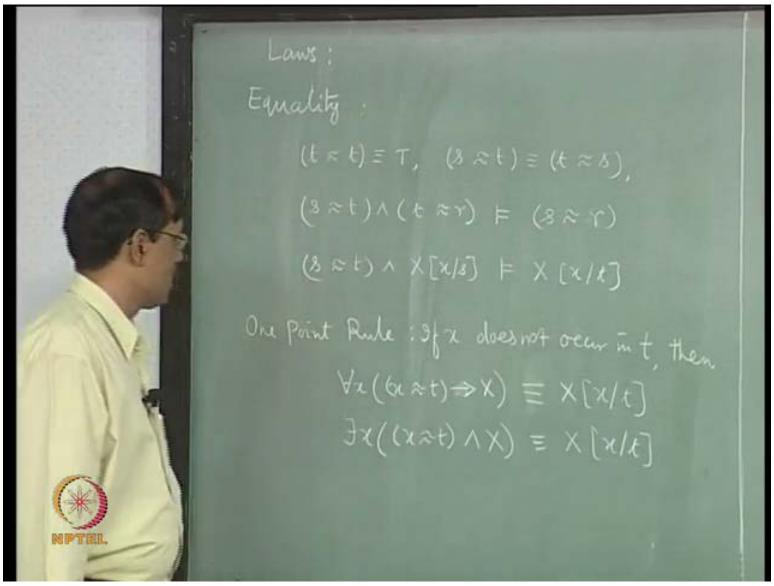
Mathematical Logic
Prof. Arindama Singh
Department of Mathematics
Indian Institute of Technology, Madras

Lecture - 29
Laws in FL

So, we had discussed six theorems. Three were almost propositional like monotonicity reductio ad absurdum and deduction theorem. Then other three were the substitution theorems. One is for the equivalence substitution, then the other is uniform substitution in a tautology, and the last one was uniform substitution in valid formulas. This requires first of all to have some valid formulas. If you have only some propositional validity, then of course, first uniform substitution in tautology itself does it. But then we need some more valid formulas for the application of the other theorem. We can get some valid formulas, clearly from the semantics. Just by using the semantics very simply, for example, you take the equality laws, they can be seen very quickly.

(Refer Slide Time: 01:08)

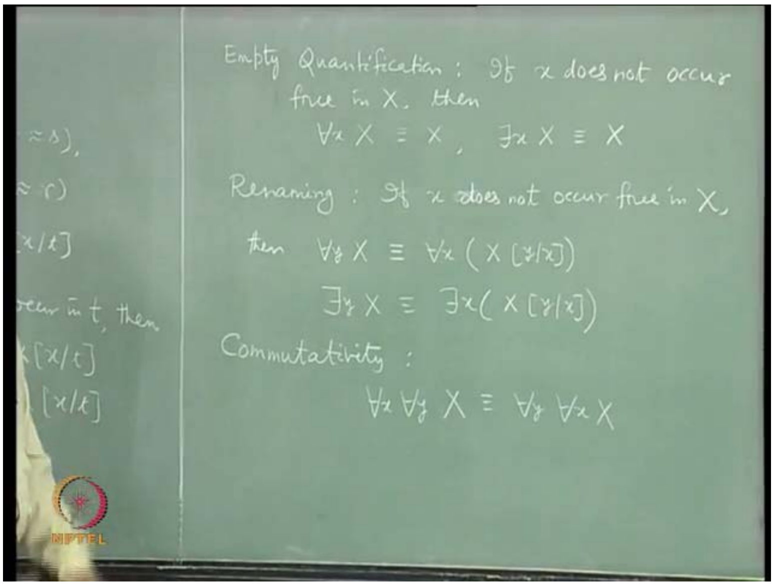


In laws of equality, we will have say, t equal to t; this is valid, right. This is obvious. Because equality symbol is interpreted as the equality in the domain. Some such simple things can be seen very quickly. For example, s equal to t is equivalent to t equal to s. There is nothing more to do. And similarly, s equal to t and t equal to r entails s equal to r. But equality, these are all the equivalence relations. It is both. Simply equality behaves like an equivalence relation. There is one more, which is required for the equality in the domain, which is its capability of being substituted. There, you have s equal to t and X x by s. This entails x by t, is it clear? All that we wanted here is, you have a formula where you have some occurrences of s. Then you replace all those occurrences of s by the term t. Then also you get, the new one is obtained from the earlier, that is what it says. That is also obvious.

Similarly, you can have some others, say, one point rule. Here, we are evaluating x at one point; it looks like that. Suppose x does not occur in t. Then you can have say, for each x, x equal to t and X. That entails X x by t, is it clear? What it says, even we can say for equivalence, why only entailment? Once you take the term x equal to t and you do not have, and it should be implies, for each x. From these, if you get X, so what happens, x equal to t, then you get X. That means to prove this X itself you have already substituted x by t. That is what it says. There is nothing about it. But this can be proved semantically. Easily, by taking a state model of left and the other one the right one.

Similarly, if you have, there is x you will have the symbol instead of implies here. This is also equivalent to X x by t. In all these laws, when you state, we assume that whenever X x by t, expression appears, the variable x is free for the term t in the formula X. That is assumed. We will not mention it once again. Well, this is about one point rule.

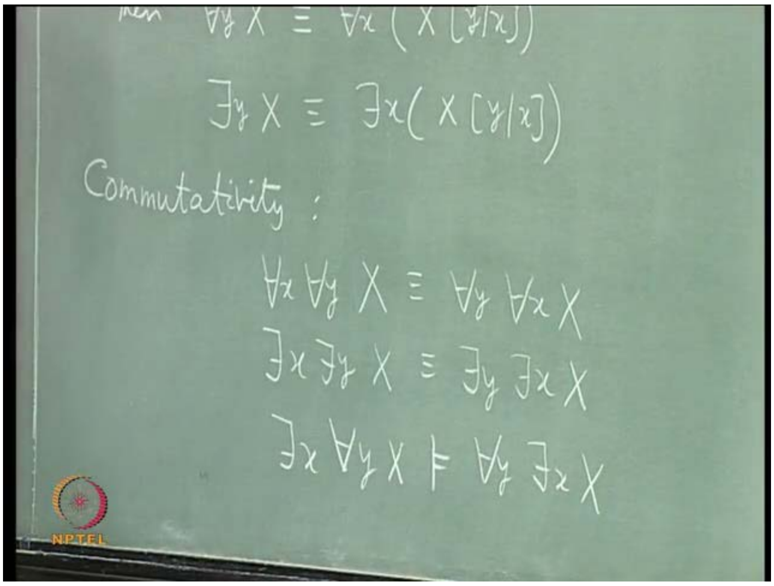
(Refer Slide Time: 05:19)



You can have some other laws, like empty quantification. It says that if a variable does not occur in the formula, then you can quantify happily without changing anything. This says, for each x, X should be equivalent to X itself. When, if the variable x does not occur free in X. Let us write this. And also for the existential quantifier, which will same thing as also X be small x substituted by t. Because vacuously, nothing will be replaced, right, small x is not free in X. Even if you write X x by t; that is same as X; it is equal to X.

And similarly, you have renaming of the variable. Which says that if you have some formulas say for each x Px, then that is equivalent to for each y Py. This is what renaming of variables means; a bound variables will be renamed. So, how do we write it? It would look something this way: for each y X will be same as for each x, in X replace y by x. But one thing, you have to take care that this variable is not getting captured, right? You should have if x occurs, does not occur free in X, then this should hold; fine. Also, same thing for the existential quantifier. In this symbolism, something is hidden. You have to read it that way: for each x Px, for each y Py is equivalent to for each x Px.

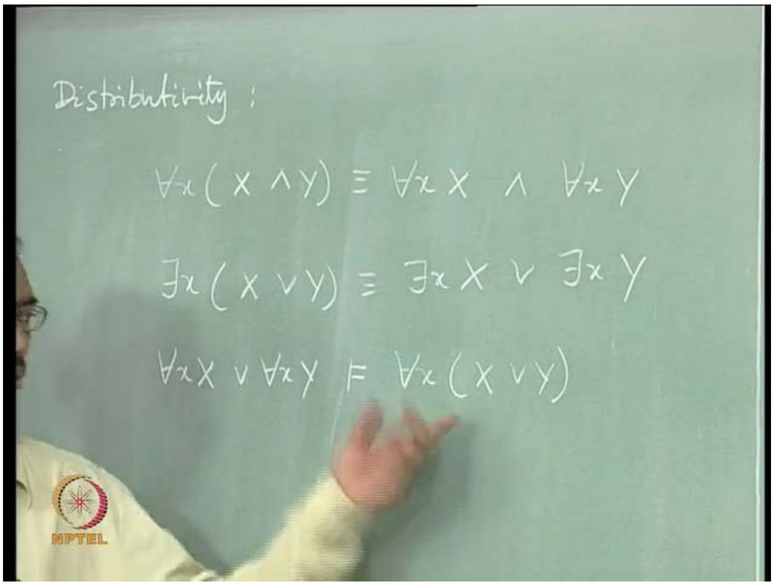
(Refer Slide Time: 08:00)



Then, you can have commutativity. Well, this says: for each x for each y and a new formula X, that is equivalent to for each y for each x; you can interchange the quantifiers. Similarly, for the existential. But both are same. That is why if you have mixed quantifiers, then something else can happen, right. You have only done this one:

there is x for each y X entails for each y there is x X. The converse does not hold; also we know, fine. That is, for each y there is x, X does not, in general, entail there is x for all y X. In some cases it can.

(Refer Slide Time: 09:06)

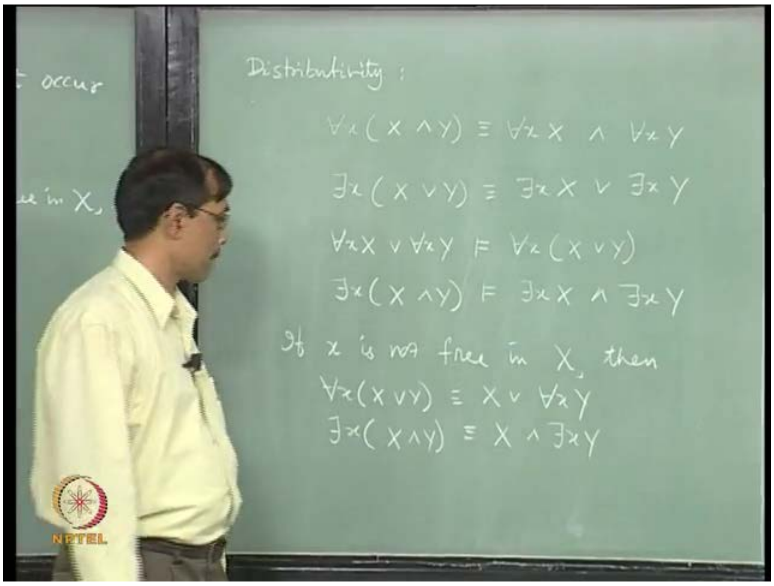


Then you have distributivity. This is a bit tricky. Let us see it. Here, you would get for each x, X and Y is equivalent to for each x X and for each x Y. Can you see why is it so? Is it clear? See, basic intention is that, when you have for each x, say, about the natural numbers, some Px and some Qx. It will say P1 and Q1, P2 and Q2, P3 and Q3; all and-ed together. So, you have to just separate it out. P1 and P2 and P3 and so on, and Q1 and Q2 and Q3 and so on. This is the basic intention.

But that, when it happens for or? We will see how it goes. With existence, similarly, or also. We have with the existence itself means, for at least one it holds, so or itself should distribute. Now, if you change this and to or, then this may not hold. But at least one side, it would hold. For each x X or for each x Y entails for each x X or Y; is it clear? Because once you say this or this at least one of these is true, in any state. So, take, in that state; suppose this one is true, then in the same state this is also true.

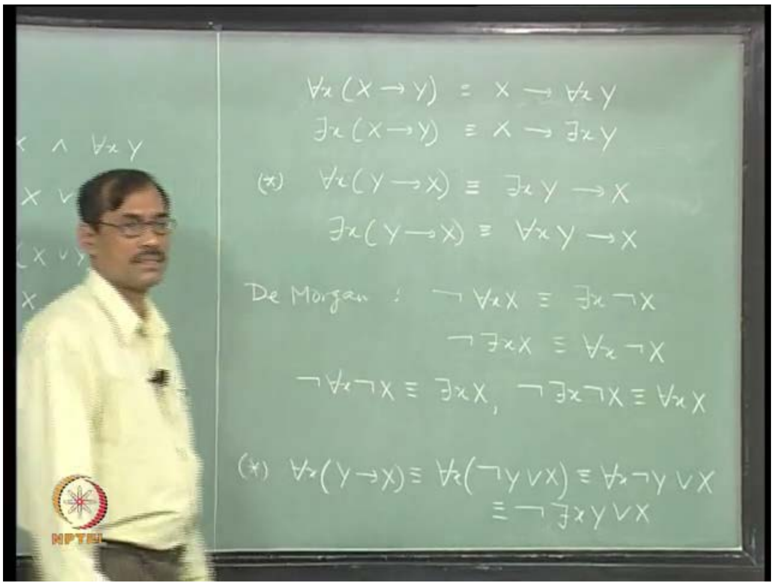
But converse will not hold. Can you give one example? It should be easy. We just take as X as Px, Y as not Px. You want to see that there is a counter example to this, fine. Our suggestion is take X as Px, Y as not Px. Then, if this is true, you would have done this. Now, interpret this as a sentence. When we interpret, what will happen? For each x, it will be, for each x in some domain D. Let us take natural numbers; P is the set of all primes. Now, this will say take any natural number, the natural number is either a prime or it is not a prime, which is true. On the right side, it says every natural number is a prime number or every natural number is not a prime number, is a non-prime number, right? Not Px. That is false. It says either every natural number is a prime number, which is false, zero is not a prime number; or every natural number is a non-prime number, which is also false, because two is a prime number; is that clear? So, this does not hold. But these hold. So, we have only one side here, entailment. Similarly, for there is x will entail this also; clear? The same Y. But converse? When it holds? Again, I think, the same technique will do. What do you want to show is, this does not hold. There is x Px and there is x not Px entails there is x Px and not Px. This says there is a prime number. This says there is a non-prime number, which are true, both are true. But the right side says there is a number which is both prime and non-prime, which is false. It is easy to see.

(Refer Slide Time: 14:05)



We have something on the conditionality. If x is a variable, which does not occur free in X, then you can have some other distributive laws. Let us take these cases first; entailment did not hold. X or Y is equivalent to X or for each x Y. We can see why it is so. Once you say x is not free in X, it means as if x is not at all occurring in X, because you have renaming. You can rename the variables to something else, fine. All the bound variables can be renamed. So, x is not occurring at all. Then it does not matter when you take it out. The same for each x, it will be renamed. That is what it says. If, just go to the details of the states, and it should follow immediately. Similarly, you have there is x X and Y; should be equivalent to X and there is x Y. Here, it is also equivalent to there is x X. Instead of X you can write there is x X, because of empty quantification; x does not occur at all. Does not matter, if anywhere you have there is x X in the beginning.

(Refer Slide Time: 15:51)



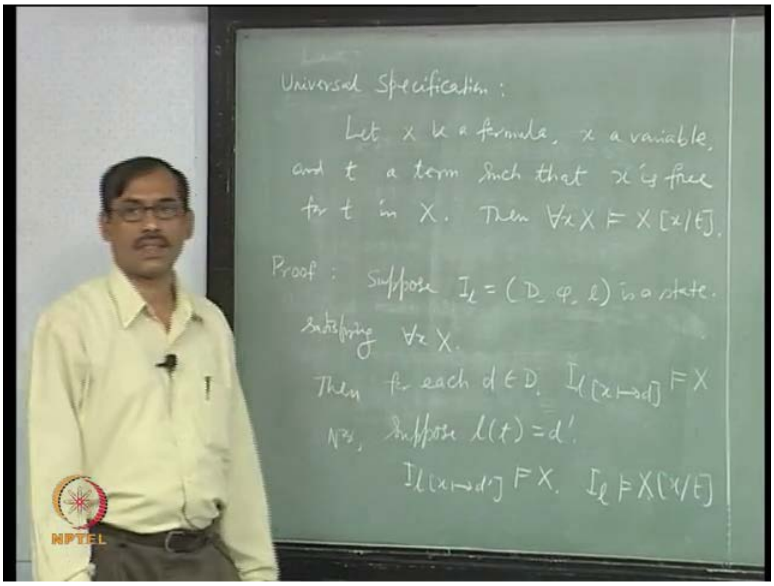
Similarly, on the other one also. You can extend it a bit, to some implication. Say, you have for each x X implies Y. Since x does not occur here, you can take X implies for each x Y. Similarly, there is x X implies Y is equivalent to X implies Y. There is x Y. What about the other type? Say, for each x Y implies X. Now, this variable x does not occur in X; that is what it says; it can occur in Y. This will not be, in general, equivalent to for each x Y implies X. It will be there is x Y implies X. Why it happens? You will realize shortly. Just, wait a minute; I will keep it star marked.

Similarly, there is x of Y implies X will be equivalent to for each x Y implies X. The key is De Morgan, which says not of for each x X is equivalent to there is x not X. That is clear from the translation itself, right? It is not the case that every man is mortal. Then you say, there is at least one man who is immortal; that is what it says. Similarly, you have not there is x X is equivalent to for each x not X; with the use of double negation. Along with it, we may get some more like not for each x not X is equivalent to there is x X.

Similarly, not there is x not X is equivalent to for each x. And if I use De Morgan here, in the star marked, this will be for each x Y implies X is equivalent to for each x not Y or X. Now, since x does not occur in X, you may write as for each x not Y or X. Now, for each x or Y is same thing as there is x Y, its negation. This is equivalent to not there is x Y or X; that is same thing as there is x Y implies X. Is it clear? It is changing its quantifier, is changing for each x to there is x, because there is a not here. Same thing is here. There is a not here, so that because for each. These are some of the laws. You will discover, when you, more laws yourself. We will not do all of them. But at least this much will be helpful for us, to go for the next ones.

Basically, the problem is for the quantifiers. We have propositional connectives and then the quantifiers. It does not show whether we have really captured all the quantifiers or not. There can be some other laws, which we will not be able to derive from these. Is it complete with all these laws? Such questions can come up. So better, we should tackle the quantifiers directly; see what happens. One of the basic things is, suppose you have a universal quantifier, say, for each x Px. If it is true in natural numbers then immediately go for P of 1 is true. In first order logic you do not have 1.

(Refer Slide Time: 20:46)



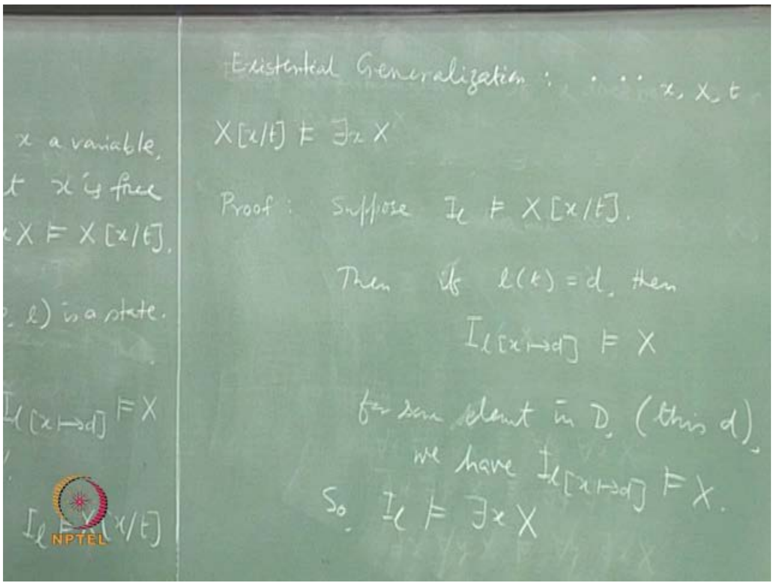
You would say that P holds for some t or p holds for some constant c. That is how we will be proceeding. Since, it is for each, it will be true for a every constant. Because every constant will be mapped by the state or the valuation. From elements of the domain, different elements of course, but elements of the domain; so for each element in the domain, it is true. Therefore, whatever constant you chose or replace in place of x that Px should be true.

This gives us a hint for the universal specification. Here it says, from our each x Px, from this, you can always conclude Pt for any term t; this is what it is. In general, you may write X. From for each x X, you can conclude X small x by t, substituted, x is substituted by t here. But then, do we need any condition? There of course, since we have the substitution X x by t, x should be free, for the variable, that variable, and should be t; for the term t in the formula X; this is what we need. Let us write it. Let X be a formula; x a variable; and t a term, such that x is free for t in X. That is the only thing we need. Then for each x X entails X x by t.

Now, we are writing only in this notation, to make it very general. Not like Px, for each x Px, then Pc; that is the only difference. You can read it that way also. The Px here, so P of t. Instead of small x, t has been replaced. Once you start with, reading it that way, it will be easier to read the symbols also. Just do not read X x by t. Sometimes, with an example, see it, what it is. How you prove it? Proof should be as simple as that; see, it is an entailment.

We will be starting with a state. We do not know whether, that on the left side, for each x X is having a free variable or not. If it does not have any free variable, it is a sentence. You can just go for the interpretations. So, let us start with a state. Now what happens, suppose, I l is a state; let us write the whole thing, is a state satisfying for each x X, fine. Then what happens, then, for each element in the domain what happens? I l x fixed to d satisfies X, by definition of the quantifier for all. Whatever element you fix to x, with that element, that formula X is satisfied; that is what it says. Now then, what we do, t is another term. Suppose l of t is equal to d prime, l is a valuation. It also evaluates every term as an element in the domain. It might be some d prime, l t is equal to d prime. Then since this is true for every d, in particular, we have I l x fixed to this d prime satisfies X. That is the end of the matter. So, this says that I l satisfies X x by t, because t is fixed to d prime. Now that is also l t, fine, l of t equal to d prime. It is same thing as taking l x to t. that is the end of the proof.

(Refer Slide Time: 25:28)



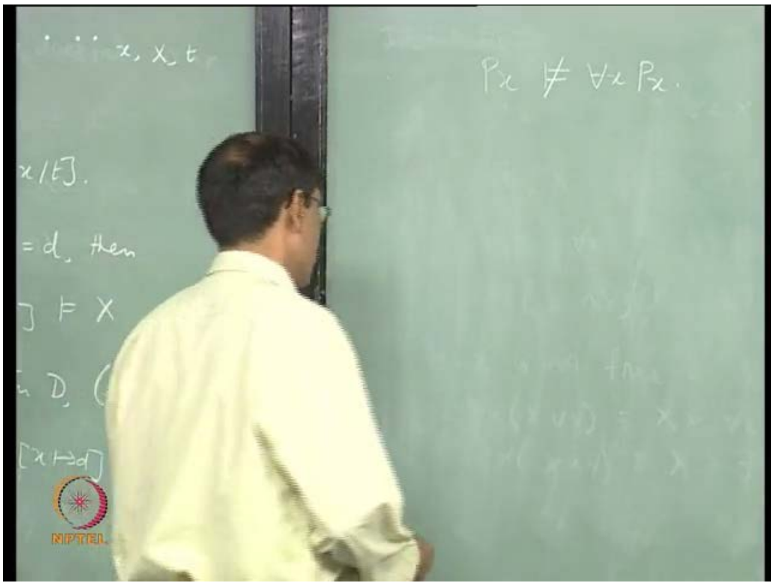
The same way we can have also existential generalization. See the duality. In one you specify, in the other you generalize. We have the same way. Let us see this. This says, with all these things, let X be a formula; x a variable, and so on. It should say, there exists x should follow from X x by t; that is what it says. Here, I am leaving those things. What is X, what is x and what is t? With the same condition like that. So, all that we have to show is x, where small x is replaced by a term t; if you have got that, then you can infer there exists a x which means your Pc. You conclude there is x Px. P is true for 1, then you conclude that there exists x Px is true, our; and that is what you are doing; is it clear?

You have found that Socrates is mortal; then you say some man is mortal; that is what basically we are doing. When you come to existential generalization, always you use that in mathematics, no? Suppose there is an existence proof. So that there exist a number like this, then what you do, you find out one number; job is over. But after that only, this law works; that is why existential generalization is permitted. We do not go for that detail in mathematics, because anyway logic is taking care of it. So, how do you prove it? Proof is similar to this.

Suppose I l satisfies X x by t. Then what happens, if l of t is equal to d, then I l x fixed to d satisfies x. That is for, that, for some d, that means for some d, d for some element in D, this D, we have I l x fixed to d entails X. So, I l satisfies there is x X; that is all. These are the easy consequences of the quantification. We have the real job when it comes to universally generalizing it. But that is also very frequent in mathematics. See, you want to prove that for every x Px. Then what you will do? Let x be a natural number. Now, do some manipulations; I do not know what they are; then finally, you find Px. Therefore, for all x Px. That is the argument you give, right? You want to show some function is continuous. What you do, let epsilon be greater than 0. Now, you construct your delta; what way you construct, I do not know; but finally, you construct. After construction you show that x minus a mod less than delta implies f of x minus f of a mod less than epsilon. Now, you conclude: therefore, for every epsilon that holds. Then f is continuous. This is very standard proof. But what it says is that, you have started with Px. Then, ended with for each x Px; fine. Is that always allowed?

I gave you that example of Sahadeva's skill in Mahabharata, did I give? No? See, it is mentioned in Mahabharata that Sahadeva was very skillful. Sahadeva, the youngest of the Pandavas. He was so skillful that even if it rains heavily, he will not get wet. He will go around it. Here is a proof. Let x be a rain drop, it can be avoided. Therefore, all rain drops can be avoided, by universal generalization. So, there is some fallacy in it. Then in which cases it is allowed? Which cases it is not allowed?

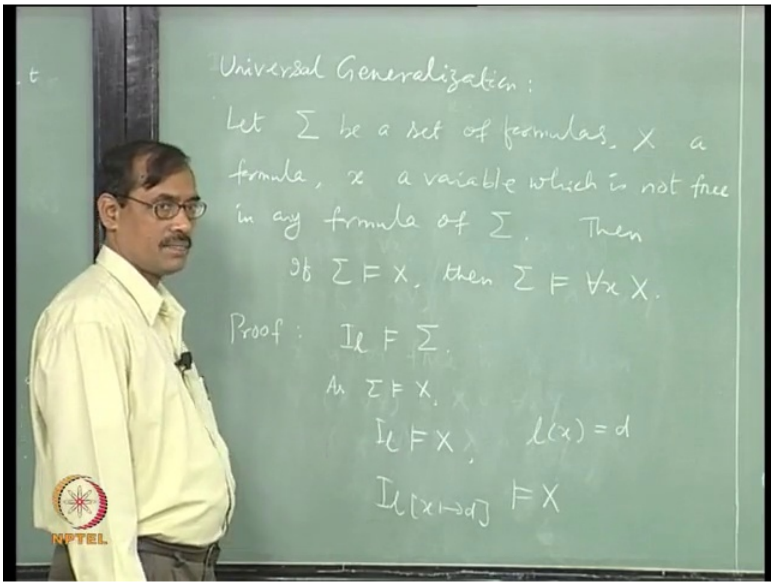
(Refer Slide Time: 30:41)



For example, we see that Px does not entail for each x Px; that is clear, the way we have defined the semantics. This does not hold; is it clear? Why it does not hold? But if you take the universal generalization of this, universal closure of this, right, that entails this; there is no problem. For sentences, there is no problem. If there are open formulas, then there is Px does not entail for each x Px, that we know, clearly. Why is it? Can you show it? This should be easy, because all that it says, you must construct a state which satisfies Px, but it does not satisfy for each x Px. Take a state, say take the interpretation first, then you have to consider the state. You have to take one interpretation, where for each x Px does not hold. Just the earlier one, say P is for prime, right? Every number is not a prime. We know now. Give one state l, where l of x is equal to 2. So, Px is now read as 2 is a prime number. That does not tell every number is a prime number. So, it is clear. Px does not entail this.

Then how to formulate this? Where it holds? We see that if they are sentences, no problem. Now, this is the premise. From this premise, we cannot conclude for all x Px, is that? But then suppose you have some premises, where x is not free. And then you can conclude, there is no problem? So, now formulate it in this way.

(Refer Slide Time: 31:38)



You should start with a set of premises; be a set of formulas; X a formula, x, small x a variable which is not free in sigma; which means it is not free in any formula of sigma, then what we conclude: if sigma entails X, then sigma entails for each x X. So that is the difference between that example and this. Here, somehow you have proved that, already sigma entails X. Even if that variable small x is free in capital X, it is really arbitrary, it could have been taken anything else also, that is the point. Because otherwise from sigma entails X where x is not free at all, how can free things come? That is the difference. You see, the constant is, suppose this is Px, sigma entails Px. Then, what it says, in sigma that small x does not occur at all, it is not free; so equivalently telling, does not occur at all; now, how come this becomes free? Here, it could have been really arbitrary. That is the meaning, when you say let epsilon be fixed but arbitrary in analysis, it says that. I do not need this as a premise, from whatever premise it may be, it does not matter, for this x also. Therefore, for each x Px, it should work. Is the formulation clear? Then we will go to prove it. Proof should not be difficult. What do we want to prove? Sigma entails for each x X.

Let us start with, say I l is a state, which satisfies sigma. Then as sigma entails X, I l also satisfies X. Now the question is, how do you say for all x, X? where from it will come? You have done I l satisfies X, where small x can be a variable, it does not matter, so if small x really occurs or it does not occur, then there is some difference possibly. It is really the interesting case, it is when small x really occurs. If it does not occur, there is nothing to prove; that you can take as a separate case. If x does not occur, then for each x, X and X are equivalent; where in a way, by empty quantification. So there is nothing to prove.

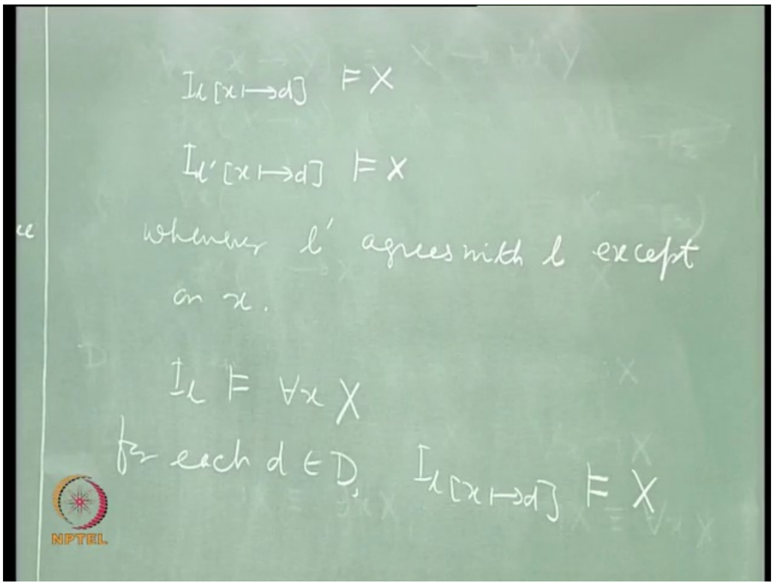
So, let us take this case: when small x really occurs in capital X. Now, when you say I l satisfies X, you have l of x in some D, in your domain; because it is occurring. Then what happens is, I l x fixed to d also satisfies X, is that, is it clear? Then you want to show that I l should satisfy for each x X. We have only shown I l x fixed to d satisfies X. Is it true for every d? Is it true for every d? If every d, then you have for each x, X, otherwise you will not get that. Can you vary d here? l x fixed to d satisfies X. We do not know whether we can vary or not. Let us take any other interpretation, any other state, say l prime. Now, what happens, if you take I l prime x fixed to d, that will also satisfy X, because all that matters is x fixed to d, whether it is l prime or l, if l prime and l agree on all the other things. You have x fixed to d. Then whenever d satisfies, this also will satisfy, this is the reason here.

Student: Sir what is l prime and l?

l and l prime are different, but l satisfies x fixed to d, l prime x fixed to d evaluate the same d, the valuation of x is under both, same d. So, once l and l prime agree on all the variables except x, then we see that l x fixed to d is the same thing as l prime x fixed to d. Because both of them evaluate x to d, is it clear? So, really you can vary and that is the point; that is the crucial point in the proof. Now, for every l prime every d also, you can take, does not matter; because you can vary this l prime.

Once we say l x fixed to d, this valuation will evaluate x to d; always. Now, you are concerned about x, to what other variables, we are not concerned. They are l or l prime; you will say they agree from the beginning. Let them agree on the other free variables; x we are worried.

(Refer Slide Time: 39:46)



Now, for x, l x to d or l prime x to d, are same; both of them evaluate x to d. This says I l x to d satisfies X and I l prime x to d satisfies X, whenever l prime agrees with l except on x. This is what we have got. We have only l x equal to d or m is, to show that I l satisfies for each x X; this is what we want to show, fine. Or, for each d in D, I l x fixed to d satisfies X, this what we want.

Student: That directly it, because I l x is.

Because x is not a free variable in any one of sigma.

Student: So, respective value of d satisfies X.

That is what we are writing.

Student: Why do we need to bring an l prime and all? But how do we write that x is not occurring free in any formula of sigma, therefore it does not matter what value you fix to x sir, why?

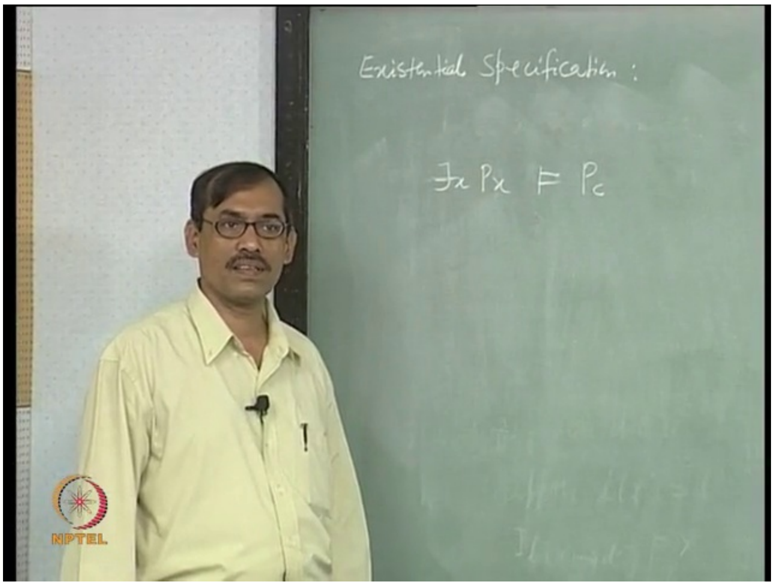
We did that to figure out, in the valuations, that is what l prime does, is it clear? That is what it is doing, nothing else. This is another way of looking at the semantics. There, if you take l and l prime, two valuations, which give the same values to all the variables except to possibly, x, they are called equivalent valuations. And for all equivalent valuations, you have the same valuation for the small x, then you have proved for each x X. So, this says, instead of varying the elements in the domain, you vary over the valuations, because anyway valuations will cover all the points in the domain D.

It says for whatever l, I take l or l prime, whatever valuations I chose, I get always I l x to d satisfies X, for each valuation l, is that clear? This says, not even d, we say that for each valuation l under I, I l x fixed to d satisfies X, then from that it follows that I l satisfies X or I satisfies X. If it is a sentence you can go for I directly; otherwise I l satisfies X. The crucial condition is this, that x is not free in sigma. If x is free in sigma, you cannot do that. Because x is already fixed. You cannot vary the valuations now, that is happening, fine?

Similarly, if you go for the existential specification it gives a lot of trouble. Not only this much, there will be some more constraints. Let us see what is the reason. Let us give an example; why this gives problem. Suppose you have, say, look at the intermediate value theorem. You want to apply intermediate value theorem to find a root of a function, some polynomial. Let us say, root of a polynomial at a point in an interval, say -1 to 1. What do you do? You say, f of that -1 is negative. Let us find; and f of 1 is positive. Let us say -1 and 1. Then 0 lies between -1 to 1. So, there is a point where f becomes 0. What is that point, we do not know. Now, how does it help you? That might be the end result; you are satisfied. But suppose that is not the end result. You want to use that point somewhere to do something else, fine? So, what is the way to go about it? Immediately, you say, let alpha be that root; because you know it has a root. Now you start: let alpha be a root. Now, use that alpha wherever it is required. Then finally what will be your conclusion? Will it have alpha there somewhere? If alpha is in there, in your statement in the theorem should be “if alpha is a root of this, then”. Usually it is not there. You have used it as a tool only, inside it. That means, in the theorem itself, alpha does not appear; somehow it is eliminated, right? And you conclude something else. That is the usual procedure, you do.

Suppose you are using this intermediate value theorem or finding this root as a tool only; that is not your aim. While proving something, you use that. You get, let alpha be a zero of f. Then f of alpha is equal to 0. Now, continue doing something. Finally, alpha is not there. So, your theorem is proved. That is the scenario of existential specification. Can you say that there exists Px entails P alpha, or Pc? That is our point. Then, how did you use it in the proof? Do you understand the question?

(Refer Slide Time: 45:57)



The question is, suppose I say there exists x Px. Does it entail Pc for a constant c? Which constant? You say there exists a constant. Then it does not help; because it is same thing as the earlier. When we say: let alpha be that root, or alpha be a root, you can say that there can be many, so if you say alpha is a root such that f of, where f of alpha is 0, you are telling it, zero, root, how do you write this alpha? Is it really a particular number? It is just a name having that property. It is an ambiguous name. We do not know what it is; but we are using it. This is the question. If there is x Px given, can you derive from it, say Pc? I think, we will end with this question.