Mathematical Logic
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Lecture - 28
Six Results about FL

So next stage, what we want is, to see similar to the propositional logic, what are the laws that can be valid in the first order logic. We will be a bit greedy in the sense that knowing some laws in propositional logic how to give analogous laws in the first order logic immediately, instead of doing them separately. For example, we know p implies p is valid in propositional logic, so does it imply that Px implies Px is also valid in first order logic? Because anyway we are concerned with the patterns only, so they are having a same pattern can we conclude from these, or we will have to go to semantics again and do it?

This is the greedy approach we will be taking. From the same from the laws of propositional logic itself we can now formulate some laws for the first order logic, if this analogy really holds. But there, you have the interpretations for the propositions, now in first order logic you have the states instead of the interpretations. So, how to connect those interpretation with the states, that first we have to see, if not everywhere, at least wherever that connection remains, we should be able to go for the first order logic.

Basically it is like, suppose you have one interpretation in propositional logic, so of any proposition it can be either evaluate into 1 or 0. Let us say, it can be there true or it can be false. Now, when you take one state corresponding to this interpretation, the state, similarly should hold for whatever formula we want to substitute in place of that proposition, for example we take the formula p implies p. See p implies p is valid, we know, right. Instead of this p, I want to write Px, so I would get Px implies Px. Now I consider a state, which will be same thing or evaluating the same way to this as the interpretation evaluates p implies p. Then immediately we can say that interpretation satisfies this if and only if this state satisfies this. That is what we want to propose first. What we propose is, suppose I have a proposition and Px a formula, in some way I am identifying this p with this Px. I have a map let us say, informally, we can write this way; we can say that we have some map, for each proposition it gives some formula there.

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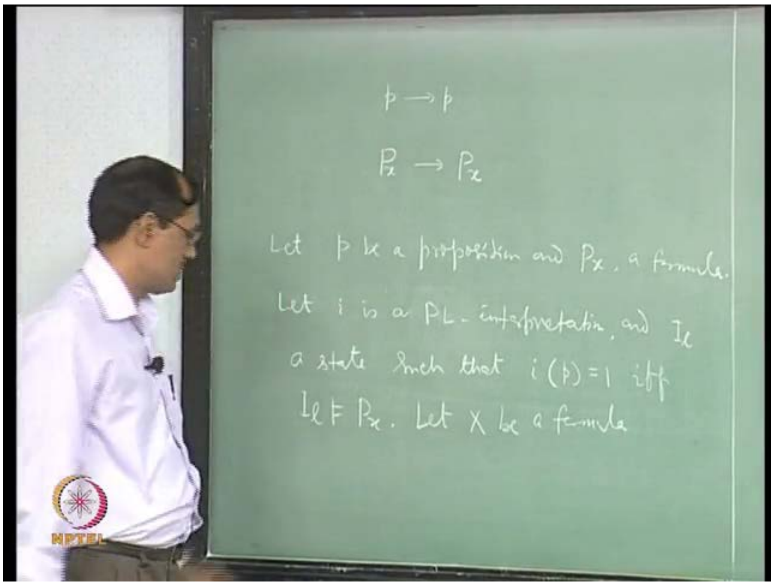

For p, I have this Px. Now what I do, suppose i is an interpretation, i is an interpretation. This is, in the sense of PL. Let us write PL interpretation, instead of just interpretation and similarly let us take I l, a state. They are, in such a way connected that whenever i of this p is 1, I l satisfies Px. What I do I take?

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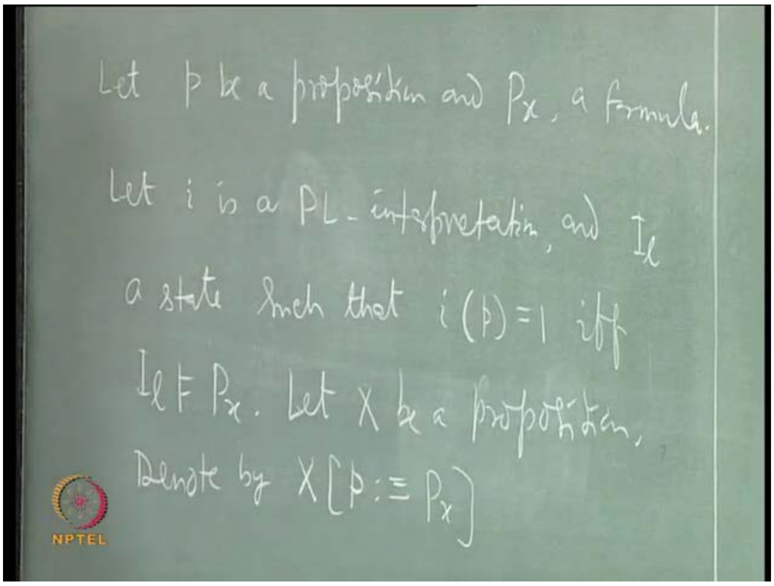

Any formula, let us say X. In X, there can be p's. Suppose there are some p's occurring in x. Then in all those places wherever this p is occurring you want to substitute Px instead.

Let us say, let X be a formula. But, we want to start from the propositions like p implies p, so we are not taking general, any formula. Let us start with the propositions where p occurs. Let X be a proposition. Denote by X p substituted by let us say Px, the formula obtained from X by replacing each occurrence of p with Px. Now what happens, X p substituted with Px, has no variable capture; because we started with X as a proposition, there is no quantifier. So, things are simplified. Here, there is no variable capture is occurring. Now with that, what we want to say is, since i and I l are connected by same way, if i satisfies p if and only if I l satisfies Px. So, what we expect is, X is satisfied by that i if and only if X p equal Px is also satisfied by that I l. That is what we want to show. If i satisfies X, then I l must satisfy Px that Px, it is X p substituted with Px and conversely also.

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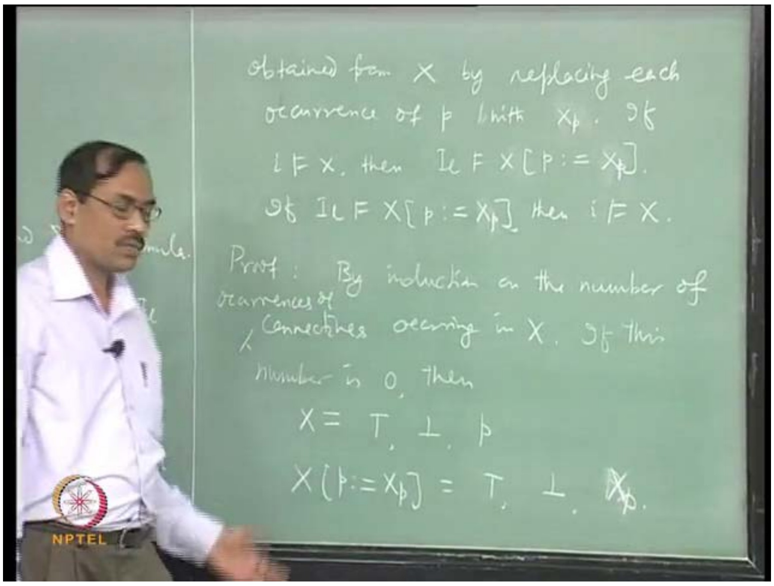

Similarly, if I l satisfies this new formula then I should satisfy X. In fact it is not limited to such atomic formulas as Px. We can take any other formula also, that should also work. Let us do that. Let p be a proposition and identify any formula corresponding to this p. Let us write it as X subscript p, instead of Px. Everywhere we should be able to substitute that, replace this Px by X p. Let us substitute; this is what we expect to hold.

It is really number of occurrences of connectives. So we wrote occurring here, but we can make it better. Suppose in the basic step, what will happen, this number of occurrences of connective is 0. Basic step: if this number is 0, then how does it look like?

It can be top, or it can be bottom, or it can be a propositional variable; that is how propositions have been formed. Once this happens, you will get correspondingly X, p replaced by Xp as top or bottom or Px; so here it is X p. Now, the things are clear; because the assumption says i satisfies p if and only if I l satisfies Xp. The conclusion and the assumption they are the same. Here, conclusion is also the same thing; i satisfies p if and only if I l satisfies X, p replaced by Xp which is now Xp, so basic case is clear.

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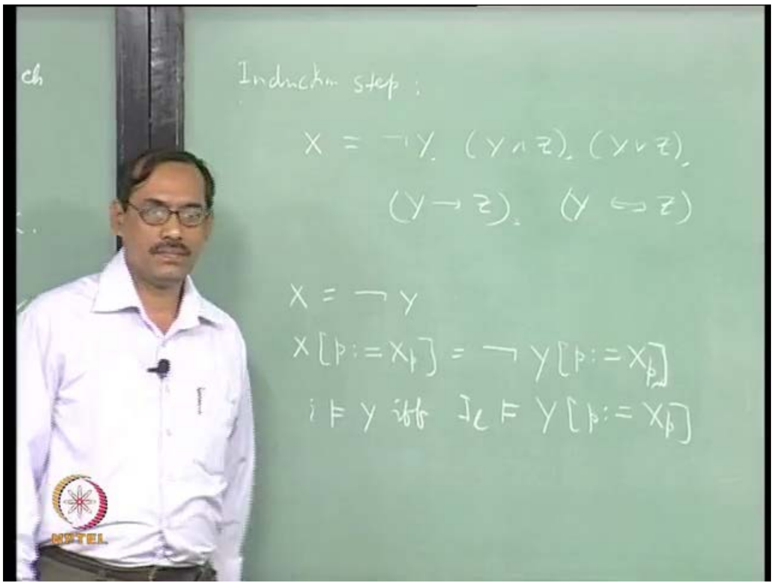

Now, in the induction step what do you do? We can say, X can be in the form not of some Y or, Y and Z or, Y or Z or, Y implies Z or, Y biconditional Z. These are the possibilities. Let us take the first case Y, X equal to not Y, X equal to not Y; then in this case X, p replaced by Xp would give not Y, p replaced by Xp.

Now, by induction hypothesis there is one connective less here, so do it exactly, how to do? Induction hypothesis is for number of connectives less than or equal to m the conclusion holds. Now, it has m plus 1 connectives then this Y has, Y has number of connectives less than or equal to m. Therefore you can use the induction hypothesis. That says i satisfies p if and only if I l satisfies, sorry i satisfies Y if and only if I l satisfies Y, p substituted by Xp; this is what the induction hypothesis says.

So, i satisfies Y if and only if I l satisfies Y, p replaced by Xp. Now conclusion is clear, fine; because of the connective not. The same way also I l tackles not as i . Now, for all the connectives, proof will be similar. You just go on writing it. Let us call it our first lemma for substitution. Now, the main theorem you can write easily where you need not connect i and I l; you have to go for validity, let us say.

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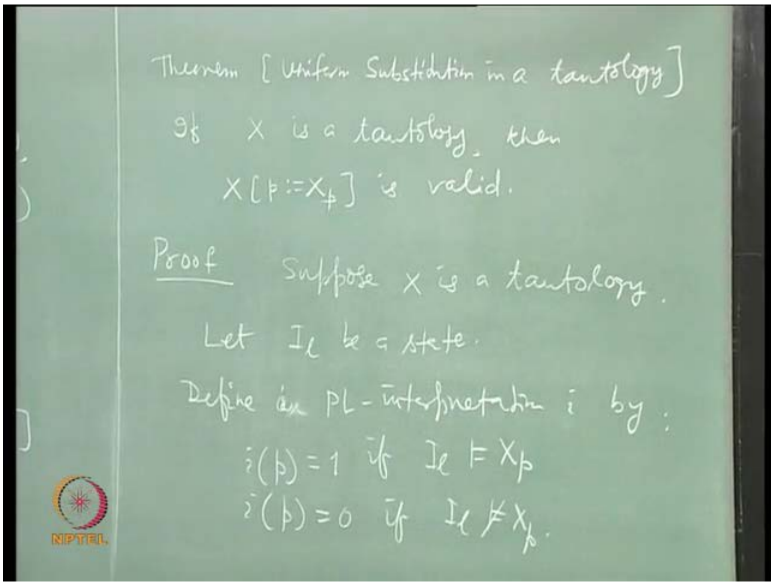

So, we can write now from the uniform substitution in a tautology, see, we are telling valid propositions in PL also, here also you are telling valid formulas. This valid might confuse, we will say in a FL, if you take a proposition which is valid, we will call it a tautology. While considering both PL and FL, we will say valid propositions are tautologies. That will not have any conflict of terminology. This is what it says: uniform substitution in a PL valid proposition; instead of telling PL valid proposition we are making it tautology. Now, it says that if you start with a tautology and then replace each occurrence of a propositional variable by another identified formula Xp, then whatever you get from that, that will also preserve validity.

We have to only write it. If X is a tautology, then X where p is substituted by Xp is valid. To write with every detail, we have to go to this first portion of the lemma. Let p be a proposition, identify Xp, a formula corresponding to this p, then denote by X, p equal to Xp, the formula obtained from X by replacing each occurrence of p with Xp there. Whenever X is a tautology, X where p is substituted by X p is also valid.

Proof is there; anything to do? X is a tautology, so that means you take any interpretation it satisfies it. Now, what do I do? You take any state; X, p equal to Xp, in this state, in this state what will happen? This p is either evaluated to 1 or 0. Whatever way it is evaluated, you find out which i is that, it, see, this is the way we will be proceeding. Suppose X is a tautology we want to show that X, p replaced by Xp is valid. So, let I l be a state. Now, I l satisfies Xp; it is a model of that, or it is not a state model of that, so define one interpretation i, or a PL interpretation i, by what you do, i of p equal to 1. If I l satisfies Xp, it either satisfies or does not satisfies or it can happen it does not satisfy. In that case, you put I l equal to zero. Either I l satisfies Xp or it does not satisfy Xp. So, whatever is the case, do that. Now then, apply the lemma. That is all, proof is over. Once it is a tautology, each i satisfies, evaluates it to 1. By the lemma, each I l will also evaluate it to 1.

This is called uniform substitution in a tautology. There is again another type of uniform substitution. If you have started with a tautology you are ending with propositions. Now, instead of starting with tautology in FL, suppose you have valid formulas can you substitute there itself, instead of from the proposition, then? That will be mentioned as uniform substitution.

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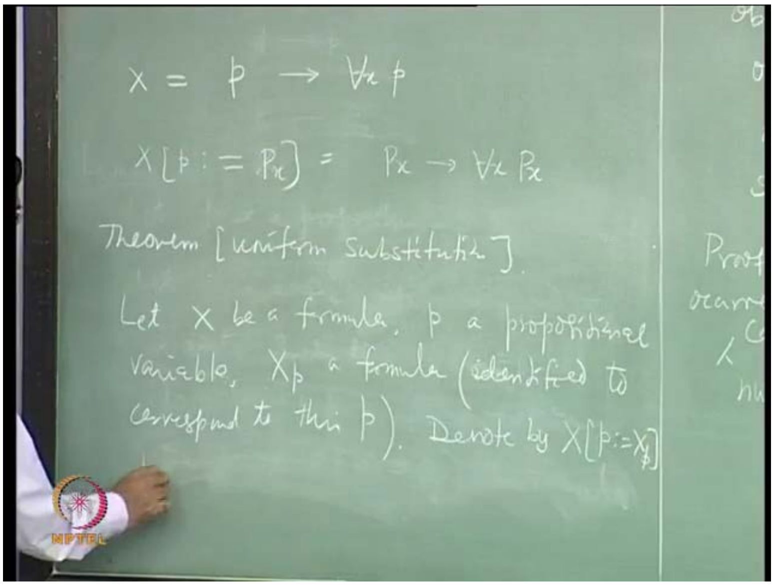

We are not starting with any tautology, here it can be any valid formulas. Our thing is that we assume or we presume that it should go correctly, so whatever formula you are getting after this substitution, should also be valid. But, there can be a problem; there can be some problem there; it is the variable capturing. Suppose you start with one formula in this form: X equal to P implies for each x, P that is allowed in FL as a formula. This is a proposition, so proposition is a 0-ary predicate. Before that predicate, you can always use one quantification, for all x, though it is vacuous. This way, x does not occur there. But, it is still allowed. Now what happens if you substitute this p by Xp or say Px, this will be equal to Px implies for each x Px; the variable is captured. One variable has been captured. I have just substituted this p by Px. Formally, this is how substitutions will work. What we get form this is, Px implies for each x Px; this is not valid. Is that? Like you take one state where x becomes 2, p means prime. So, in the set of natural numbers it will say 2 is prime implies every natural number is prime, which is false. So, this is not valid though where from we have started is valid, because there is no x there. So it is vacuous; it does not matter, whether x is there or not, that is valid. You should have some constraint on the capturing of the variables. Once you generalize it to valid propositions or valid formulas, that is exactly the proviso, that is exactly the constraint we need.

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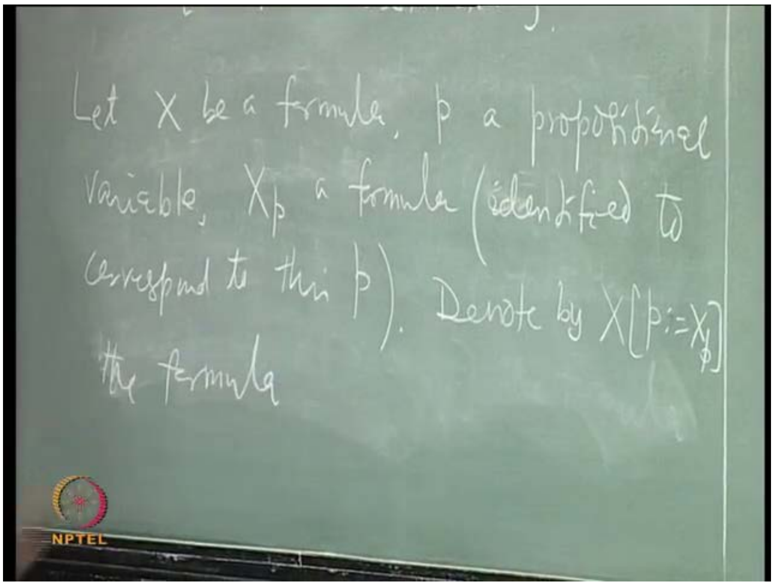

Let us formulate the theorem of uniform substitution. We start with, let X be a formula; p, a propositional variable; then Xp, a formula corresponding to, our identification, to correspond to this p. In fact, this is only for our mention, formula is enough, this is what we are going to do. We denote by X, p substituted by Xp as earlier. The formula obtained from X by replacing each occurrence of p with Xp. Now we need something more; we have to give a constraint. Suppose no free variable of Xp becomes bound or is bound really in X, p replaced by Xp. This is the condition we need. If X is valid then X, p replaced by Xp is also valid. Again proof will be similar.

But, you have to formulate first its lemma then proceed. The same proof almost will work because corresponding to each i, we will get one I l which are so related: p is satisfied whenever that Xp is also satisfied in the corresponding state model.

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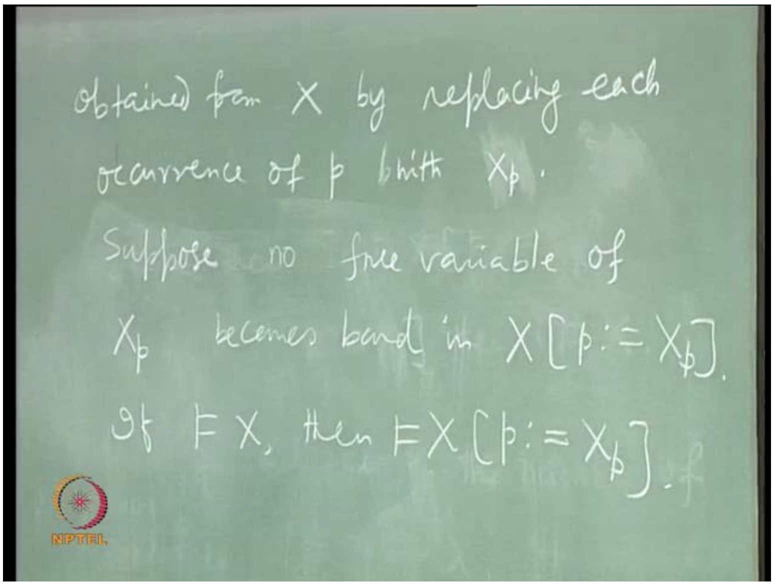

Then proceed in proving that lemma. Now induction hypothesis or induction step will involve one more thing that whenever you started with your X that can have a quantifier, earlier it was a proposition see what, now quantifier. Now, when we start with X itself there can be quantifiers, so two more cases will come that X is in the form for each x Y, there is x Y; in that place you will need this proviso to prove it; that you can do as an exercise.

These are the two things which will be useful in getting many more theorems form the propositional logic first, then from the first order logic itself. There is another kind of replacement which we have done earlier, which is the equivalence substitution. If something is equivalent to another, you can always replace it, which is having equivalence; so let us formulate it first. Let A and B be your formulas, X a formula, so what do you want to say is, that if A is equivalent to B then in X you replace the occurrences of A and B, you should end with some equivalent formula; that is what it says. Suppose A is equivalent to B. Denote by X, A equivalently substituted by B, what substitution will give now, the formula obtained from X by replacing each. But, do we need each occurrence here? They are equivalent, even some will do. So that each or no or some occurrences of A by B in X. What we conclude is, X should be equivalent to the new formula or you say, the new formula is equivalent to the old one.

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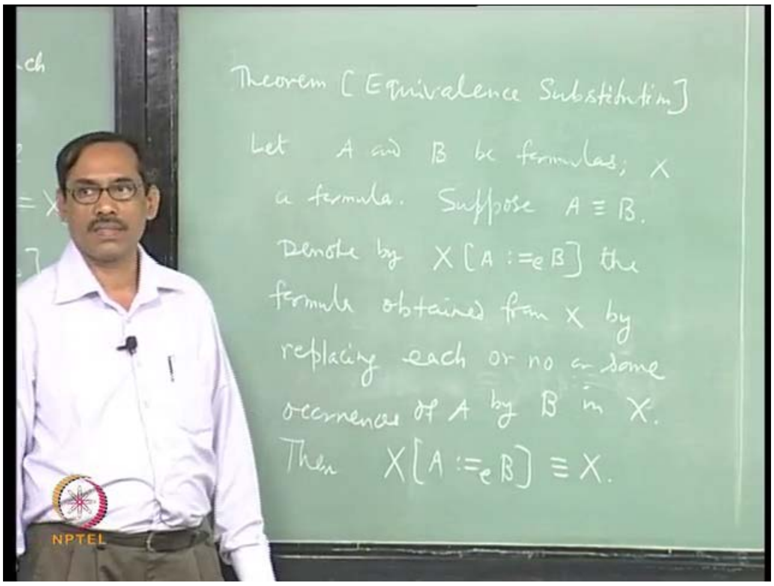

So, proof is again conceptual; there is nothing more to it. You will say, A is equivalent to B, you take any state model or any state, then in that state whenever A is satisfied, B is also satisfied, and conversely. That is the meaning of equivalence. Let us start with any state I l. We want to find out whether I l satisfies this new formula or not; or in the old formula, let us say I l satisfies X or not.

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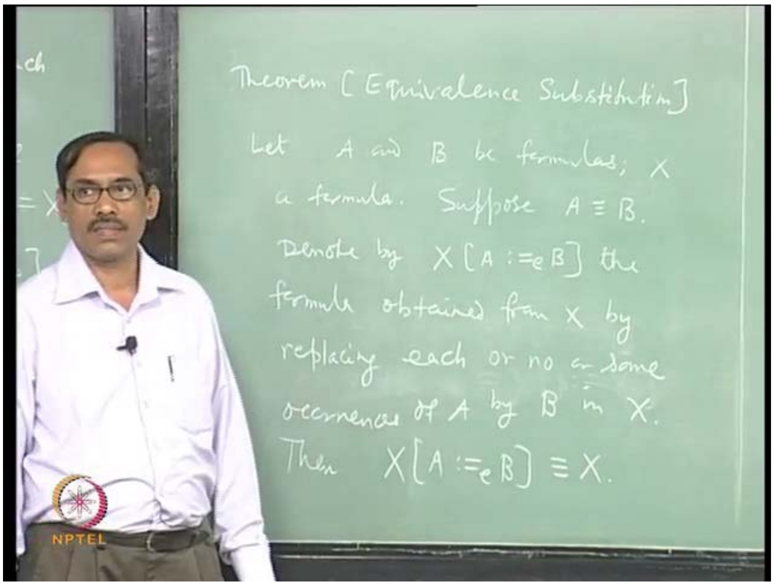

Now, how to find out whether I l satisfies X or not? Go for our scheme. Our formal semantics simply reduces satisfaction to the level of satisfaction by a predicate conceptually or not? Suppose I have a formula X which looks like Px implies Px, for simplicity. Let us say whether I l, we want to find out, whether I l satisfies Px implies Px. How do you proceed? Our formal semantics asks us to do the following.

You say I l satisfies Px implies Px, if, next step is, we will use the semantics of implies, that connective arrow; so that says I l does not satisfy Px or I l satisfies Px. It will come to this. If it is something more, let us say, for each x Px implies Px, this is what we want to do, then this step will be: does not satisfy for each x Px or I l satisfies Px. There is one more thing to be done, for the quantifier. We go on. Next, we write if and only if for some d in our domain, right, what happens? I l x fixed to d does not satisfy Px, is that for each x it does not satisfy? That means there is at least one element for which it does not hold. Then what happens, you write this, then, or I l satisfies Px, that is what we had done. Finally, it is reduced to some state either satisfies that atomic formula or it does not satisfy the atomic formula.

It is reduced to that, inside something, for some, for all, or, and something is written. You are now thinking that conceptually. Suppose I start with any state that satisfies X. My formal semantics says that I should go on writing, types like this, in our meta-language English. Go on writing like this. Finally where something will be written Px, not Px or something will come. Atomic proposition, atomic formulas will be coming. I l satisfies some atomic formulas or I l does not satisfies some atomic formulas with some or all these kind of expressions will come there. Now what happens, whenever I l satisfies is occurring there, in that big reduction, I l satisfies A, you can simply replace I l satisfies B, because A and B are equivalent.

Now, doing that whenever whichever occurrence corresponding to the replacements A and B you replace I l satisfies A as I l satisfies B, or I l does not satisfy A as I l does not satisfy B. Now, you get a big expression like this which will simply prove I l satisfies X A replaced by B. That is the proof, conceptually. The doubt is that we are not writing directly. If I l satisfies X then I l satisfies X, p equal to Xp. The reason is I do not know how the interpretations of p and I l of Xp are related, in your lemma, which you proved for you substitution in a tautology, you have that connection.

So, it can happen that one interpretation satisfies p, but the corresponding state, I am considering, does not satisfy Xp. In the basic step, it fails. Is that clear? But then to prove this uniform substitution, you have to start with a lemma. With that constraint, corresponding to i of p, you first construct I l, the corresponding I l, which will be of the same way with Xp. But, anyway ultimately our result is this; which we will be using; not for any particular state, validity or satisfiability, that is what we want.

Similarly, here you are starting with A equivalently replaced by B. And it goes to the level of states, but since A is equivalent to B always you can write I l satisfies that, I l does not satisfy that. So, proving or formulating another lemma is not required here, because of the condition of equivalence.

So, these are the three substitution theorems which we will be using to obtaining many more laws from the propositional logic and also from the first order logic. Let us see some examples, whatever here, laws in propositional logic, one is the example we have already given.

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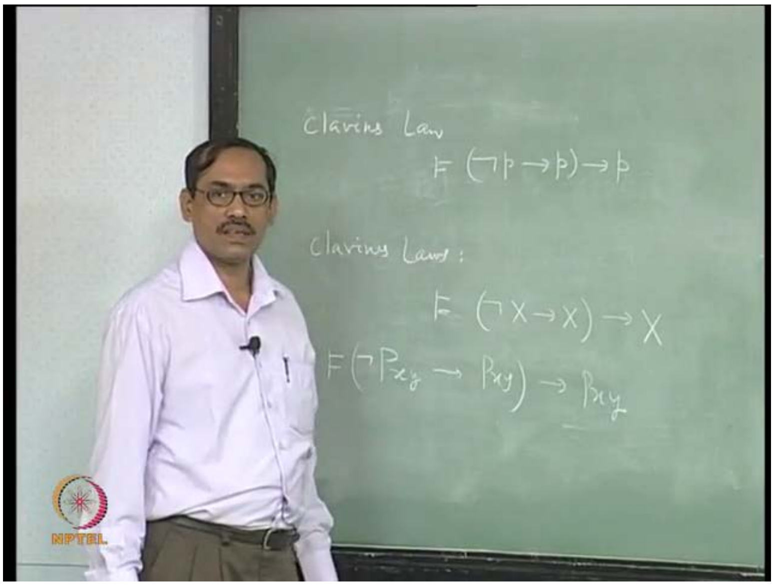

Suppose I start with Clavius' law. Which says, not p implies p therefore p. This what it says. Then the same thing I can now write in first order logic; which will look like whatever be the formula X, it will say, not X implies X, from this you can conclude X. Now, X is any formula, right? If you have taken X as Pxy, you get not Pxy implies Pxy; this implies Pxy; this is valid; and with a lot of, infinitely many.

So, it is a law. Replace this X by any formula. Now you can translate or rewrite all those laws from PL to FL. Whatever you did, we are not going to write them again. Now, there are three theorems in propositional logic. We want to see whether those things still hold or not. For example, you have deduction theorem. If you apply deduction theorem, here in propositional logic, it will look like not p implies p therefore p.

You can take it as a consequence. Now also it will come as not X implies X entails X. But then you have used deduction theorem. This part is easy, because modus ponens. The other side: from these to conclude not p implies p implies p is valid. This is really the main part of the deduction theorem.

The question is, whether deduction theorem holds in FL? Whether monotonicity holds? Whether reductio ad absurdum holds in FL?

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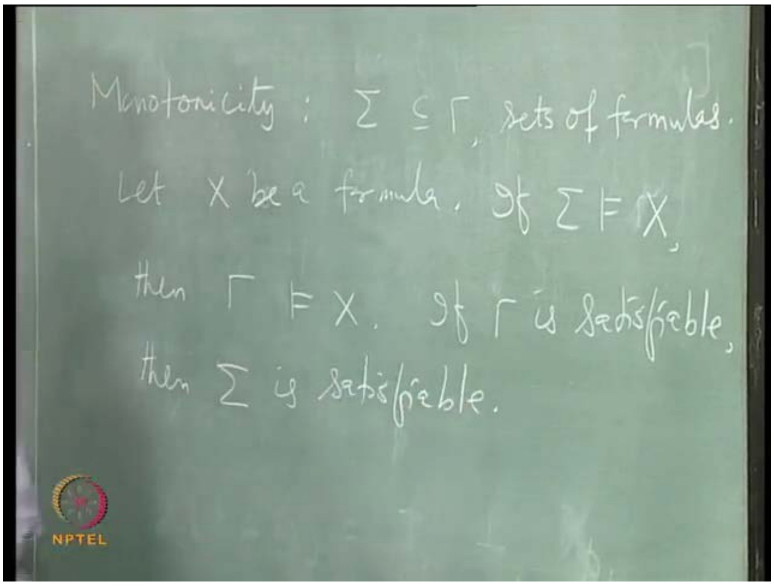

Let us see monotonicity. How does it look like. It says, suppose sigma is a subset of gamma, sets of propositions, now you write sets of formulas, FL formulas, and then let us take another, X is a formula, be a formula. It says if sigma entails, gamma, sigma entails X, then gamma also entails X. And there is another part, it says if gamma is satisfiable, then sigma is satisfiable. Since gamma is satisfiable it has a state model, so you have to start with a state. So, let I l be a state. When we say I l is a state, we are not writing all the details, we should have written first I equal to D phi is an interpretation. Then, let l be a valuation under this interpretation, so that I l becomes a state. All these things are assumed here. So, suppose I l is a state. Now then what we say that if I l satisfies gamma then what happens, by definition, I l is a state model of each proposition in gamma or each formula in gamma, now.

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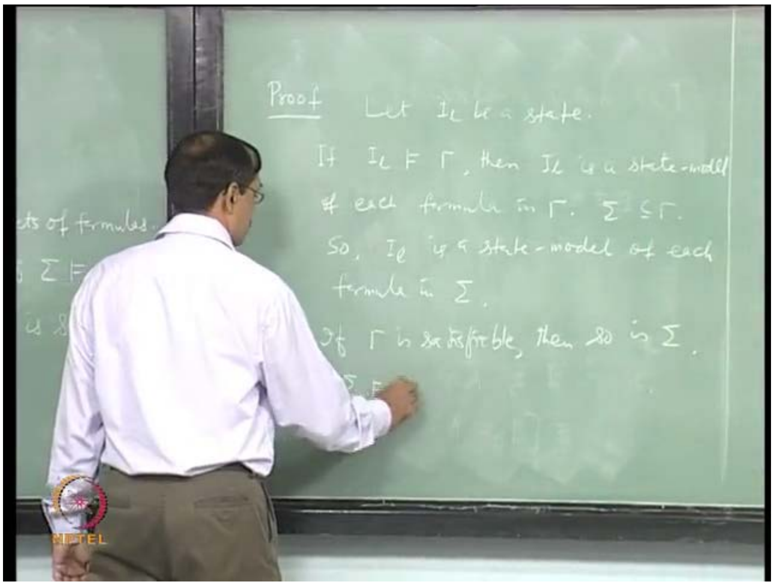

So, just the detail I have to write I l is a state model of each formula in gamma, now sigma is a subset of gamma, so I l is a state model just vary mechanical of each formula in sigma. So, this says any state model of gamma is also a state model of sigma. That much is enough; those two propositions are reformulation of this.

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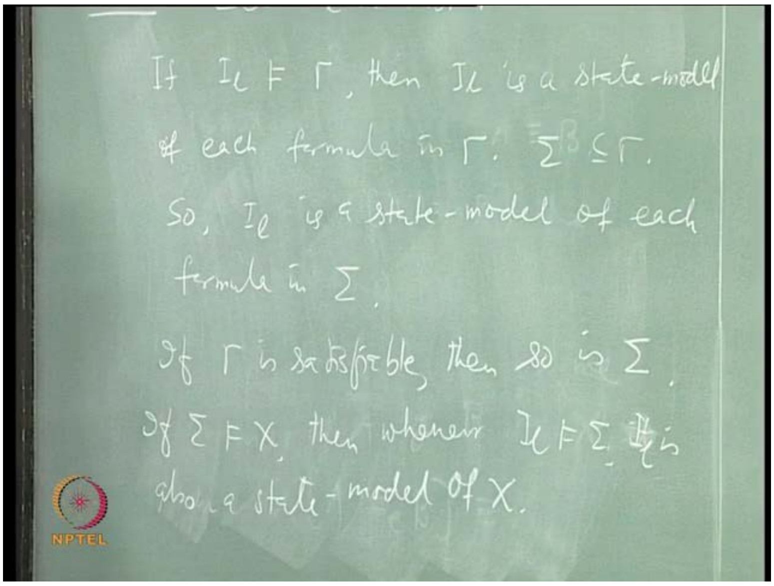

It directly proves that if gamma is satisfiable, then so is sigma. If it is satisfiable, I l satisfies it therefore I l satisfies sigma therefore sigma is satisfiable, so second thing if sigma entails X.

Now, the other one. If sigma entails X, then whenever I l is a model of sigma, it is also a, is not it? I will say I l is also a state model of X. This is given. We want to prove gamma entails X. So we start with a state model of gamma. But, we have already shown any state model of gamma is a state model of sigma. So I l satisfies sigma. Now since sigma entails X, I l satisfies X; that is the end of the proof. So monotonicity is easy.

Now, you see if you have taken just i as in propositional logic, instead of this I l, is it not the same proof as that of propositional logic? It is really propositional. In that sense, the proof is also propositional. It does not go to the detail of quantifiers because state models perform the same way as interpretations in propositional logic. Then what about reductio ad absurdum or deduction theorem?

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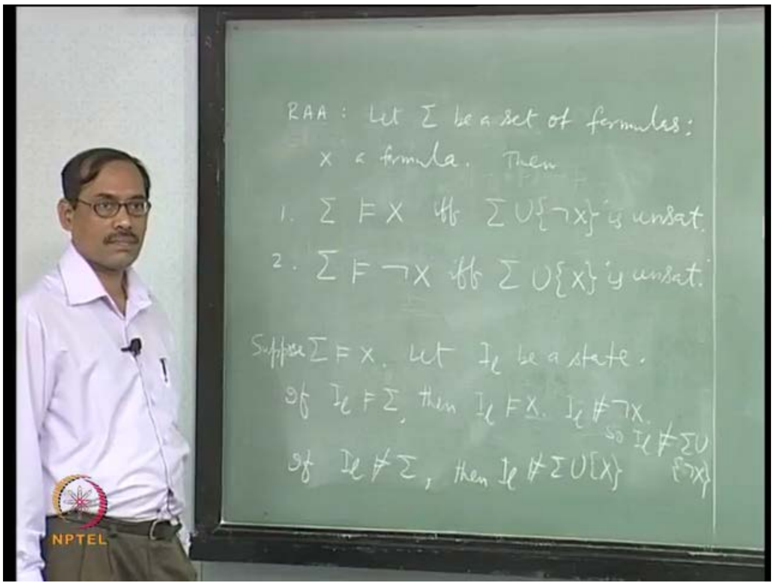

Just you have to formulate it. Let sigma be a set of formulas; X a formula. Then it says, sigma entails X if and only if sigma union not X is unsatisfiable. Also another version. Sigma entails not X if and only if sigma union X is unsatisfiable. Now, what happens, here first one; let us say. Sigma entails X. You have to prove sigma union not X is unsatisfiable. First, suppose sigma entails X; to show sigma union not X is unsatisfiable; how do we proceed?

Let I l be a state. If I l sigma, there are two cases. I l satisfies sigma or I l does not satisfies sigma. If I l does not satisfy sigma, then there is nothing. I l does not satisfy sigma union not X also, whatever that I l may be, monotonicity. So, this becomes unsatisfiable. What about first one? The, let us copy them. I l does not satisfy sigma union X. Either this, what it is telling in this case? What happens, if I l satisfies sigma then since sigma entails X, I l satisfies X. So, I l does not satisfy not X; so I l is not a state model of sigma union not X, is that okey? Now, is that clear? First one? if I l satisfies sigma, then I l satisfies X, because sigma entails X. Now, I l does not satisfy not X because of connective not. By monotonicity, I l does not satisfy sigma union not X; and also by definition, because not X belongs to sigma union not X. So, one of them will, does not satisfy, therefore does not satisfy the set.

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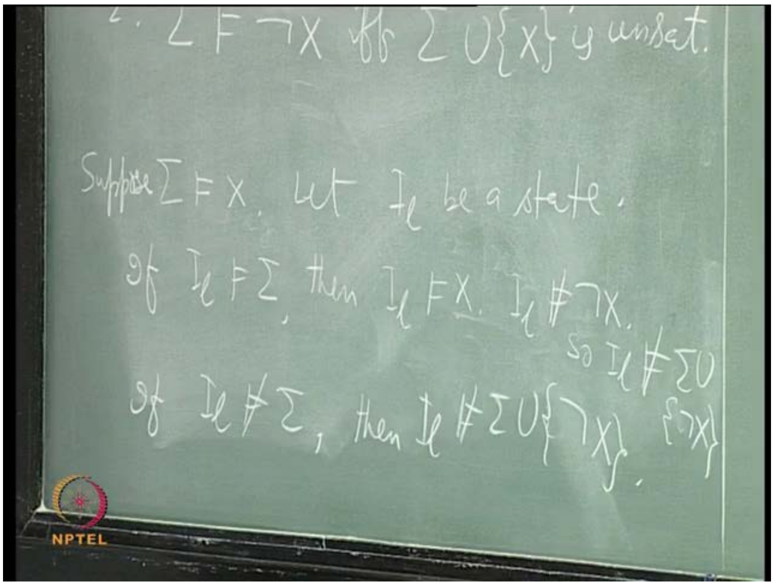

On the other hand, suppose I l does not satisfy sigma. Then I l does not satisfy sigma union not X. Because in sigma there is one which itdoes not satisfy, so also sigma union X. But, we need sigma union not X. Let us take not X. Any set, we could have taken; sigma union not X. So in any case, whatever state I l you start with, it does not satisfy sigma union not X. Now, give that argument. Therefore, sigma union not X is unsatisfiable. This says I l does not satisfy sigma union not X whatever be this I l, so sigma union not X is unsatisfiable.

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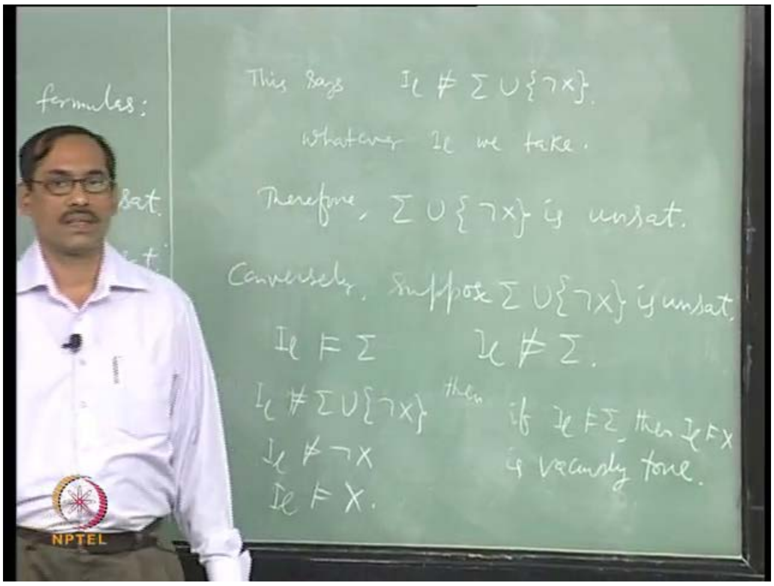

Conversely, what you do? Conversely suppose sigma union not X is unsatisfiable. Again there can be two cases: I l satisfies sigma or I l does not satisfy sigma. If I l does not satisfy sigma, then what happens? All those I l, for all those I l, the statement if I l satisfies sigma, then I l satisfies X is true vacuously. That is what we want. sigma entails X. So, when I l does not satisfy sigma, then.

If I l satisfies sigma, then I l satisfies X is vacuously true. In this case when I l satisfies sigma we know that I l does not satisfy sigma union not X, because sigma union not X is unsatisfiable. Therefore, this I l must falsify not X. at least one of them is falsified, but all those in sigma are satisfied, so the other one is falsified. This means I l falsifies not X. Therefore I l satisfies X. That is what we wanted. That means if I l is a model, a state model of sigma, then I l is a state model of X; is satisfied. That sentence is true in both the cases; that sentence is true. Therefore, sigma entails X.

The same proof will hold for the second one also. What you have to do is, replace every occurrence of X not having not there, with not X and every occurrence of not X as X. Simultaneously you replace X by not X and not X by X. Simultaneously in this proof itself. You get the proof for the second one.

Now, deduction theorem you can prove. Similarly, they are also propositional. This is really propositional proof. We have not done anything with I l; you could have started any interpretation itself.