Mathematical Logic
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Lecture - 26
Relevance Lemma

So, let us start with one example first; see how the quantifiers are interpreted.

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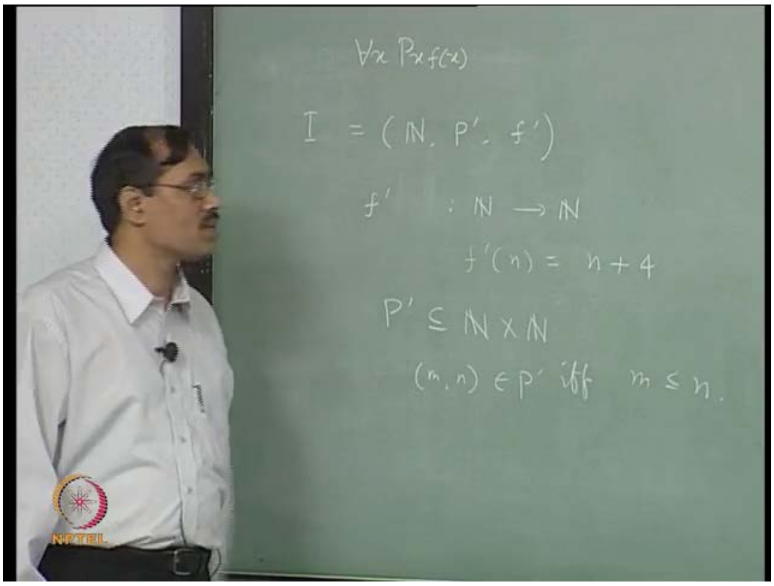

Let us take, say, for each x, P x f of x, so was, this is the formula. We have only one quantifier here, and it is very, looks very simple. P x f of x. f is one function symbol, P is one predicate symbol; it is a binary predicate. We start with one interpretation. Say, let us take natural numbers again. And in this, you have to tell how this P is interpreted, how f is interpreted. Sometimes, instead of writing phi, here, and then writing phi of P is this, and phi of f is this, you just write, say P prime and f prime. Implicitly telling that these are the predicates to be interpreted. That P is to be interpreted as this P prime, and that f is to be interpreted as this f prime. You have to tell what are this P prime and f prime. Let us say, f prime means, f prime of n, f prime is a unary function symbol, only one argument it has, so f prime, which corresponds to this function symbol, should be a unary function, unary partial function, let us say.

We will write this as on D n to n; it should be defined on the domain; now define by, say f prime of n is equal to, usually do not write f prime of n, in the beginning. Then we give this value at n. We write this, let us take it this way, where f prime of n is, say, n plus 4. Somehow, we have to define it; that is, it is n plus 4. Now, what is this P prime? P prime has to be a binary relation on the natural numbers; it is a subset of N cross N, which subset we have to tell it. Now, which one we will take? Let us take, say, P prime is less than or equal to. We say m n belongs to P prime if and only if m is less than or equal to n. This defines a function phi of f which is n plus 4, and phi of P, which is less than or equal to. For all, you have to write so much, what this is, what it says.

Now then, we will see how this one is interpreted. As it is, if you see uniformly, it says that for every natural number n, n is less than or equal to n plus 4; that is what it says. But then, let us look at how the formal semantics takes care of this. Because we have to go to the states and then from the states, we have to come to interpret this quantifier. Let us start with one state. The state will be starting with a valuation. There is only one variable occurring here. But, you can define for other variables, that does not matter; they will not come into discussion at all.

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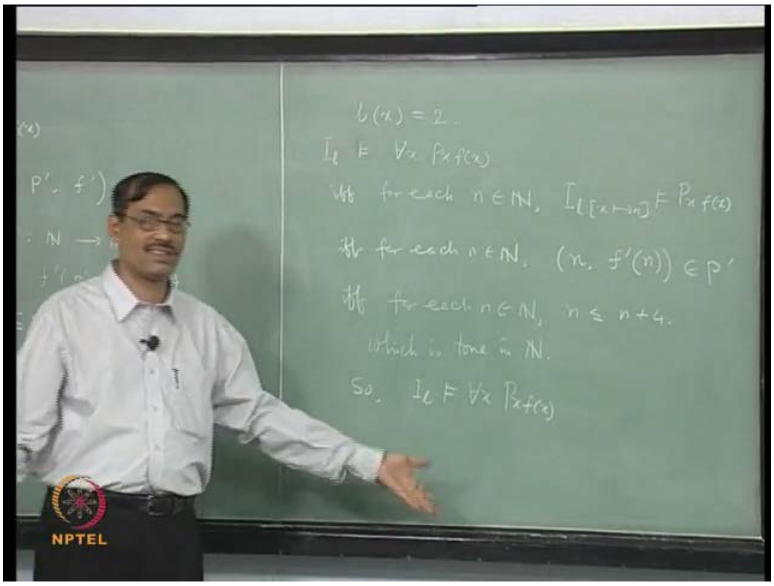

Let us say, l is one valuation which is under this interpretation. Under this interpretation means, of this formula, that formula is also there. There is no constant symbols occurring; only one variable is there; you can just give l of x equal to some number, natural number, and continue. Let us say, l of x is equal to 2. You are just taking arbitrarily and see what happens. Now, we have this state I l. So, I l will satisfy for each x, P x f of x, if, what happens? You have to go for the definition. If, for each element in the domain, we go for the obtained valuations, where l will be changing. In this, l x will be fixed to something some n. Let us, for each n in natural numbers I l of x fixed to n; this should satisfy the formula without the quantifiers P x f of x. Let us go on writing it. For each n in N, what about this? When does it satisfy P x of x, that we have to see. By definition, it will satisfy when the corresponding pair belongs to the corresponding predicate.

Now x, instead of x we have to take l x fixed to n, is that so? Once you take l of x fixed to n of x that is n. But, we have taken x to any n. The same n will come. We will be writing n and then l x fixed to n f of x; f is interpreted by I directly, phi of f, right. Now, this will be phi of f of l x fixed to n of x. That is f prime of l of x l x fixed to n of x. That is again n belongs to P prime. You have omitted three steps there. If you go on slowly writing, you have to write l of x fixed to n and so on. This says, if and only if for each n in n, n is less than or equal to f prime of n, which is n plus 4. Which is true, which is true in N. So, we conclude that I l satisfies for each x, P x f of x.

Now you see how the quantifiers are interpreted, so that they are overriding the valuation itself. You have never used l of x equal to 2 anywhere. Because ultimately we will be getting to l of x fixed to n and x is fixed to n. The old l of x equal to 2 is of no use. But let us see another example, where it is not a sentence; see what happens.

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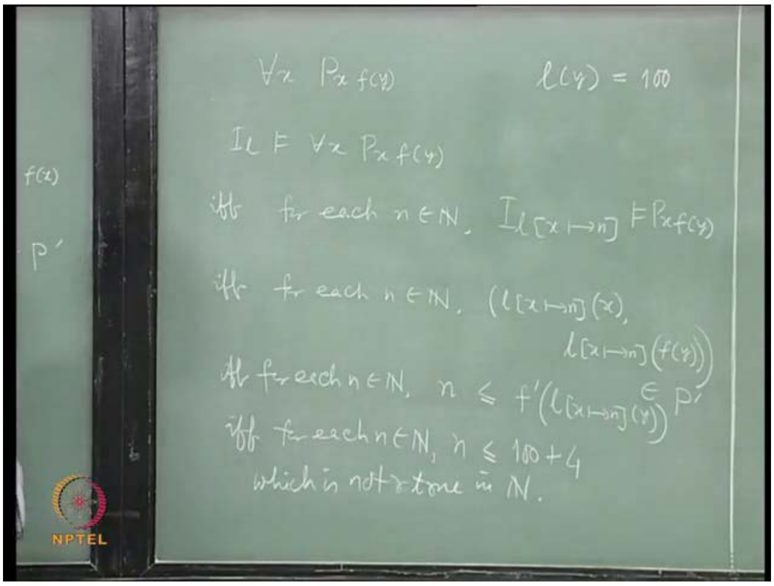

Suppose I say for each x P x f of y, so here y is a free variable; it has not been quantified. If the same interpretation, let us try and fix l the same way also. Now, it is I l satisfies for each x P x f of y; but we need to interpret y here, because l of y, you do not know what it is; you have to fix that, then only it will count, so let us fix. Say, l of y is equal to some 100. Suppose this is our l which maps x to 2 and y to 100, now then this will be, if and only if, for each natural number n, I l x fixed to n satisfies P x f of y if and only if for each n in N. Now, I l means P is less than or equal to anyway, so x will be taken as n in this new valuation, x fixed to n, so that it is n. Then this is your P and what is f of y? f is again f prime, it may be something, plus 4. Now, it is y, l x to n of y, that will be used not l x fixed to n of y, but since l and l x fixed to n, I agree on y, that will be same as I l of y. They can only vary on x, because x being fixed, all the others are the same as earlier. So, l x fixed to n of y is equal to l of y and l of y is equal to what? 100. So, it will be 100 plus, this f is working, 4 or I take some more steps, we will write 1 or 2 more lines.

If we go for that, it will look like if l x fixed to n of x, l x fixed to n of f of y this belongs to P prime. Then this is so, if for each n in N. Now, l x to n is n p is less than or equal to, then you have to evaluate this l x fixed to n of f of y, so that is again f prime of l x fixed to n of y. That gives for each n in N, n is less than or equal to, now, f prime means just whatever its argument plus 4. Now, this l x fixed to n of y is l of y which is equal to 100, that is why this is 100 plus 4. This does not hold, not true in N. That is why I l does not satisfy for each x P x f of y. It does not satisfy.

But then you see how this l has been used for y earlier; it was not being used at all. Now, it has to be used, because it is occurring free. Once it is free it will give some particular element in the domain, is that clear? This really leads to something. Can you tell why? Well, what you are telling is, each time when you choose different value for y, your l changes for different valuations, the corresponding state may or may not satisfy, but it has to be the same. So, the interpretation itself does not interpret directly this open formula; it is a state which interprets. But in the other case, when it is a closed formula, a sentence, whatever l you would have chosen, does not matter. The interpretation directly interprets the sentence. Even if it is defined through the valuations, finally it does not matter; the quantifiers really overrides these valuations.

Then it tells something about the difference between closed formulas and open formulas. You really need states for open formulas; you may not need states for the closed formulas. And the clue is the set of free variables occurring in it. Suppose you imagine there are, there is a formula, where you have some free variables and this valuations l or l prime, let us take two valuations which agree on all the free variables; then what will happen? Finally, if I l satisfies, will I l prime satisfy and so on? This is what these two examples hint at. This would be, it looks, this should be right, and that is exactly your relevance lemma. It says that whatever relevant to that sentence you need to be concerned about, that all that things, you need to forget.

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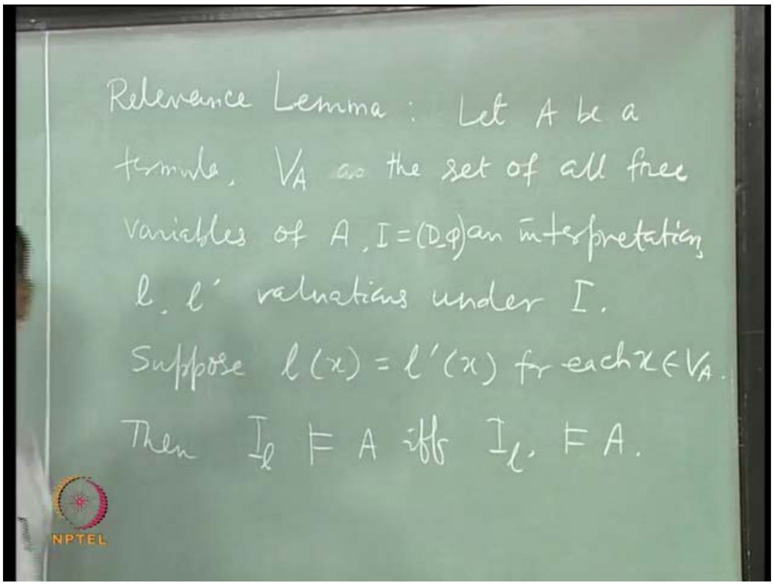

Let us formulate it. Let A be a formula and let us write V subscript A as the set of all free variables of A, and let us take I to be an interpretation. You may write this I as an ordered pair, with domain and the map say I is D, phi, an interpretation and let us take two valuations l, l prime, valuations under this interpretation I. What we say that if this two valuations agree on V of A, all the free variables, then this would evaluate the same way; that is what it says. Suppose l of x equal to l prime of x for each x in V A, then the states I l and I l prime should evaluate the formula A the same way. So, I l satisfies A if and only if I l prime satisfies A. This is what we conjecture.

How do we prove this? Well, it seems there is nothing else, but induction. Because our semantics itself is defined inductively, starting from the base cases where there is no connectives, no quantifiers and then slowly introducing connectives and quantifiers. So, let us have the proof by induction, on the number of connectives and quantifiers in A taken together.

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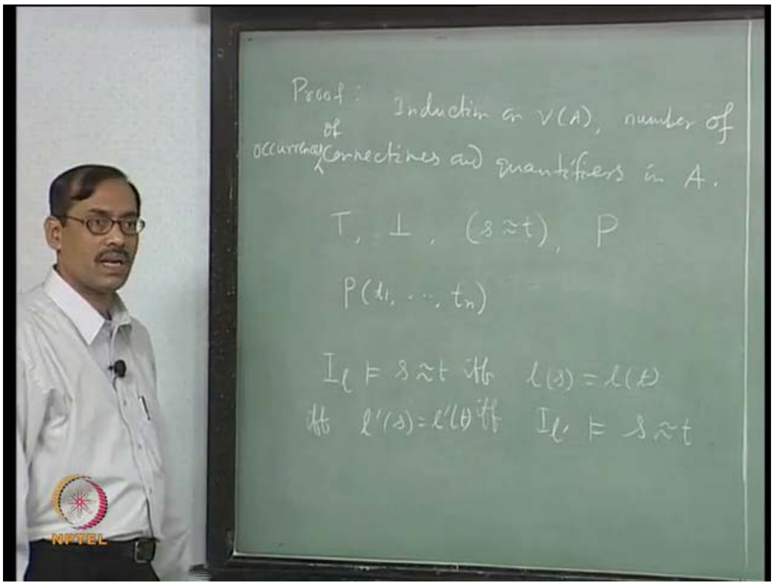

Suppose that is nu A. We use induction on nu A. This is the number of connectives and quantifiers in A. It is really number of occurrences of, this is not number of, there can be, there may be only one connective occurring 100 times, in that case, 100 will be contribution to nu A, not 1.

You should write number of occurrences. What is the basis case? No connectives, no quantifiers; then how does it look like, A look like? Yes, there is no connective, no quantifier beginning or it can be bottom or it can be of the form s equal to t. But, before that it may be some proposition without anything 0-ary, P is 0-ary, or it can be P of t1, tn. These are the basis cases. Is that? That is how we have given the syntax of the language. So, it can be top, it can be bottom, it can be some 0-ary predicate, a proposition itself, or it can be binary with equality, or it can be any n-ary predicate; there might be function symbols inside this, t 1 to t n, they are terms.

But, no connectives, no quantifiers, that is important. That is our nu A, so nu A is 0, that is our basis case. Now, in all these cases, just try to see what is the conclusion. I l and I l prime should be evaluated by the same way. Is it happening? This case is easy. I l of top is always same whatever the l, whatever I maybe, right. That is satisfied. This is also, what about this proposition? There is no variable at all, no variable at all. So, l or l prime does not matter; they evaluate the same way, is it? Now, what about s equal to t, variables can occur, but also constants can occur. But, constants will not give any problem because l of any constant equal to l prime of the constant. Constants are interpreted by, directly by I, I l of those things come from the definition of I itself. So, l of any constant c is equal to l prime of any constant c, equal to phi f c. They directly give rise to the elements of the domain. So, constants give no problem, variables can, right? If s equal to t and the variables are interpreted the same way, so l and l prime interpret the same way. It is there again; it is an inductive step on the number of variables occurring in s and t.

That is again another inductive step, if you do it formally. But that is clear. There are variables in X, so one variable x is occurring, then what happens? l of s given the evaluation of l of s, which element is it? You have to go for l of x; then all those instances of l of x will become substituted by l prime of x, because they are same. So, finally it will come to the same element and finally, s equal to t is interpreted as the equality itself, the same as relation. So, it will come this way. I l satisfies s equal to t if and only if l of s equal to l of t, because this identity or equality symbol is interpreted as equal to in the concrete domain, whatever your domain is. Then this will give l prime of s equal to l prime of t.

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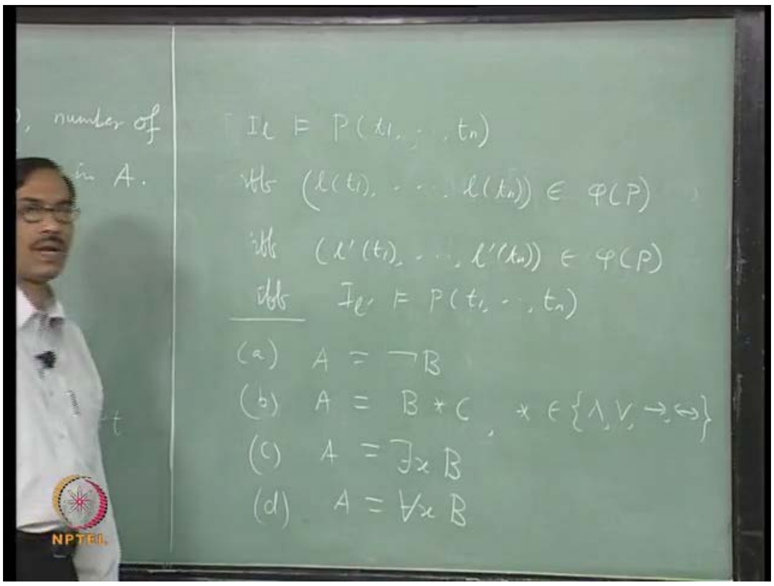

Since on the variables they are interpreting the same way whatever, with the variables, whether it is originally quantified or not quantified in A we are not bothered, there is no quantification. Here, occurs then, this happens if and only if I l prime satisfies s equal to t, this case, what? P t 1 to t n it is similar to this. Instead of the equality we will have P. Here, let us do it; just you have to write some more lines. We will start with I l satisfies P of t 1 to t n if l of t 1 to l of t n belongs to phi of P. If they are so related as the corresponding relation to P, and then next two lines, similar things; l of each term equal to l prime of each term. You just go to l prime of t 1, l prime of t n belongs to phi of P; and that is exactly what you wanted. I l prime satisfies P of t1 to tn. So, basic step is clear.

Now for the induction step, suppose number of occurrence of connectives is k, or up to k; that statement holds then for k plus 1, so that it holds or not? Now, if it is having number of occurrences of connectives and quantifiers taken together, k plus 1, how will it look like? Now again, structure, because that is how we have defined our formulas. So, then that structure will have not in it, not is one connective possibly, or any binary connective, or it will have for each x, for some variable x, and then a formula, or there is x and another formula. You will essentially get four cases. In the induction step, we may have A equal to not B or A will be say B and some binary connective C. So, this star may be in and, or, conditional, or biconditional; one of these. Of course with parentheses, or it can be in the form: there is x B, B is another formula, x is a variable; or it may be in the form A equal to for each x B. One of these four. Then in each case, let us see.

Now if A is having k plus 1 as A nu, nu of A is k plus 1, then nu of B is k; one connective less. And if A is B star C, then B, C will have nu, less than or equal to k. That is why we need strong induction. You assume for all less than or equal to k. This case, connectives may be same, but one quantifier is less. In B again, nu of B will be less than or equal to k, equal to k. Here similarly, for each x B will have again equal to k. So, you can use induction hypothesis on these and then go for the induction step. That is what we will be doing. Let us take not B case. If A is equal to not B, then you start just like this; and A is, the definition of satisfaction. Say, case a, we will say I l satisfies not B if and only if I l does not satisfy B, by definition. Now, use induction hypothesis. B has only nu less than or equal to k. So, our conclusion holds. I l prime does not satisfy B. Then, if and only if I l prime satisfies not B, it is too mechanical. Similarly, case b, just the connectives, propositional rules.

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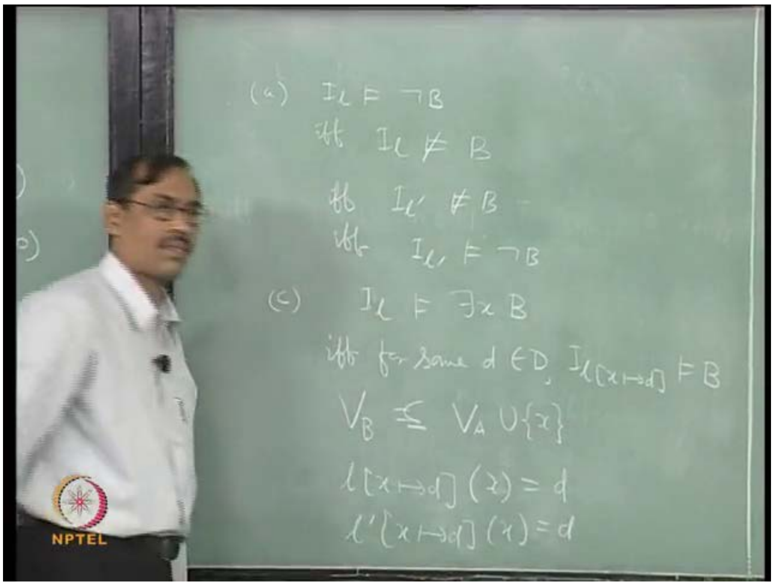

Now, let us see the quantification case. Suppose A is there exists x B. Now you say I l satisfies there exists x B; this happens if and only if, just the definition, for some element in the domain, for some d in D,d I l x to d, x fixed to d, satisfies B. That is what it is. How to use induction hypothesis on this? That is our point.

Now what happens, l x to d, this new variable x is occurring in B; probably we do not know exactly, and it might be occurring free there. But, in A it is not free. It was there exist x B, it is in the scope of that quantifier there exists x. It was not free. We observe that the free variables of B can be an additional free variables of A union x. We have l and l prime agree on the formula, then we can use it. Our induction hypothesis is on this statement, where we have l and l prime agree on V A. That means, on V A only, l and l prime will agree. What about this x? We need that to apply the induction hypothesis. See your problem? It is not straight forward here. For x also, we have to see what is happening. Now, let us see. l x fixed to d of any variable in A, we are not worried. Only for x, we are verifying; if you take any variable y that, of course comes as it is, then of x is equal to d. What about l prime x to d? That is also d, though l and l prime may not agree, we do not know what they are, but l x fixed to d, l prime x fixed to d. they agree.

Now, we are not going to see whether l and l prime agree, but whether these two agreeing or not. Even on the other variables. Because there we are applying the induction hypothesis. We have to verify all these valuations on any y. But that is not difficult, because l x fixed to d of any variable y is equal to y and l prime x fixed to d of y is also equal to l prime of y, and l x l y and l prime, all agree. So, these two also agree, fine. Therefore, these two valuations agree on all the free variables of B. So we can apply induction hypothesis on l and l prime, when x fixed to d, not directly on l, l prime. We will write it again.

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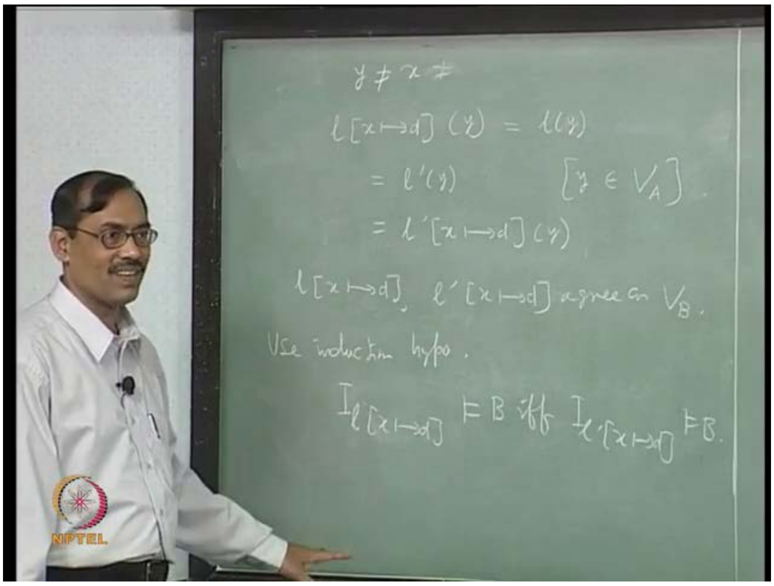

Now you verify; l x fixed to d of y where x is not equal to y or let us say y is not equal to x, so this y equal to l of y by definition, because x is only fixed to d, all others are as they were. So, this is l of y and l of y equal to l prime of y, y because if y is not equal to x and y belongs to free variables of B, then y belongs to free variables of A. V of B is a subset of V of A union x. Here, we gave a comment. Because y belongs to V A, due to this, then this is equal to l prime of x fixed to d of y, only x is fixed to d, others are same. That means, l x fixed to d and l prime x fixed to d, these two valuations agree on V B. On V B. Because in V B, you can have free variables from A, for x, x we have verified, and free variables of A are, here which are not x, is that so? These two valuations agree. Then use induction hypothesis, it is here, hypothesis to conclude I l x fixed to d satisfies B if and only if I l prime of x fixed to d satisfies B, nu of B is less than equal to k, that is why you are able to use it; all those three are required. Once that is done, you come back to this statement. This says, for some d and d I l prime of x fixed to d satisfies B, therefore I l prime satisfies there exists x B.

Let us take one nice corollary of this. Suppose A is a sentence. what do you conclude from relevance lemma?

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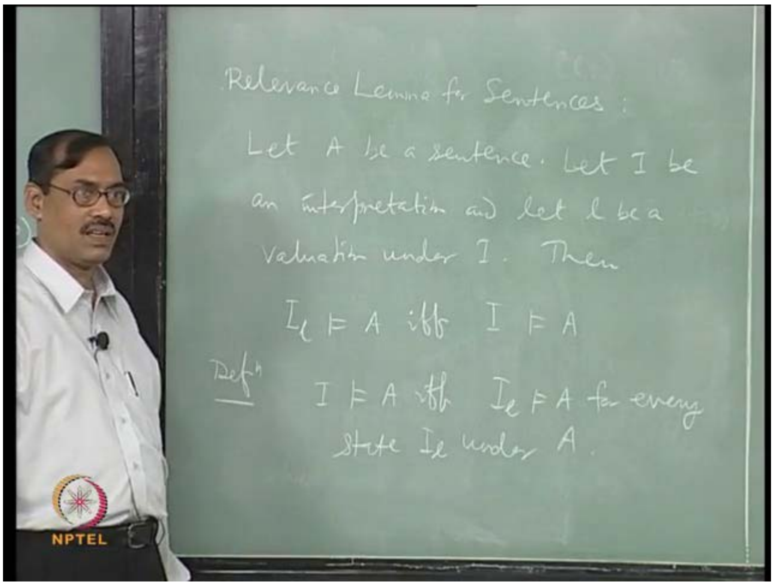

No free variables; so we will be starting with a sentence. Let A be a sentence. In that case free variables A is empty, V A is empty, there is no free variable. Then taking two valuations. You take l and l prime. They will agree on V A, whether, there is no case of difference, so once they agree your conclusion will hold. Conclusion is I l satisfies A if and only if I l prime satisfies A. So, any valuation l, now let us write it. Let I be an interpretation, let l be a valuation under I. Then I l satisfies A if and only if, you take any other valuation, l prime, then I l prime also will satisfy A. So there, we will write I itself satisfies A, we will give the definition of; definition says I satisfies A if and only if I l satisfies A for every valuation, for every state I l under I.

We have only defined satisfaction from states point of view. How a state satisfies a formula. Now we are defining how an interpretation satisfies a formula. An interpretation satisfies a formula if and only if all states under the interpretation satisfy it. So, this now says any two states you take, they either satisfy or does not satisfy, do not satisfy together. If one of the states satisfies, then everyone satisfies; conversely were everyone satisfies then that one also satisfies. This is what it says. That means, you do not have to go to the states to interpret the formulas which are closed. A sentence can be interpreted exactly by an interpretation; no states are required. This was our intuition; was, and A is there. Even if you say there is a sentence and there is a state model of it, it is equivalent to telling there is a sentence it has a model. There is a nice corollary of this again. You can see that if you take any interpretation of a sentence either the interpretation satisfies it or the interpretation satisfies its negation; it should be clear from this now.

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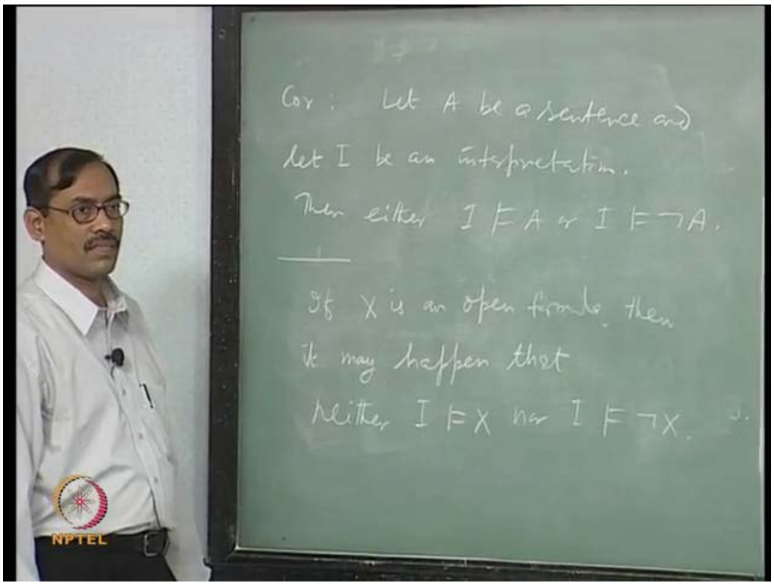

As a corollary, we will have this. Let A be a sentence and I be an interpretation. Then either I satisfies A or I satisfies not A. For the formulas it may not happen. We will come to that. Now, see how to prove this. It is because you take any state l, I l, so take a valuation under this interpretation. Now, I l either satisfies A or it satisfies not A by definition for the states, now due to relevance lemma for the sentences, it says I l satisfies A if and only if I satisfies A. So, you just substitute with the variable; that proves it. But it says something more; proof is easy, there is nothing. It says that if you take any interpretation, it either satisfies the sentence or it makes the sentence false; one of them will happen. That if any arbitrary formula is there, not a sentence, it may do neither, right? So, that means what we are telling is, if X is an open formula, then it may happen that that neither I satisfies X nor I satisfies not X; that can happen. Example? To show this we need an example.

So, definitely you are going for an open formula for the sentence. If you go, you do not get the answer. Now we have to consider one open formula, where there is at least one free variable, it means that. There is a free variable. Let us say P x. There can be more also. Suppose x is the only free variable. Now, can you construct an interpretation, where that interpretation neither satisfies Px nor satisfies not Px? Suppose Px. Well, first domain. Let us take I equal to, you are thinking in natural numbers, so P, only 0 will be excluded, if you take positive integers, only 0 is excluded, let us try. Say, D is Z plus, that is our P. Well, it does not show exactly. Let us write say P prime and P prime is Z plus, set of all positive integers. Now, Px, how Px is interpreted? It is never interpreted, we need a state to interpret it; l of x should be specified, so what is l of x? One case. Let us take 0, another. Let us say l prime of x is 1. Then I l satisfies P x if and only if what does it say? l of x belongs to P prime if and only if 0 belongs to Z plus, which is not, which is not true.

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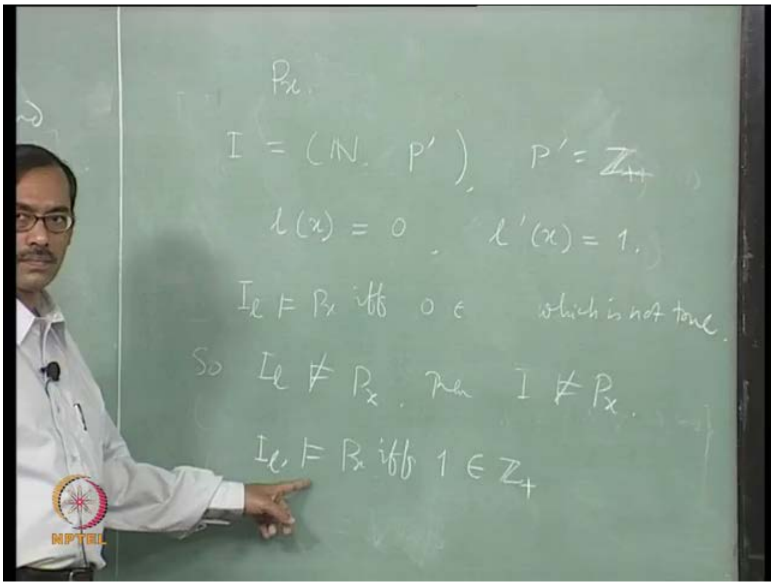

So, we conclude I l does not satisfy Px. Then I also does not satisfy Px, because by definition, I satisfies a formula if and only if every state under it satisfies it. There is one state which does not satisfy. Next, what about I l prime? Let us verify. I l prime satisfies Px if, what happens? l prime of x belongs to P prime, so l prime of x is 1. So, 1 belongs to Z plus. It is. So, this is not the example. Well, at least one more element we need, which should not be there. We can construct P prime, to be everything bigger than or equal to 2. Then take l prime x equal to 1, is that okey? Or, to see it better way, let us take Z plus as, sorry not, here P prime, this is sufficient. See, our aim is to say that I does not satisfy Px. Now, how do we show at least for one state under I, I l should not satisfy A? It is for I l, so I does not satisfy this; again Px, Px, for not P x, we want for not Px now.

So, let us try that way. We want to show: I l does not satisfy not Px; this is what we want, this is what we want.

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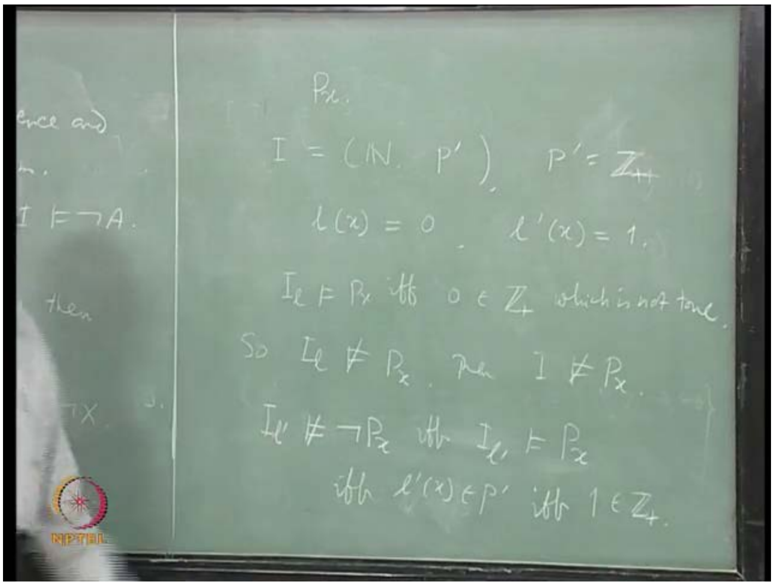

Here, we have shown that I l does not satisfy Px. Now, you want to show I l does not satisfy not Px. But that will not happen. Let us take I l prime, because l prime Px, we have taken as 1. So, let us try I l prime does not satisfy not Px. Is it true? That is what we want to verify. So, this happens if I l prime satisfies Px, not P x, so l prime satisfies Px.

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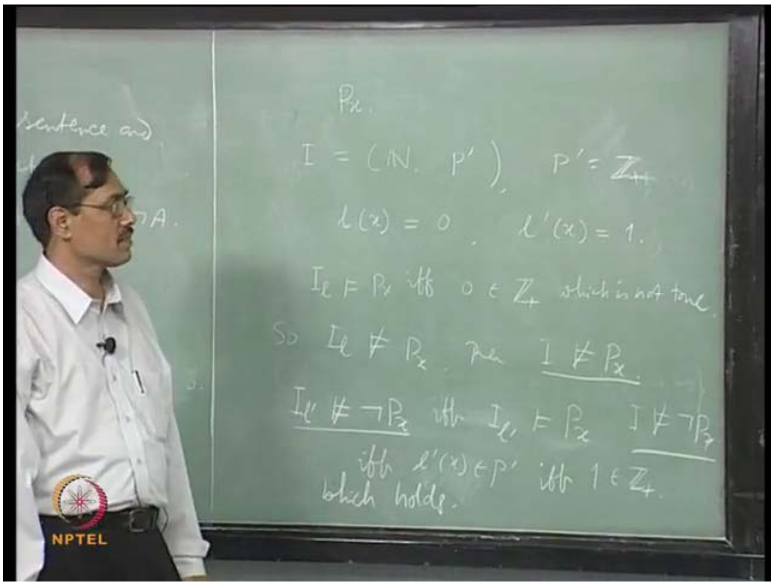

So, this happens if and only if l prime of x belongs to P prime. This gives, if and only if l prime of x equal to 1, so 1 belongs to Z plus, which is true. Which holds. This says I l prime does not satisfy not Px. Therefore, I does not satisfy not Px. So, both the things are done. I does not satisfy Px and this, from this way, you conclude I does not satisfy not Px, this is what we wanted. Clear? Neither of them is satisfied.

This gives us another way of looking at the open formulas. See, for the closed formulas, you do not need the states; interpretations enough. In fact, if you read the closed formulas or sentences directly through I, then you get one fact statement in your domain. Now, you have to verify whether that is true or false, according to the mechanism of domain itself; this is the procedure. But then, if it is an open formula, you cannot do that. You need to go for the states, that is. So you have done it. But, once this is done for the sentences you have another alternative; then you can think of getting some sentence from the open formula and try to see what happens.

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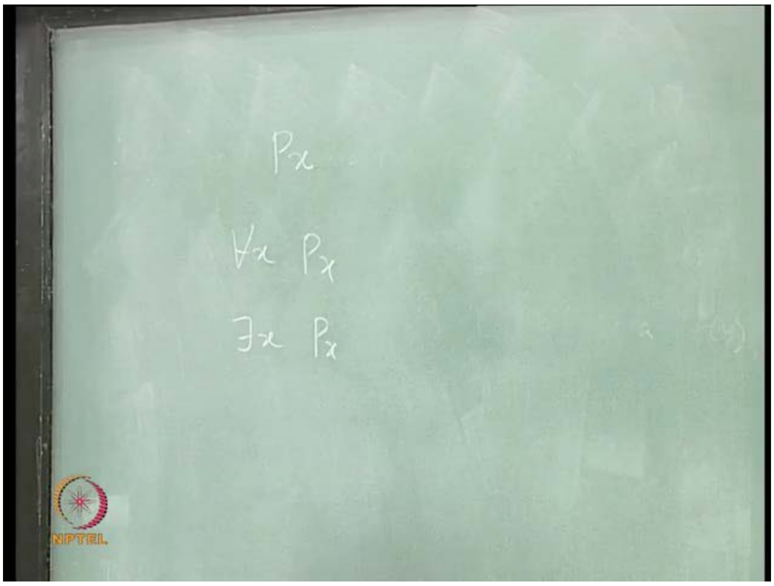

See, suppose I have Px. Here is, x is one free variable, this is an open formula. Now, I may consider two sentences, this. I can interpret in any domain I like. This also, I can interpret as a sentence. But, this is not a sentence; it cannot be interpreted as a sentence. So, I do not know the proof of it, but this two I can interpret. I know whether that is true or false, in any domain, given. Now, from the truth of this, can I conclude something about the truth of Px? Well, up to some extent, we can conclude. This is what we will discuss next time.

What we have done today is, only relevance lemma, and then we concluded that it is meaningful to define when an interpretation becomes a model of a sentence; but not of a closed formula, not of an open formula. Now, to consider open formulas, we have that either you take for every x of all those things, or there is x of all those things; then try to see what happens. This, we keep it open now, next time we will discuss.