Mathematical Logic
Prof. Arindama Singh
Department of Mathematics
Indian Institute of Technology, Madras

Lecture - 27
Validity, Satisfiability & Equivalence

So, last time we had discussed the relevance lemma for the first order logic. We have stated it in a way that only one interpretation is fixed; under that interpretation, if you take different valuations which agree on all the free variables of the formula, then it does not matter whether you consider this state with one valuation or the other. If one state fixing the, one of the valuations, is satisfying the formula, so is the other, and conversely. But relevance lemma can also be generalized a bit. Like, you consider the same interpretation. Then there might be some predicates, some function symbols which are not occurring in the formula.

Informally it looks that whatever value or whatever relations you associate with those predicates or whatever functions you associate with those function symbols, which are not occurring in the formula, it should not matter. In other words you can think of different interpretation, I is one interpretation, J is another interpretation with the same domain D. Suppose the maps phi and psi in different interpretations, now agree on all the predicates and function symbols that are occurring in the formula. Then you can show that any state l under any one of them by fixing the variables will either satisfy or not satisfy accordingly. That is easy to see, because phi P will be the same relation as psi P, so it does not matter whether inside the proof you write phi of P or psi of P, anytime. You can just exchange between them. That is the generalized form of the relevance lemma.

But then we have told that for sentences something else happens, for relevance lemma. Then, if you take the formula Px whether you go for there is x Px, for each x Px, they will be sentences from Px by quantifying over the variable x. So we need and we told that it might be up to certain extent. For that up to certain extent means what? We have to specify it. So, for that reason we will go back to our usual flow of presentation of the first order logic. There, we have come across one idea that state can be a state model of a formula, but we have not defined how a formula is satisfiable or it is not satisfiable. As regards propositional logic that was our scheme. We go for interpretations, then models then see how a formula is satisfiable and how it is valid and so on. These are our main targets. Let us start with that today.

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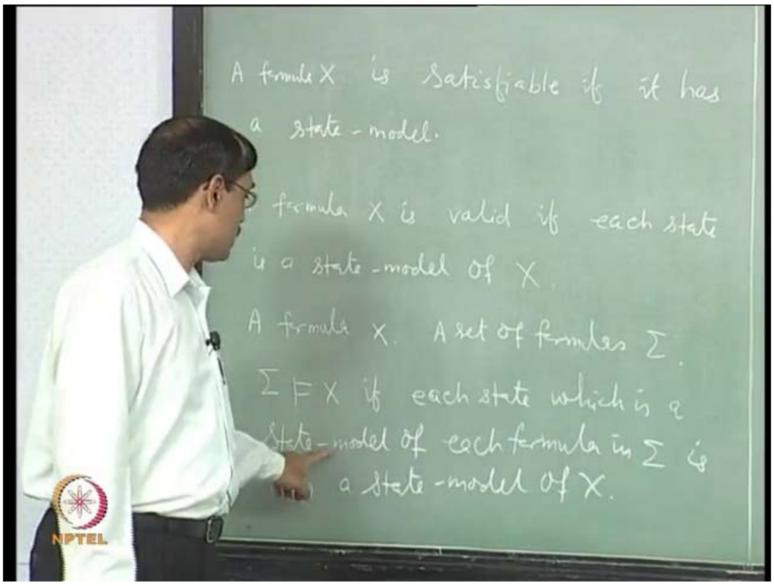

We say that a formula X is satisfiable if it has a state model. This means, it will be called satisfiable if there exists one interpretation I, there exists a valuation l under I such that the state I l satisfies the formula X, is that so? That is what we mean by telling that it has a state model. Otherwise you will say that the formula is unsatisfiable. That means you take any interpretation with whatever domain whatever map phi, does not matter, and take any valuation under that interpretation. Then the ensuing state I l will falsify it. In that case only we say that the formula I X is unsatisfiable. Similarly, we can define validity. A formula X is called valid if each state is a state model of X. Just like your interpretations in propositional logic.

Now, we are dealing with the state models or interpretations which are states. In fact, states, we are not telling them as a interpretations, is that; then you say that a formula is invalid if this does not happen. Which means, you can always find one state which falsifies X, then you say it is invalid. Then we can really generalize this to the consequences also. Once, we have come across this. You say that a formula X is given, and a set of formulas sigma is given. You want to find out when sigma entails X. For sigma entails X what we need is, let us go back to propositional logic. It says you take any interpretation which is a model of all the formulas in sigma or propositions in sigma. Then such an interpretation should be a model of X; that is what it is. Now, we will be dealing with state models. So this one, any state which is a state model of all the formulas in sigma is said to satisfy sigma. Such a state which satisfies sigma has to satisfy X. Then we say that the consequence is valid. We say that sigma entails X, here each state which is a state model of sigma, state model of sigma is a state model of each formula in sigma, is also a state model of X. If such a state model of sigma exists we also say that sigma is satisfiable; lets write it separately.

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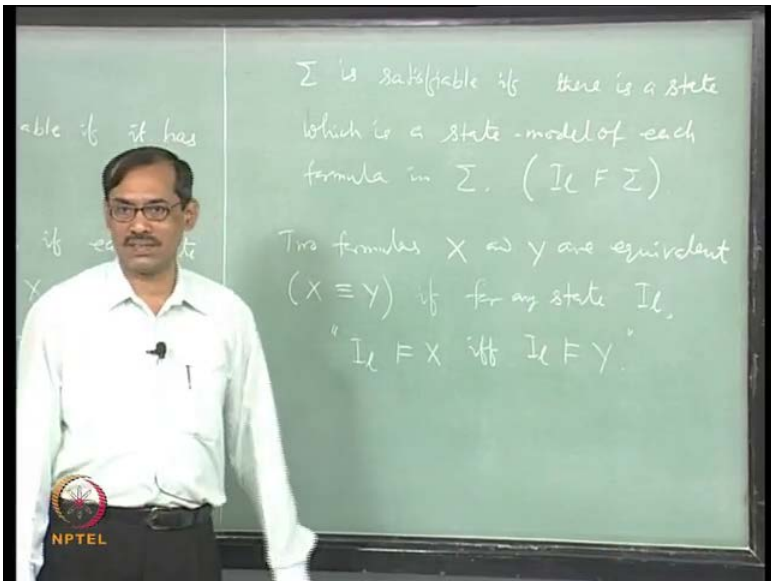

We say that sigma is satisfiable if it has a state model. That means if there is a state which is a state model of each formula in sigma. In such a case we also write the same symbol say this state is I l, we will write I l satisfies sigma, this the way we will be writing. It says, you consider any state I l. If I l satisfies sigma, the I l must satisfy X. In that case you say sigma entails X; just like your propositional logic. Instead of interpretations I, you are writing there, you are now writing states I l.

If you take two formulas X and Y, X and Y are called equivalent; so again we write the same symbol X is equivalent Y; now, you should be able to do it; if X entails Y and Y entails X. Which means you consider any state, that state either satisfies both X and Y or falsifies both X and Y; is that so? That means for any state I l, either I l satisfies both of them or it falsifies both of them, yes? You can write it this way. It is a criptical way of writing because here we are following the same way, if than and everything we are writing. So, I not if and only if. We go with this. Which means that you take, consider any state; it is either a state model of both of them or it is not a state model of both of them simultaneously. Let us see some examples, how it goes.

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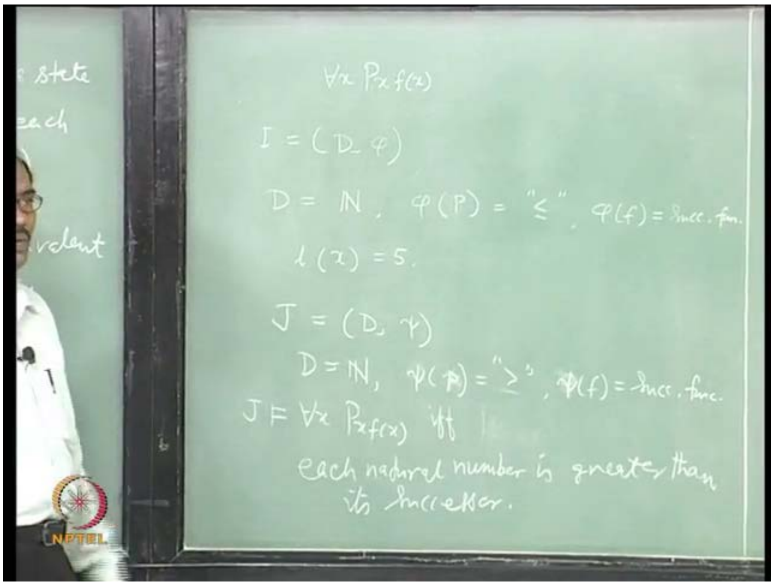

Consider this formula. Now is it satisfiable? It should be. We do not see any contradiction there. But then to show that it is satisfiable, we have to really construct one interpretation, one valuation and then show how this happens. Let us go for some interpretation where we will take, say, I is equal to D phi. Let us take D equal to say set of natural numbers, we are fond of it. Now then, say, phi of; we need only for P and for f, say, phi of P is something and f is as we have taken earlier, say successor function. Now you can read that sentence in this interpretation. This is all that we need, for the sentence to be satisfiable or not one interpretation is enough; we do not need to go for the valuations.

But, you can keep the valuation, does not matter; due to the relevance lemma. As it is, the sentence now reads: every natural number is less than or equal to its successor; that is true. So, it is a satisfiable. But you have to really go for the states. Relevance lemma now says that if a sentence is satisfiable, then you need not consider the states, you can simply consider the interpretation. Any state, that interpretation, are the same things for the sentences. For the sentences, you can really redefine, a sentence X is satisfiable if it has a model, not even state model. It has a model because once it has a state model it will have a model wherever the interpretation is, the state is, that interpretation will also satisfy. But, if you want to see, let us say, l say, l of x equal to 5 does not matter. Once you go through everything, it will be translated as the same sentence. This 5 will never occur in that. That is the satisfiability, what about invalidity or validity?

It does not look like that it will valid, there is no big structure there. Now, for invalidity what to do? Again we have to construct one interpretation. You just take say P is greater than or equal to; you have done already, a successor. Once you say J equal to D psi, where D is the same natural numbers, and phi, or psi, I say greater than or equal to. Now we will take greater than. Just take the opposite of that, compliment of that is greater than or equal to, works then phi of f same successor function.

Student: Sir, where psi of P is what.

Psi of P is greater than or equal to; and here also psi of f is the successor function. Then what happens in this interpretation, we say that J satisfies for each x Px f of x, if what happens? If, the sentence, the sentence will be for each, for each natural number, the natural number and its successor should be related by psi P. So, write it as a sentence directly: if, each natural number is greater than its successor. If this is true; this is not true. Therefore J is, J does not satisfy for each x Pxfx. So, we say that if you consider states, again does not matter. You just take l x equal to something and then J l will be translated to the same sentence again. Therefore, this is invalid.

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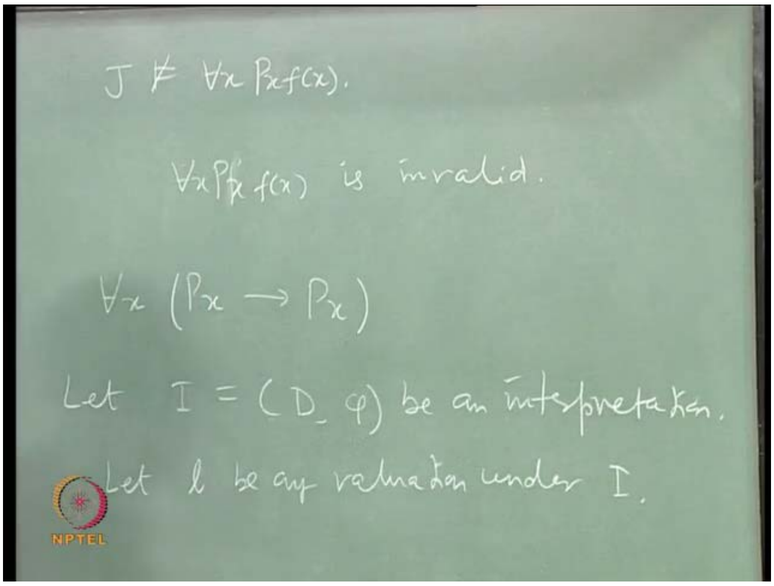

What about this sentence? For each x, let us say, Px implies Px. It should be valid, but to show validity if you take only one interpretation that will not suffice. Because validity requires you take any interpretation, any state under it, that should satisfy it, so you have to really start abstractly.

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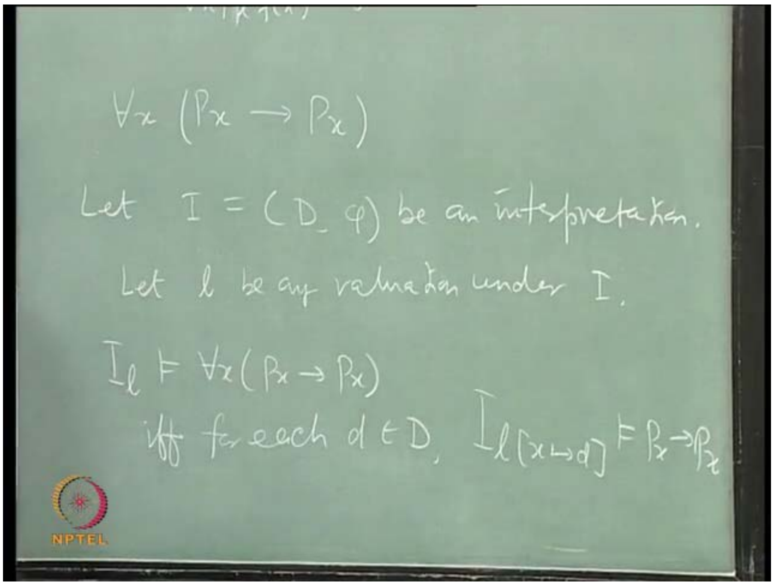

Let I be equal to some D phi. We do not have to specify what this phi is. What this phi is, something; it is an interpretation. In fact, here also you need not consider states, because it is a sentence. Let us start with this; go by the definition.

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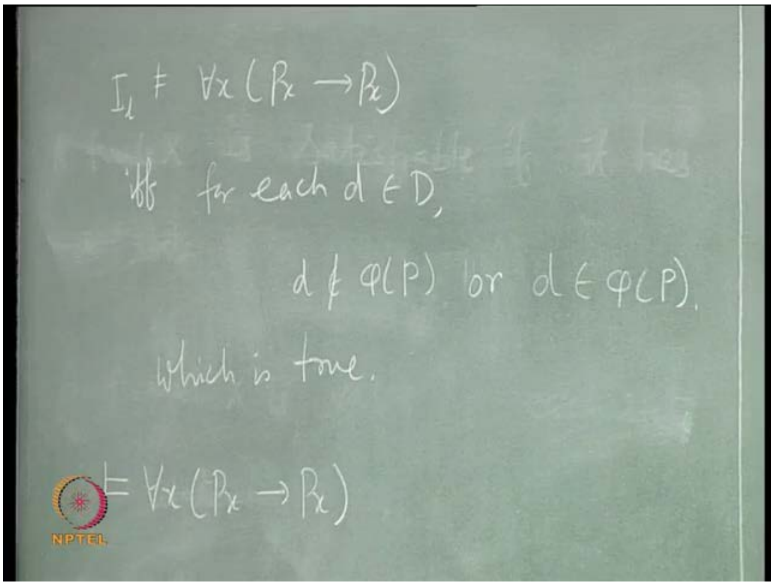

So let l be any valuation under I. Now, I l satisfies for each x Px implies Px if by definition, what will it give? Just go by the definition, what does it say? To look at it formally, if and only if, for each element d in the domain D, when you go for the states, new states, l where x is fixed to d, so we have to see I l x fixed to d should satisfy Px implies Px. Then it will, it will come to I l satisfies for each x Px implies Px if and only if for each d in D, now this phi will be written in terms of phi; so, you say that I l x fixed to d satisfies, you want Px implies Px; so it will go to what? l is Px to d now. It will say, if it is translated to English sentence: if x belongs to phi of P or d belongs phi of P then d belongs to phi of P; that is how it will look. Is that so? That means you may write this d, if d belongs to phi of P, then d belongs to phi of P, because x is fixed to d and p belongs phi of p under I.

So, I l x to x fixed to d satisfy, when this happens and implies become if then in English; in fact it is not if then, which I would have written, if d does not belong to phi of p or d belongs to phi of P, according to our formal semantics. Let us write that way, or that is how we have written; this implies, which is true. Therefore we conclude that for each x Px implies Px is valid. Again, when it is valid we will write the same, whereas in propositional logic will prefix that with the symbol FL, so this is valid.

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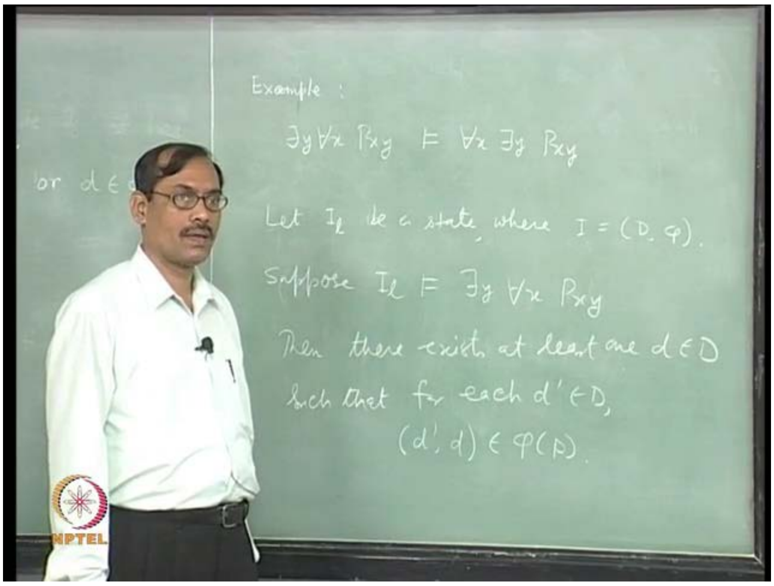

For equivalence, let us take one more example. Again does it happen? This one says, in abstract terms, that there is one d in your domain, there is some particular element d, let us say, so that whatever x you take, does not matter, x and d will be related by P. This only asks you, whatever element you choose from the domain you will get some d corresponding to that so that x and d will be related. So, you take the 7, d as earlier. It should work. But then how to prove it? You have to start with interpretation.

Let I l be a state, where let us say l of x, l of y are given. You are not writing it because we have not specified it, where I equal to D phi. Let us start with that. Then suppose I l satisfies there is y for each x Pxy. Once this happens, we want to show that the same I l satisfies the other sentence. Now, when I l satisfies this, what do we get? It will say, there exists at least one element d in our domain; let us write it; then there exists at least one d in D such that for each d prime in D we have d prime, d belong to the relation phi of P; this is what it says.

See, we want to show I l satisfies each x there is y Pxy. That means, this there exists at least one d and for each d prime, somehow it should come out. Now, how to bring in that? If we can visualize this d, for example let us see as natural numbers, then it will look like 0 d belongs to phi of P, 1 d belongs to phi of P, 2 d belongs to phi of P and so on. Then we can say whatever the natural number is that d works, that is how we are thinking. So simply we would say that this d does not depend on d prime. Here, that is what it says. You can say it is a vacuous dependence, the same d works for every d prime.

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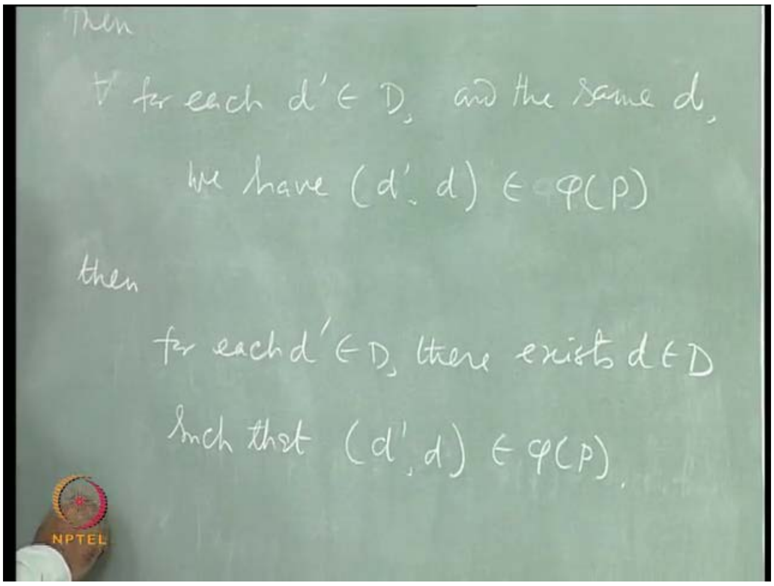

Then we see that for, then, for each d prime in D and the same d, we have d prime d is in phi of P. We have just restated it, to make it understandable in a better way. Which says that for each d prime in D there exists d in D such that d prime, d belongs to phi of P.

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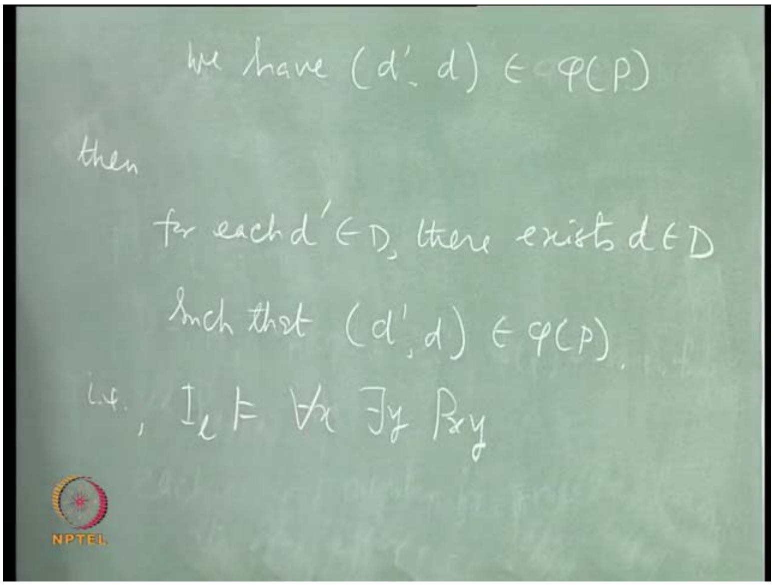

Which means I l satisfies for each x there is y Pxy. Now, then it also says something why the converse does not hold.

Writing this way might give you some problem that they are just commuting. But they are not really commuting. There is some understanding, it is going on along with that; it is not just formal writing. Here how to show that this does not entail this? If we just go on writing like that, it will not. We have to give one particular interpretation where one sentence holds the other sentence does not. That is what we need to show. This is not just writing like this and then say the same d prime, the same d make not work for all d. That will not give the answer. If it really, give one example. Now to show this, what we do, take I equal to one particular interpretation, we want and let us say, and some phi with, see phi of P is.

Student: Equality.

Equality? Less than or equal to will do. Then this interpretation I, I satisfies or each x there is y Pxy. Because? What is the reason? It is simply translated to some sentence in this interpretation. It is a sentence, right?

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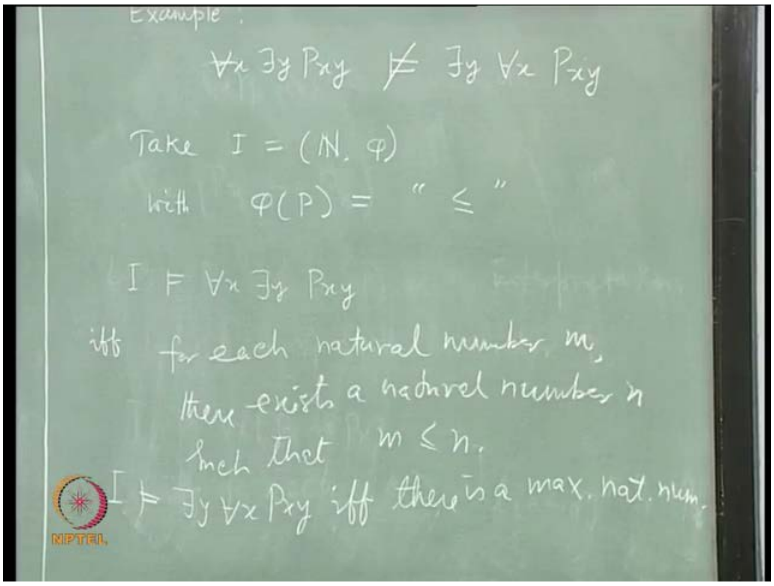

We do not want to go to I l. Here it is. The translation is what? For each natural number there exist a natural number, say for each natural number m, there exist a natural number n such that m is less than or equal to n; that is correct.

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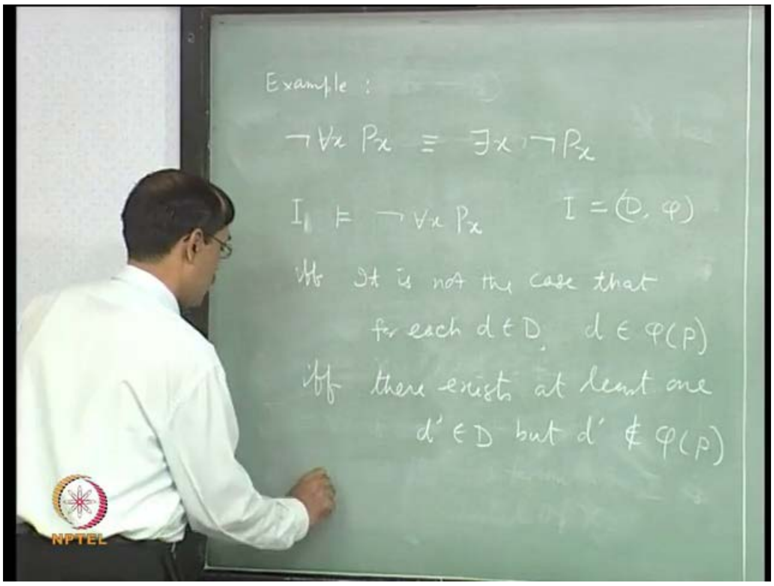

For each natural number there is a greater natural number; that is what it says. For each natural number m there exists a natural number n such that m is less than equal to n. But, now I satisfies the other sentence: there is y or each x Pxy if and only if what happens, it is there exists a natural number n such that, whatever natural number m you take m will be less than equal to n. There is a maximum natural number, which is wrong. Natural number, which is false. Therefore, this does not entail the other one.

Let us take this equivalence. It some such law, of De Morgan, right. Because for all or for each means for each d and d in D, should be true; which is something like P of d1 and P of d2 and P of d3 and so on. And there exists one means for some d, P should hold. That means Pd1 or Pd2 or Pd3 and so on. This is something like your De Morgan's rule. Now, can you show it? Well, we have to try with some interpretation say I l satisfies not for each x Px. We want to show validity, so no example will do. You have to start abstractly. Here, let us take I equal to D phi; l, we need not fix. Here, we can start with I itself due to relevance lemma; so let us try that; if it goes like this.

Now, I satisfies not for each x Px if, what happens, it is not the case that, that for each element in the domain Pd holds; d belongs to phi of P. See, phi P is unary here, so phi of P must be a subset of D. This says, it is not the case that for each d in D, d belongs to phi of P. Which means there is something beyond phi of P. But, it is in D, so, if and only if there exists at least one d prime in D, but d prime does not belong to phi of P. It is just the same thing as that, which does not belong to, it belongs to phi P complement, that corresponds to not. So, if and only if, I satisfies there is x not Px.

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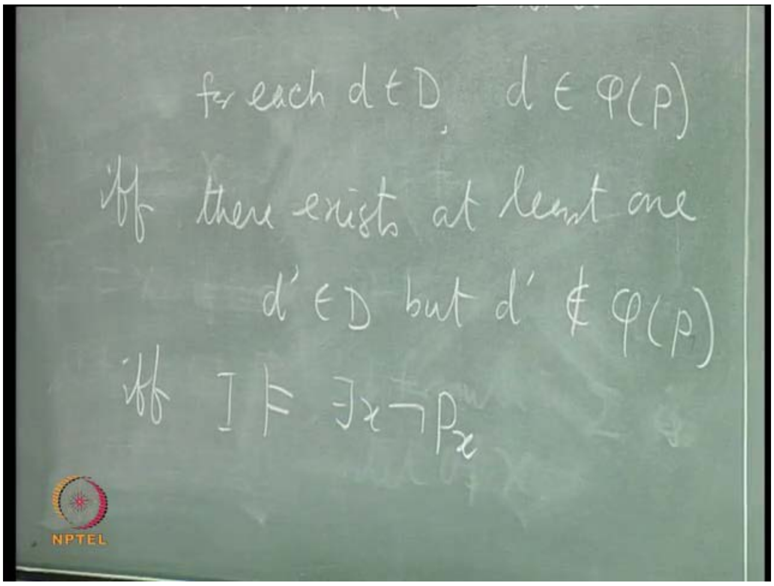

Let us go back to our discussion on the relevance lemma. Where, we had one formula Px and we are thinking of which, where to look at it? Suppose Px is satisfiable. Now what will happen to for each x Px or there is x Px? These are the question. Let us try Px is satisfiable. That means, there is one state which satisfies Px; it is a state model of Px. l x, we do not know what it is. Let us write it as d in our domain. Here, I take I equal to say D phi. When you say I l satisfies Px, it means P l of x.

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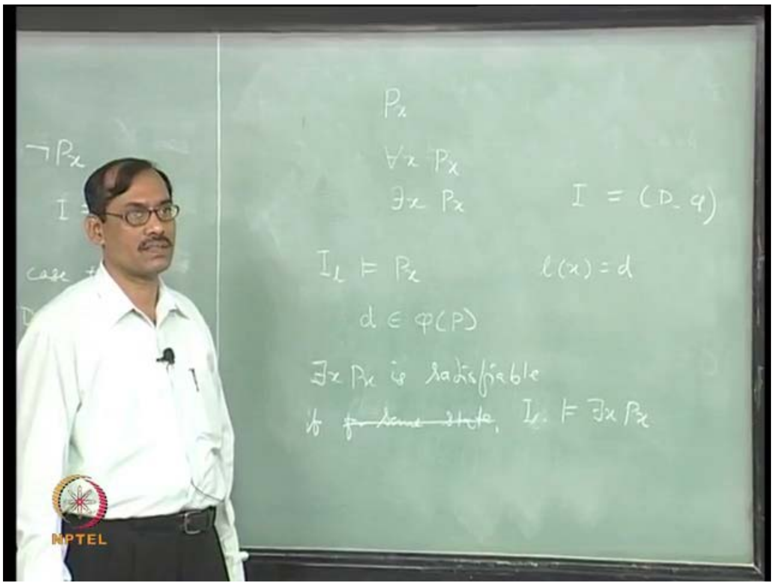

So, d will be coming up; d belongs to phi of P, that is what it will say. This particular element, to d which l assigns to x, must be inside your relation, subset of that relation, subset of that domain D. Will this be true for each x Px? It will say every element is in phi of P, P is the totality. P is the whole of D, that is what it is says. But, will this be true because it requires there exist one element d which is in P, that is exactly. This is that or if you look at this, you say I of, you say there exists x Px is satisfiable.

If for some state say I l prime, I l prime satisfies there is x Px, that is our definition, now I l prime satisfies the x Px, is the same thing as telling I satisfies the x Px because of relevance lemma. So, you may have forgotten the l prime. Forget this state. We simply write I satisfies there is x Px for some interpretation. It may not be that interpretation, some interpretation. We know I satisfies there is x Px. Now, this is done when there is some element in the domain which is in the relation, which corresponds to P, that is what exactly told by this I also.

It says, if I start with this I, I of d belongs to phi of P. Instead of any other I, I can take the same I, same phi, so d belongs to phi of P, satisfied. That means Px is satisfiable if and only if there is x Px should be satisfiable. But if you consider validity case, it will be different. It is not there is x Px, it should be for each x Px; let us see why it is so.

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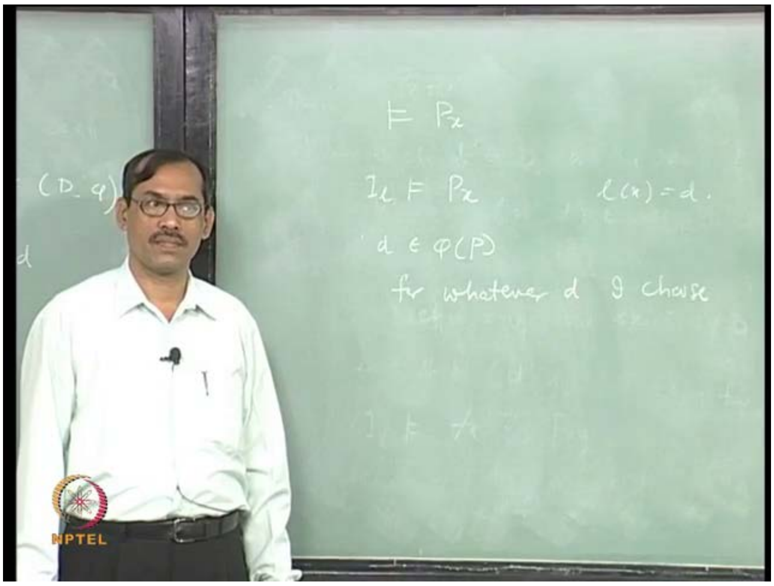

Suppose we take that Px is valid. Once we say Px is valid, it means whatever interpretation you take whatever state you consider under the interpretation, that state satisfies Px; that is what it says. Now, we say I of l satisfies Px for whatever state I chose. Suppose l of x is some d. It says d belongs to phi of P, for whatever d I chose. Because this is true for every I l, every state. I choose the l's, so that every element is, now obtained by, some l. Then that is equivalent to varying this d also. Whatever d I chose, d belongs to phi of P. So, it is same thing as telling for each d in D, d belongs to phi of P. For each x Px is also satisfied by the same I. But why same I? Any I, I choose from the beginning, the same way it will proceed. If I chose that, I start with that, I find that the same I satisfies for each x Px. And whatever I, I choose, does not matter. The same argument still holds. That means Px is valid if and only if for each x Px is valid. This argument, really, we can generalize a little bit. Instead of having only one free variable we can have many free variables. We have to go accordingly. Let us give a definition.

Suppose X is a formula with all its free variables as x1 to say xm. Here we are not telling that it has, these are the variables which are from the beginning, that x0, x1, x2. We are just making it abstract; some x1, xm. Or, you can write y1 to ym, if it is confusing. With the syntax, then we write X as X square bracket x1 to xm; just to say that these are the free variables, these are all and only free variables in X, nothing else is there, just to give that information we write this way. Now, we define that the existential closure of X is, there is star X, which is equal to there is x1 there is x2 there is xm, x1 to xm we are just giving another notation, because we do not know what are the variables.

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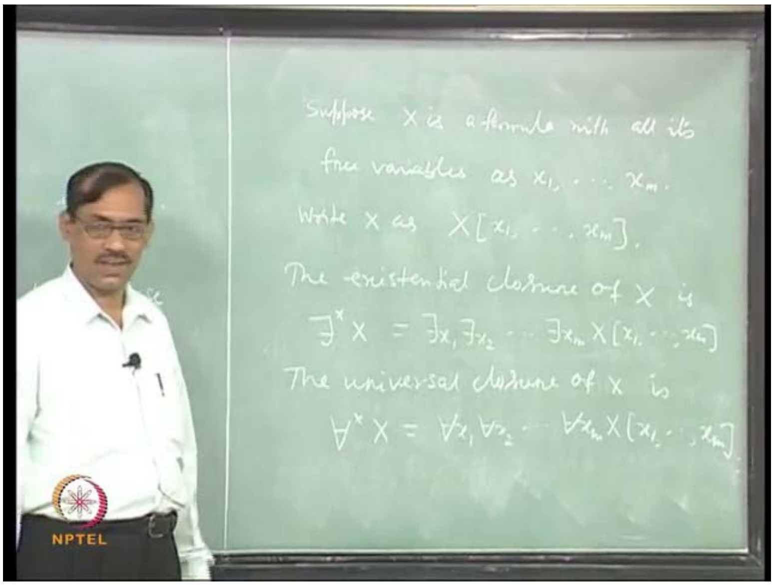

So, write it this way. There is star X, which means you take any free variable there, existentially quantify over all these free variables, whatever sentence you get, that is the existential closure. Similarly, we say universal closure. The universal closure of X is for each star X, which is for each x1 for each x2 for each xm, X of x1 to xm. Here, as we have seen for Px, there is x Px is the existential closure; and for Px, for each x Px is the universal closure. And all these quantifications are done in the beginning, not inside anywhere. Of course, inside will be arbitrary; where to do it? So this fixes it. Our observation can be really summarized. What we have observed from this examples is: X is satisfiable if and only if its existential closure is satisfiable; and x is valid when its universal closure is valid.

In fact, the way we have introduced the semantics, had a bit of confusion, now that is removed by this theorem. The confusion is this. You consider the formula Px implies Px. If you translate, it will look like, if x is a P, then x is a P; something like this. Suppose P stands for x is some man, or something. See, if x is man then x is man. But, we do not know what x is, so without having a particular thing for this x, it is not a sentence. Also how do you say it is valid? Some philosophers will not be able to accept it at all, because x is a variable it is a named gap; there is nothing there, how can you say that this P, that is, P should not be allowed, that is what they say.

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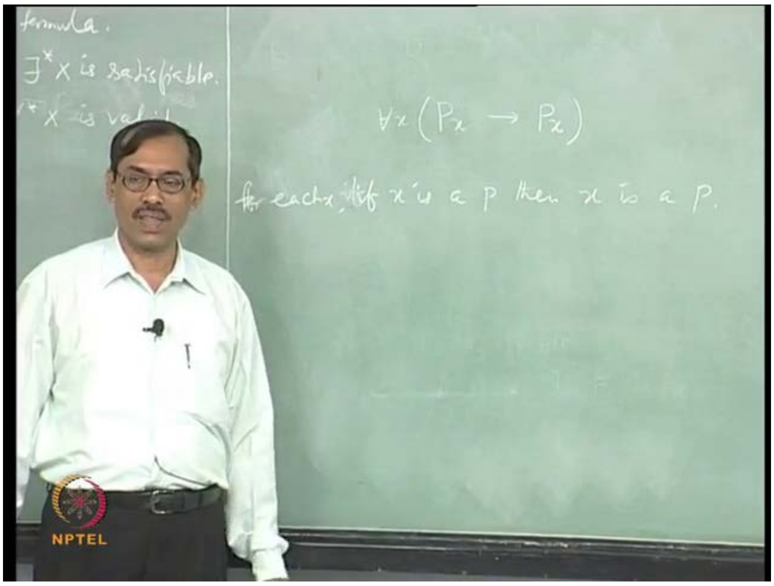

But, our semantics says, yes, it is valid; because you take any state, you are now in an interpretation; any state I l, that l of x will become some concrete person, in some domain, a concrete element. And this P will become a concrete relation. So, x is a P will have a meaning. Now it will say d belongs to phi of P. Under any state, it will say if d belongs to phi of P, then d belongs to phi of P; so it is allowed. Interpreting open formulas by states is philosophically doubtful, but we need them in programs. So, we have to do it. But now this theorem says that whatever we have done, whatever confusion we have brought in, can be resolved.

In fact, you look of, look at them as sentences by universally quantifying over it. If it is validity, then there is no problem, it becomes a sentence. Now, it will look like this and we just say for each x, which is a sentence. And this really gives an alternatives semantics. There, what you do, you just interpret the sentences, do not interpret the formulas at all; interpret only the sentences by translating them into some concrete domain. You know how to deal with the concrete domain. There is some concept of truth there, so translate it; verify the truth there.

Now, sentences can be interpreted. But then how to go for formulas? Through this theorem, you say, formula, any open formula is satisfiable if its existential closure as a sentence is satisfiable. You do the other way around. That also can be done, if needed. And validity, similarly, can be just is this. We will not prove it, because we have already proved it. This particular case Px and for each x Px, there is x, there is x Px and Px is satisfiable, The proof is same. That is the crucial step; which will come in the inductive step. It will be done by induction, now on the number of variables, free variables in X. We will just give an outline.

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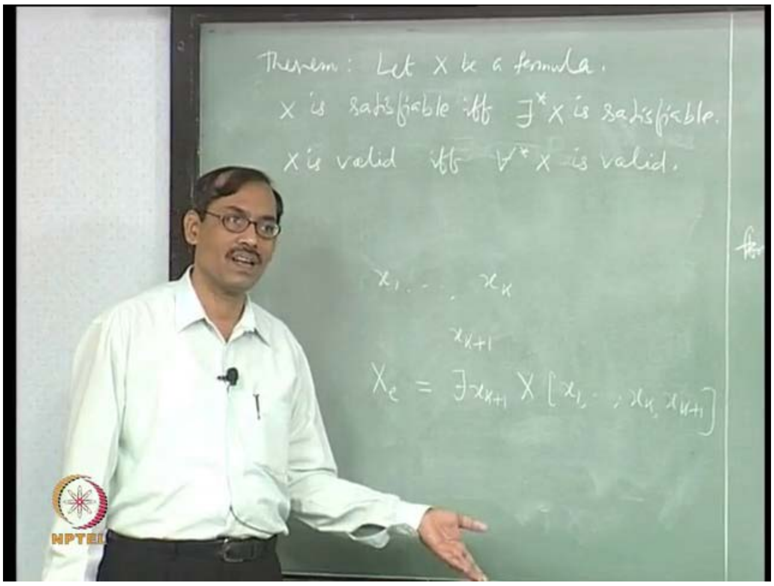

Suppose there are x1 to xm, X, k is number of variables, of k, if it is 0, it is a proposition; it is a sentence. There is nothing to do. Suppose for k free variables, it is alright. Now you have another free variable, that is the inductive step. Then what you do, first write Xe equal to there exists x k plus 1 and then all the others x1, xk, x k plus 1, only one variable I quantify. What happens, Xe is having only k free variables. Your induction hypothesis will apply on x e. All that we have to prove is, X is satisfiable; this X is satisfiable, if and only if, Xe is satisfiable.

Once you do this, you use induction hypothesis and then go for the conclusion on Xe. And the proof is that one, which we have done: Px, there exist x Px, there is only one free variable, one existential quantification there. So, same thing we will proceed. The proof is not a big thing here.

All that we have done today is, given the satisfiability, validity, invalidity and consequence, equivalence. Then we have connected the satisfiability and validity with the relevance lemma. We see that sentences can always be interpreted directly. This gives rise to one alternative semantics, where you first define satisfiability or validity for sentences through interpretations, without going to the states. Then open formulas are satisfiable or valid according as they are so, the existential closure or universal closure for satisfiability or validity respectively. So, states are not required; that is what the simplified semantic says. You can directly interpret this way. But, I told that this is up to some extent, so that some extent is, till we are concerned with satisfiability or validity alone, nothing else. But you may require something else. So, states really required for that purpose. That example you can take. If there is a program and something is happening you have to work till termination, that decides satisfiability or validity. But inside what is happening, you cannot be, you will not be able to analyze. If you have a state concept, you can analyze, in, what is happening in the next step, even if it is not terminated, that is that advantage. So, we are going with this semantics.