pushed to the right; make one indentation to remind that it is a sub-calculation we are entering; after this will come the original.

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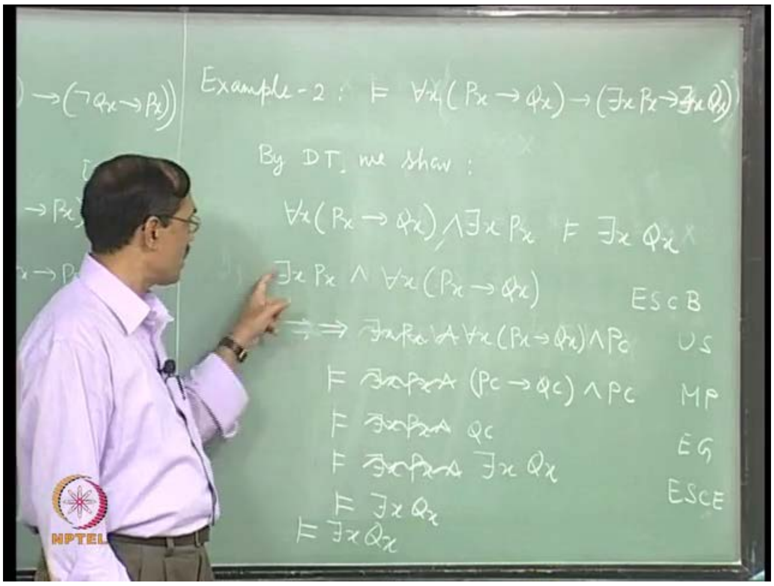

We will see an example. How it is used. Say, so this, we want to prove. Now, you see this X can be out of this, because here it is a sentence. That itself, where x is not free. So, you could have written this x outside also; does not matter. Which means we can take, remove this bracket; that is what it says. Suppose you want to prove it by calculation. Then first, we will be using deduction theorem. Because it is a serial implication. Something implies B implies c. You say, by deduction theorem, we show, due to deduction theorem it is enough to show: for each x Px implies Qx there is x Px entails there is x Qx, which is same thing as taking and here also.

Till now it is propositional; only deduction theorem. Then we have to give a proof of this. How do you go? Where to start with? You may be starting with there is x Px, because that c, I can use it here to specify for each. Let us write this way. There is x Px and for each x Px implies Qx. I start from these premises. Now, you want to use existential specification. I write Pc, but I cannot write Pc, because from there is x Px, Pc does not follow. But I can use Pc. So, extra premises I have to take. Then I say that my existential specification starts here with this c; begins with this new constant c.

I say, this implies. Let me indent it here, implies. I can add there is x Px or for each x Px implies Qx and Pc, which is something like in your deduction theorem, in informal proofs you take one other premise; then finally, you finish. Come to that. That added premise implies your conclusion. So, this is an added premise, I am taking. Now, with sigma, I take Pc. Now, sigma here does not include there is x Px. If it is really confusing, you may not start, even, with there is x Px. Start with Pc itself. But that will not give you any hint, why you are starting with Pc. So, let us keep it as it is. Now then, what we do? We go for universal specification here. So, write here this entails there is x X; and this, we write Pc implies Qc and Pc.

So, this universal specification we are using by taking x as c; x is substituted by c here. This one is equivalent to writing c is new. I am going to include your c; which is a new constant. Now, from these two, I will use modus ponens to get there is x Px and Qc, is that right? From Qc, we will go for existential generalization. So, this gives there is x Px and there is x Qx from Qc. Now, what it says, this new formula what you have got, has no c. There is no c here. Since, there is no c here, our original formulation of existential specification says, if there is no c finally, which is our Y, now that Y is there is x Qx here; then you can say sigma union there is x X entails Y.

So, this gives you entails. You can say there is x Qx. This is, and anyway, but this and is immaterial. We never use it anywhere at all. It will say that there is x Px and this one will entail this there is x Qx, is that so? What we do? Here, our existential specification ends. So, you repeat it. And we say entails there is x Qx; telling that it is existential specification with c ends here, how you are going. So, this indentation will tell us that from this step to this step, it is only a sub-calculation. You could have done it outside. Inside when doing it, there is x Px, also you can just start with Pc and conclude there is x Px; after that you say there is x Px; and this entails there is x Qx.

That is what we are going to do. That means, in every step, you even could have deleted this, there is x Px, because it is a sub-calculation. See, you start with Pc and then continue, find there is x Px. So, all it says, this is your sigma, sigma union Pc entails there is x Qx, therefore, sigma union there is x Px entails there is x Qx. Can you read the proof? Now, you have to read the theorem of existential specification. It says, from this indented sub-calculation, it says sigma union Pc entails there is x Qx. Therefore, sigma union there is x Px entails there is x Qx. Side by side, you can also give informal proofs. Now, you are matured; we can do both the things at a time.

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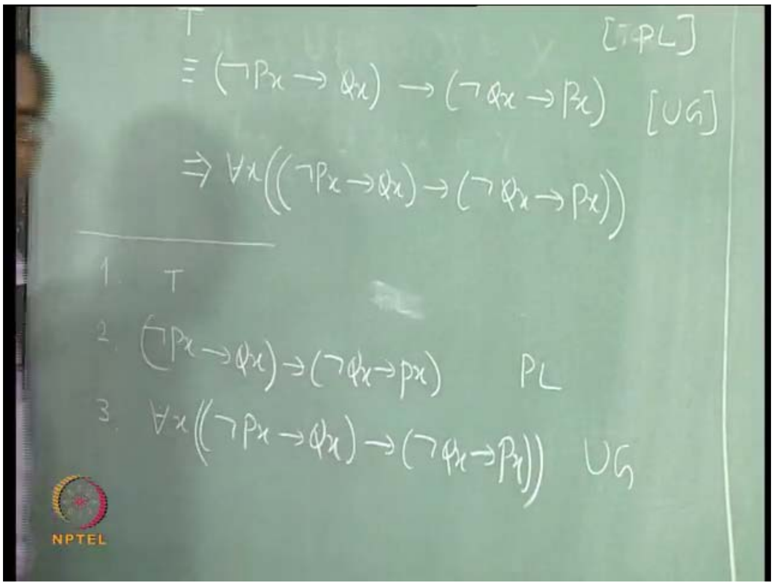

So, this one we could have started with top; and just rewriting it to see that is an informal proof. Then by PL it follows not Px implies Qx implies not Qx implies Px. Now, informal proofs, we will not start with this step; we can show it is valid. So, top entails this. That is how we are doing it. Top is this, then this justification will be written here instead. In calculation we write previous step. That is what we are going to do. It says, now here we are giving justification, not, we are going to do what we have done. Here we have simply used PL tautology. Then third one, we simply write for each x not Px implies Qx implies not Qx implies Px, by universal generalization.

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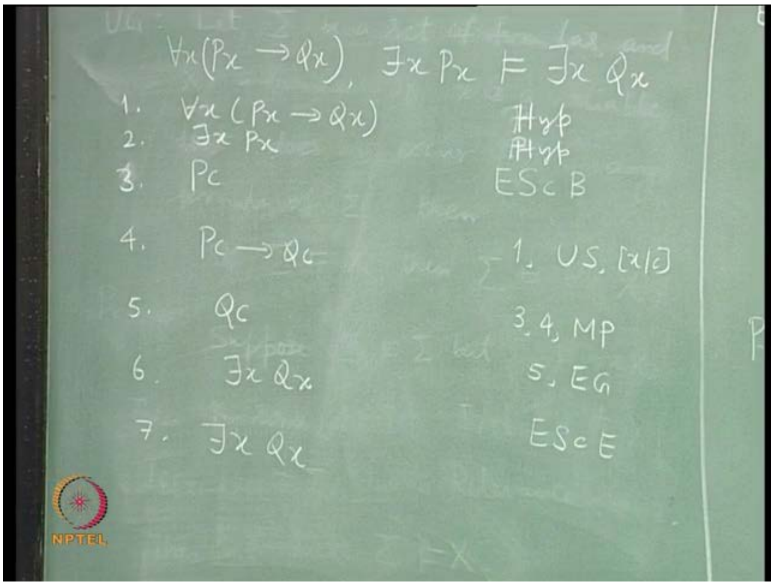

This is what we wanted to prove. Our proof says that we start with Pc; we say here justification as existential specification with c begins, which can be one extra assumption. Before this, if we want to list all our premises, we can keep it; informal proof allows that. So, its wiser to start with all the premises listed. Instead of this you start with one, for each x, Px implies Qx; two, there is x Px; all these are premises. Or, let us write hypothesis rather than premises to not confuse with our P. Yes?

Student: Sir, this is informal because you are writing Pc.

Informal proof has some formality; it does not use the symbols entailment or equivalence symbols.

Student: That is also not completely formal.

Conclusion is also not formal; but conclusion can be made formal. Similarly, informal proofs; they are just different styles. Problem is, in calculation you have to connect with the previous lines, in PL. Informal proof has some more informality. We can just use any premise anywhere, does not matter; that is the difference. Here in calculation, you can show, what is equivalent to what, what is entails from where, and so on. In informal proof it is not equivalent; it is only entailment; that is another difference.

Let us see, how does it proceed. This will be our third line. Here it starts, because there is x Px is there, we start with Pc. It does not follow from it. So, you have to write it specifically there. It does not follow, but it is one extra premise you are adding due to existential specification. Then, we instantiate; from one, it says Pc implies Qc, one, universal specification. If you want to mention how you specified, you may write x by c, x has been substitute by c. This to say what is the specification. Then we use modus ponens as earlier. So, three, four modus ponens. And then, six, there is x Qx from five, existential generalization. Now, you get a formula Y which is independent of this ambiguous name or new constant c. Therefore, we say, once more we have to write, there is x Qx; it follows from the premises, where your existential specification ends, is that okey? That is how it will be proceeding in informal proofs.

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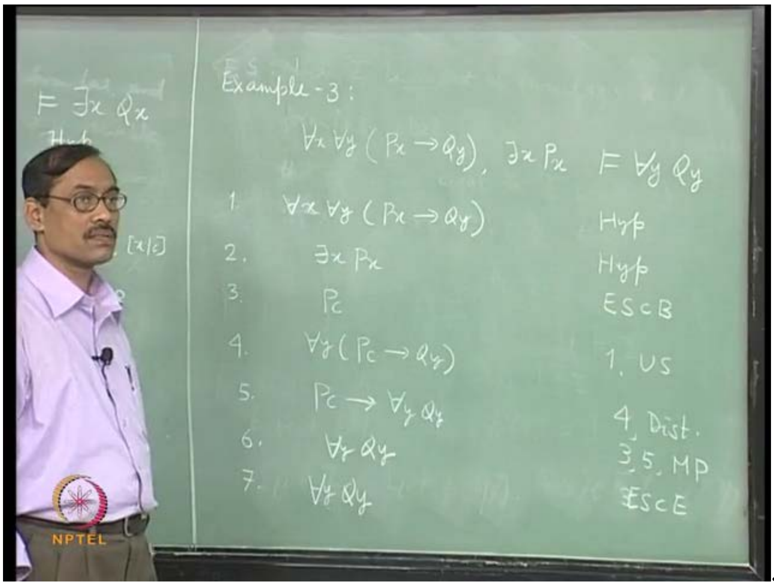

Let us take one more example. Let us see whether this has a proof or not. Can you try informal proof first? Let us see. What is our plan? We want for each y Qy. So, you should be able to get Qy from somewhere. How to get Qy? From this anyway. This will come from there is x Px. So I can have Pc. Then I can instantiate c. With this, I get for each y Pc implies Qy. Then? Pc is not having y. So, I can use my distributive law; take that out, I get for each y Qy directly. I do not need universal generalization here. Also if you do not want, you can instantiate that y. If you do not want distribution, instantiate that y, get Pc; apply MP, get Qy, for each y Qy.

Let us write it. We start with this one. Premises. It is a premise. Second, there is x Px is a premise. Next, I want to use existential specification. So, I say Pc. Fourth, for each y Pc implies Qy, from one; universal specification. Then I say Pc implies for each y Qy, distribution. Four, distribution. Next, Pc and this gives for each y Qy. Three, five, modus ponens. Next? Since you got it, you want to stop it here. If you stop, it will say for each x for each y Px implies Qy and P c gives for each y Qy, because this step we have used an extra premse; it has to go. So, next is for each y Qy. We will write Pxc. Now, in order that this is concluded, you should have the premise that there x Px. If it is not a premise, then it cannot be concluded. Do you see that? So, along with this, better write what premise we have used. Similarly, here EScE, you are using that as a premise: there is x Px. But in a calculation, anyway you have to see it without referring it back. In an informal proof, you can read it, where it is really used.

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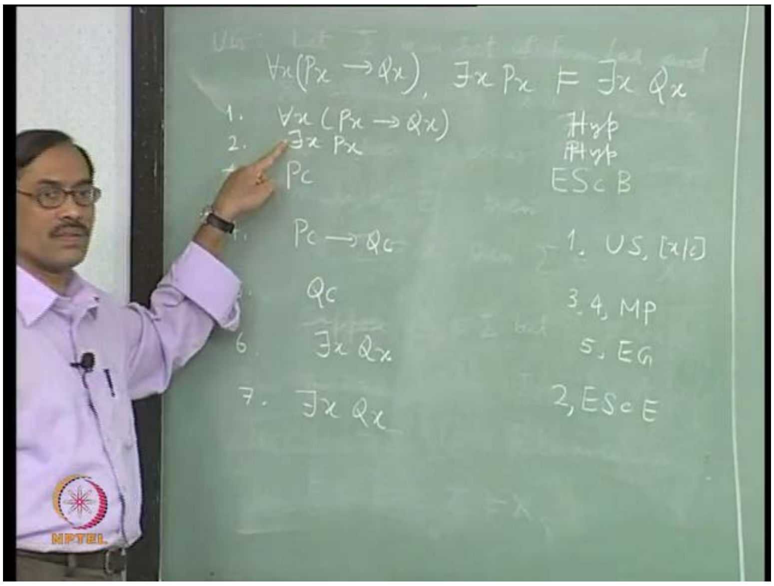

For example, here, when you write there is x Qx, better use two; you are using that as a premise; that is why it is followed. Otherwise, it does not follow. Is it clear? Here, for example, you have for each y Qy. This follows with Pc as a premise. Now, when you use existential specification with Pc and this to conclude there is x Px; with this, give you for each y Qy. That means there is x Px is used as a premise. Yes?

Student: For each x Px used for three itself.

No.

Student: You say Pc.

It is not used, because there is x Px never entails P c.

Student: You have used.

That is the whole part in the existential specification itself; otherwise you would not need existential specification in that form. Since it is there, we are taking this as the extra premise; here it is extra information, it does not entail. All it says is, one along with three, gives you six, therefore one along with two, gives you seven; that is existential specification. Is that clear? So, you mention two, ESc.

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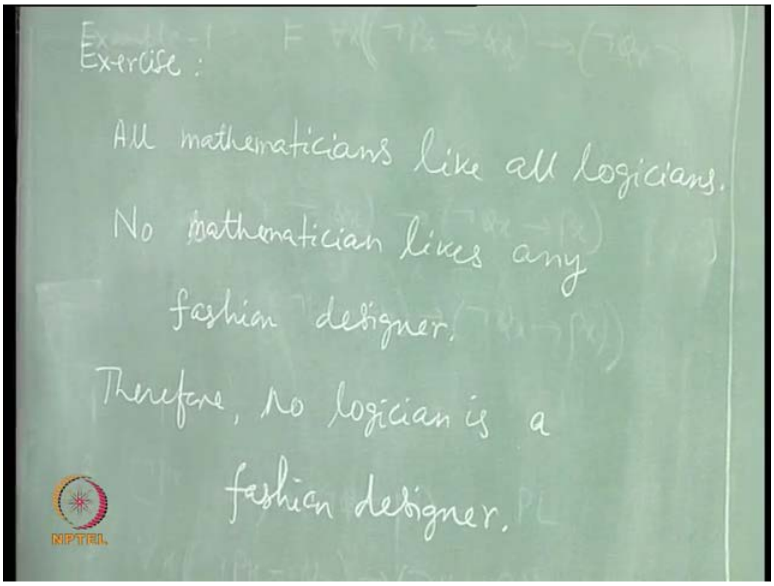

First symbolize this, and then try to give a proof, whether you can give proof or not? So, all that we have done today is use the laws, this four laws and earlier one. There is one thing you can try in the proof itself. For the last one, instead of using distribution here, try to use the laws; these quantifier laws, and still conclude it; which will give you some confidence that "I do not need to remember so many laws". These four quantifier laws along with the PL laws will be enough.