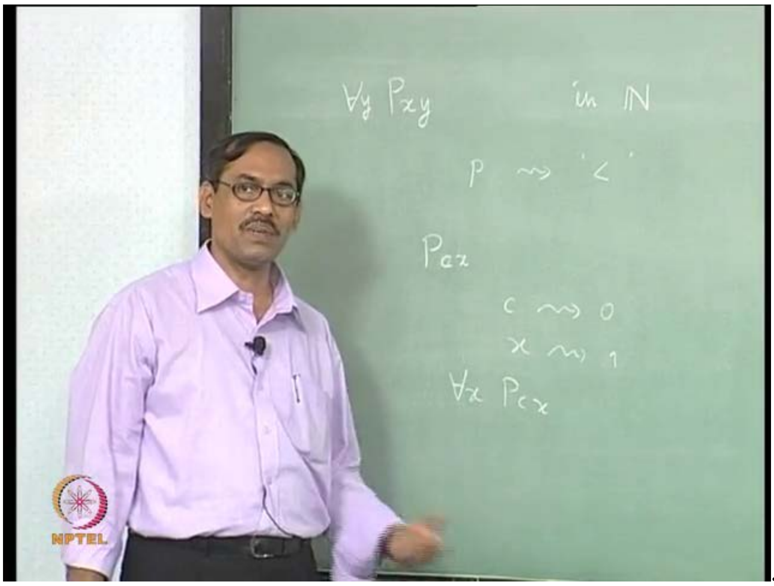
Mathematical Logic
Prof. Arindama Singh
Department of Mathematics
Indian Institute of Technology, Madras

Lecture - 25
Semantics of FL

Let us consider how to interpret or how to use these formulas, interpreting some other formulas. Suppose, I take one example, say Pxy, you have quantification over y.

(Refer Slide Time: 00:29)



Now, consider this formula. Here, suppose I want to interpret it in the domain of natural numbers. Suppose we have natural numbers, we have to interpret it. Now, while interpreting, first thing we have to do is, think of this P, the predicate which is a binary predicate as some binary predicate or binary relation over the natural numbers.

This P is now, mapped to, in somewhere, some binary predicates, right or binary relation. Let us take less than. It is less than, which has meaning in the set of natural numbers. We will formalize all these things, but now let us look at it informally, how does it look. Then, what is the next step? If you translate this sentence, it will look like, for each y, x is less than y. That y will be varying over natural numbers. When you say for each y you have only universal, natural numbers, nothing else. For every natural number the sentence will be coming like, for every natural number x is less than y.

Now, what about this x? Because this is not a sentence, we are not able to say whether this sentence is true or false in natural numbers. To make it simpler, let us take P0x; 0 is not in the first order language, you have to interpret it somehow. So, you cannot say 0, P0x. We will say for example, Pcx. Now, what do we do? This constant c should be associated with our 0. Now, the sentence will be 0 is less than x. Is it a sentence? That will be the translation in the interpretation; 0 is less than x. x is just a named gap. It is a variable. Of course, it is a named gap; in the set of natural numbers it can take some values from the natural numbers.

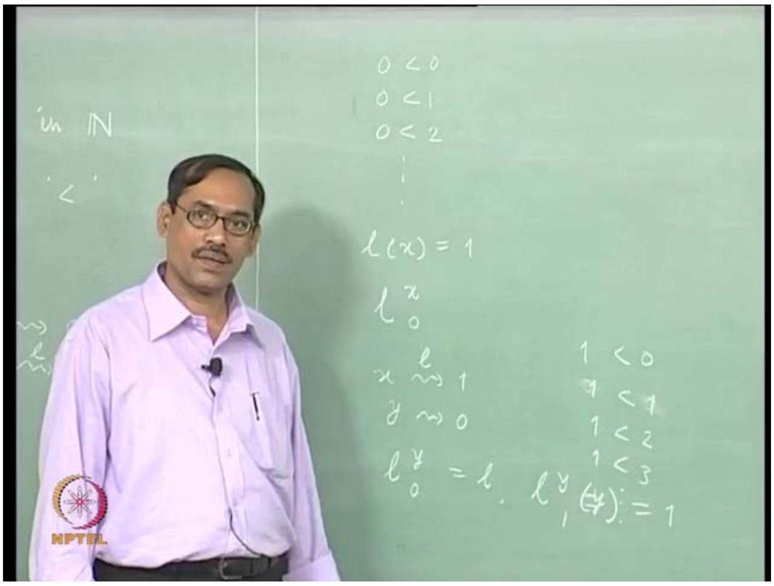
Even in that case, how do you say 0 is less than x? x is a natural number. This is not a sentence also. If you say for each x, well, it is a sentence. If you say there is x, then it is also a sentence. But as it is, it is not a sentence. Now, what is to be done? Either you have to take one of these two, or devise some other methods. If you take any one of these two, it does not look how to justify it. Why for each? Reason is not for there is x or otherwise. So, what we do is, you take a medium, a middle path. We say this x variable itself is associated with some number. Then that also becomes a sentence. That is another way of looking at it. It will give rise to some problem; we will see what the problems are.

Suppose we associate this x to 1, now I can read the sentence. It will look like 0 is less than 1, which is a sentence in natural numbers. And I associated x to 0, that is again a sentence: 0 is less than 0; it may be false, does not matter, it is a sentence. This is what we are planning to do. Not only associate the predicates but, associate the variables and associate the constants, so that you will get some sentence in our interpretation, in our domain of interpretation.

Now, if you associate variables with this; go back here. How do you write for every y Pxy or say, from this sentence for each x Pcx. You say x has been associated with 1, c has been associated with 0, so this we can read as 0 less than 1. Then what is for each x? There is problem again. The thing is, how to associate this and also interpret for each x Pcx? That is our problem, now is it clear? You want to associate the variable, fine, but then we also want to interpret for each x Pcx.

What we do here is, this association x to 1; now if you look at this sentence, which is interpreted in natural numbers, it will be an infinite number of sentences. Just look at the meaning of for each x, as we understand. This says for every natural number 0 is less than that number. That means we have all these sentences 0 less than 0, 0 less than 1, 0 less than 2.

(Refer Slide Time: 05:50)



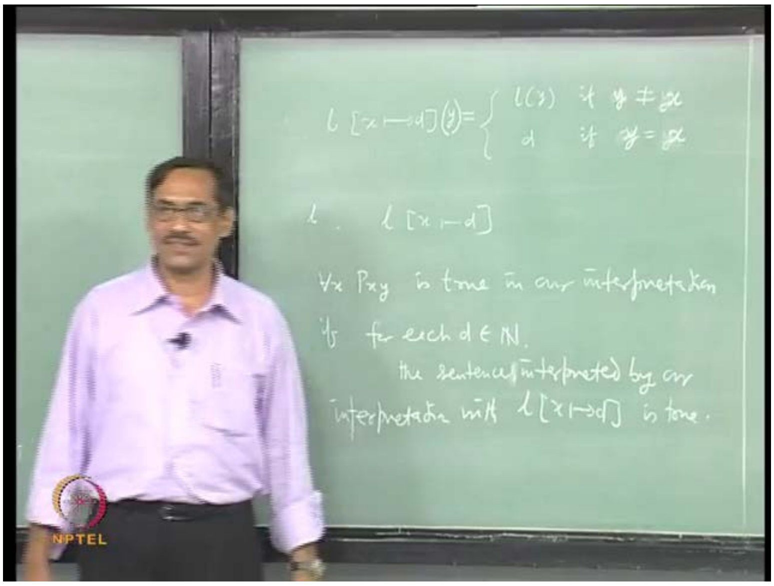
So all these sentences are there. If all of them are true, this sentence will be true otherwise not; that is what it says. But if x has already been associated with 1 how to bring this other one: 0 less than 2? What we do is, 0 less than 1, we are able to get because x has become associated with 1.

When you come for each x Pcx, somehow you have to free that association: x to 1. And consider again association x to 0, or x to 2 and so on. Suppose you say this association is l, fine. We have l of x equal to 1. Now what do we do? From assignment itself we construct another assignment where x is related to 0, which will be just like l but, fixing x to 0. Is it clear what we are doing? When you come to this formula: for each y Pxy, this has for each y, for each natural number y, x is less than y. Now, what happens, we want to associate x with something, y with something, suppose you associate x through, by this l to 1, and y to some 0.

Now, without this for each y you will read it as 1 is less than 0, but when you come to for each y Pxy, you will be considering again infinite number of sentences, 1 is less than 0, 0 is less than 0, then sorry, 1 is less than 1, then 1 is less than 2, 1 is less than 3 and so on. All these have to be brought somehow, not only 1 is less than 0. Is it okey? Because you want for each y.

What we do here is, we take, construct l y 0 let us say, y is already fixed to 0; so l y 0 will be equal to l itself. There is no chance, but if you take l y 1 this will be equal to, different from l. So how to specify this map? We will say this takes y to 1. That is the only fixing you are doing. This notation says y is fixed to 1, is it clear? This fixing, but what about x? See, x is free here, so x has already been associated to 0 or 1, x is already associated to 1. We are not worried about that; we are only worried about how to take care of this for each y. In l we have 1, where x is 1 and y is 0, in l y 1, y fixed to 1. You will say x is associated to 1 as it is, as in l, but, y is changing to 1. So, that means these are some new assignments or new valuations which will fix this variables to the subscripted 1.

(Refer Slide Time: 09:25)



You can give some other notation. Let us say l x fixed to d is equal to l of y, if x is not equal to y; and it is d, if x is equal to y. This is evaluated at y, is it clear? If you want, you can write the other way, y is not equal to x because at y you are verifying. Let us write that way. y not equal to x and y is equal to x. What we say here, when for each y Pxy is coming, now we will transfer our responsibility to these valuations. We will say that for each d, a natural number, what will happen with this assignments, or new valuations, the corresponding sentence would be true in natural numbers.

So, 'for each natural number' is coming now; it is making that x redundant. What about that? y was assigned earlier by l; that becomes redundant, for each y takes over. Is the mechanism clear? What we say is that if you take l and then consider l x to d, we say for each x Pxy is true in our interpretation if for each d in the set of natural numbers. You say the sentence interpreted by our interpretation with this new valuations l x fixed to d is true. In fact, this is not the sentence. There will be many sentences: each d you are taking, right in that sense, but you are writing each.

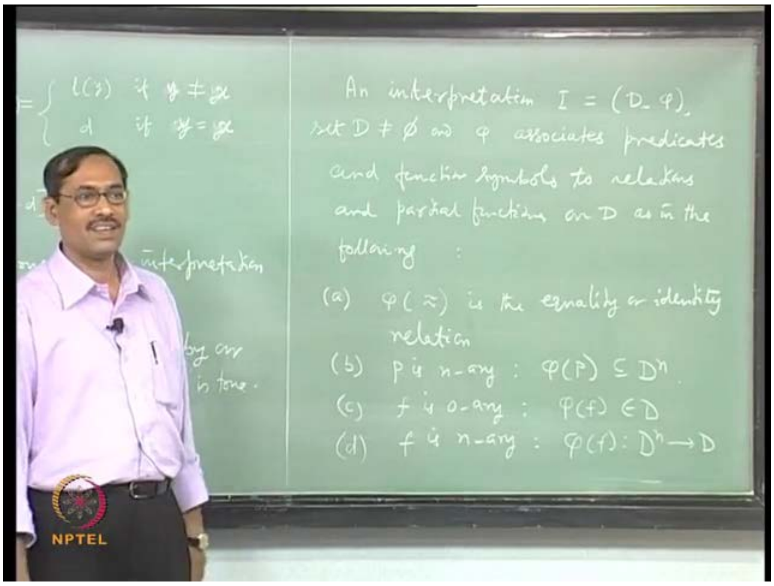
All those sentences you obtain by assigning this new variable x, x is this y to x here. Let us go for this x. For each x Pxy will be taken as true in the domain. You just vary over that assignment l x to d, and then say for each d in the natural numbers. In this new valuations the corresponding sentences will become true. What it says is that l of y is kept as it is, suppose l of y is already fixed. It is fixed with the same ones; so the values are fixed already, someway. Then what about the bound ones? If it is for each x, you will say that for each natural number d, something should be satisfied. But what should be satisfied? l of that, right? Whatever we assign to get a sentence; so that is taken care by, here, because l y is the same as earlier.

In fact, we are coming to many things now; so many things are involved. First, we have one interpretation. In the interpretation there should be one nonempty set, like natural numbers. Then we should tell how these predicates are mapped to relations. Let us say there is a map phi which tells which predicate is becoming which relation in our interpretation. Then there will be constants. For example, c here, c becomes associated to 0. And in general, there can be function symbols, not only constants, because terms can be involved. So, we will also say that the same mapping phi associates function symbols to real functions on the domain. So functions and predicated, they are defined now.

Now, coming to the variables, what we do, we take one valuation l which assigns a variable to a particular element in the domain. If it is a free variable, it goes well. You just interpret them as the elements in the domain. If it is bound, then what we do, we change the valuation l, which fixes variables to elements. By changing we get new valuations. And now we say for all those new valuations, the corresponding sentences will be true, then it is 'for each'. When it is 'there is x', you say at least one of those will be true. Is it clear informally?

We will come to formal semantics. What we will do in the formal semantics, first, we have to start with a pair having a nonempty domain d and one map phi, which associates predicates and functions to your predicates and functions in the domain itself.

(Refer Slide Time: 14:57)



So, an interpretation I is a pair; this, you write it as D phi, where D is a nonempty set and this phi associates predicates and function symbols to relations. And these function symbols will take the general view instead of, total functions we will take any partial function also, and partial function and D. How does it associate? We have to really give the details. So the details, let us give as in the following.

First thing is, we have to write for the predicates then for the function symbols and so on. All those things you have to specify. A predicate can be 0-ary predicate to begin with. So, 0-ary predicates are prepositions or you may say atomic prepositions. Then there, what should happen with the 0-ary predicates? They are already prepositions. They, when translated in an interpretation will give you simply sentences directly. They will not give just relations. They are just sentences, because no argument is required to make them sentences. That is why they are 0-ary. If it is some unary, then one argument is there. So one object is filled, then it will become a sentence. In the interpretation, we do not have to say anything for them; we will take care of that later. They can be true or false, that sentences.

Now, you are giving just the association of this phi; how this phi associates things. First thing is, you have phi of the equality symbol. So, phi of the equality symbol will be interpreted as identity itself, equal to, the same ones. It is a binary relation. When we say x equal to y, it will be interpreted as x equal to y, or x is same as y, in the domain. This is the equality or identity relation. Then, if it is not equal to symbol, then how to do? Let us say P is an n-ary relation or n-ary predicates. In that case we will say phi of P is one n-ary relation. So, n-ary relation means it should be a subset of D to the power n. If it is a binary relation, it is a subset of D cross D, right? So, 1-ary, is a subset itself, like x is prime; so the set of all prime numbers inside N, that is a subset of N. If it is ternary, you will say it is a subset of D cross D cross D and so on. In general, we will say, if P is n-ary, phi of P is a subset of D to the power of n. If we want to include another, that is the real thing, phi of P, where P is 0-ary, you say it is just a proposition, just a sentence in the natural numbers or in the domain. You have started with the any function on D. It speaks something about objects in D, that also you can include.

Then, let us come to function symbols. If f is a function symbol, it is 0-ary, then what should we do? It is a constant, a name, something like Socrates. It should have been translated back as some s, s is 0-ary function symbol. Now, that s should be associated to Socrates or Plato, somebody you have to associate or in the natural numbers you may say phi, 100, 0, something. That means this will give you that phi, that phi of f should be relevant in D itself. So, we say phi of f is an element in D.

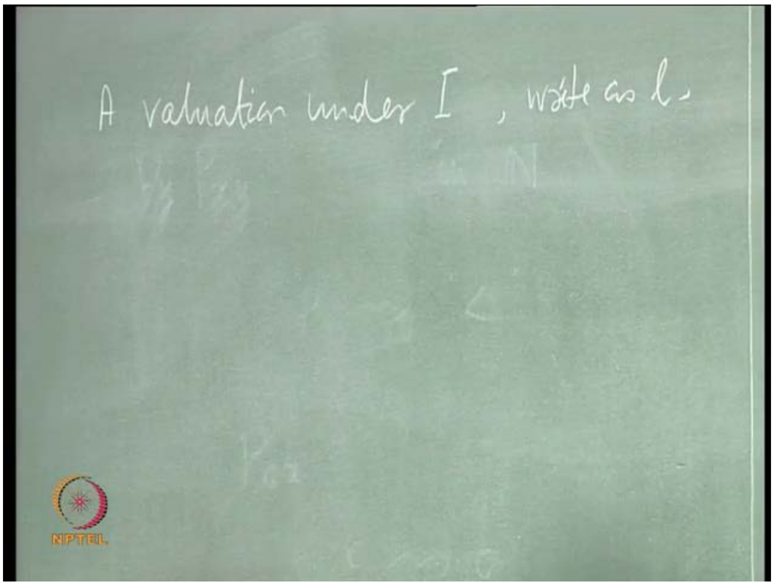
Now, if it is n-ary, then it should be a function of n arguments, mainly partial, it can be a partial function, but of n arguments; argument should be same. We say that phi of f is a function from D to the power n to D; it is a partial function from D to the power n to D.

Student: The set D, what is that?

See, we have started with the interpretation I which is having two components. The first component is a nonempty set, just a nonempty set, nothing else is there. There can be some structures, which will be useful later. At this moment we are taking it as a nonempty set. Then there is one map. This map associates predicates and function symbols to concrete relations and partial functions on the domain D. Though you have done there, if P is binary we are taking, let us say, it is associated with the relation of less than, which is also binary. Arity will have to be preserved. You cannot write P of something and then ternary: between something. That is not possible you cannot interpret the sentence. You are keeping arity, as it is, same. If P is n-ary, this is D to the power n.

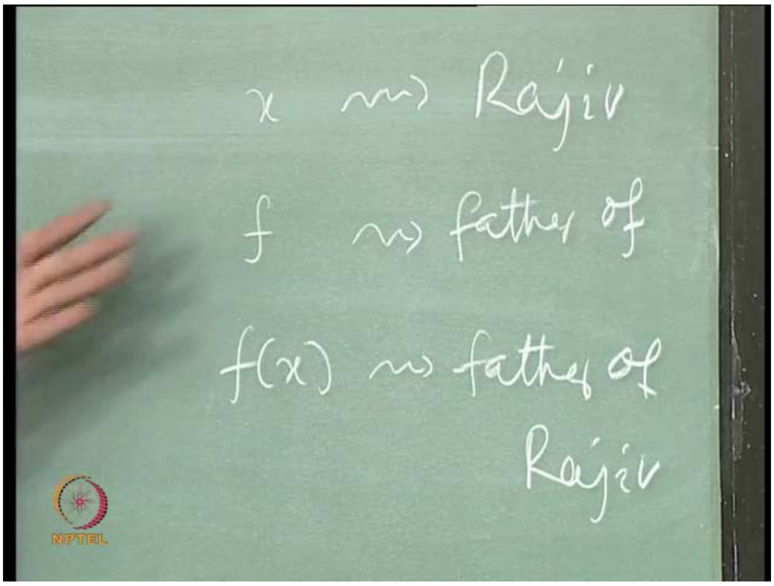
Similarly, function symbol, n-ary, it will become a partial function of n arguments. But once you put in those n arguments you should get to complete elements from D itself. Because there, a definite description that describes an object finally. It is a father of, mother of, sister of, so and so that gives you so and so later; one person, right? it is not from D to the power n to D square, to D itself. Is it clear?

(Refer Slide Time: 22:14)



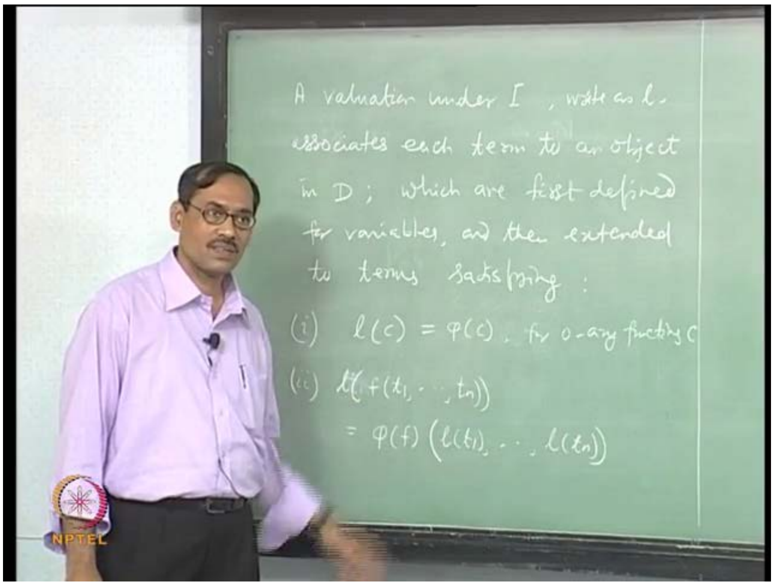
Then what we need is, assignment of the variables. That l, l type of things. So, what we say? We define a valuation under this interpretation I, because valuation will take the constants or anything, variables, let us say, to elements in the domain, so that D is important also. We say that a valuation under I, here we will write as l. Let us say, l, we can give any notation later; such a valuation should give variables to elements. So that you have variables to elements. Now we say, say, x is a variable, which has been given to Rajiv Gandhi. Now, f is a function which is father of. Now, f of x will be what? Interpreted as what? Feroz Khan. It should be father of Rajiv Gandhi. Suppose you have x which has been associated to Rajiv, Rajiv Gandhi, let us say. And f is associated to by phi 'father of'. Then f of x, we require this associated with father of Rajiv Gandhi, it should happen like this. That means your association should be extended to terms not only remain as variables. Because all the terms are only definite descriptions, they should also point out to some persons in the domain, some objects in the domain D.

(Refer Slide Time: 23:18)



So, we should be able to extend these; that is what we are going to define. That, this is one; these valuations are some mechanisms or maps which associate every term to elements in the domain D; but then it is not arbitrary. It has to take care of this; so you will be defining it for variables first, and then extending it to terms.

(Refer Slide Time: 24:34)

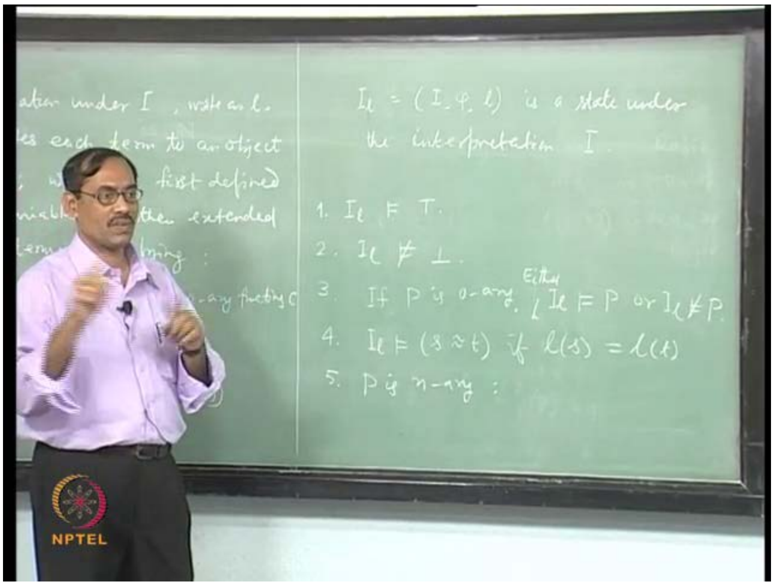


So, this valuation under I associates each term to an object in D. And these are first defined for variables, and then extended to terms satisfying. What should it satisfy? This is our requirement, you need this. Suppose I write l of x as Rajiv, l of x equal to Rajiv, now this f is associated to father of by what map? It is not l by phi, right. So you will get phi of f equal to father of. Now, what we should get is, l of this should be equal to father of Rajiv. That means l of f of x should be equal to phi f of l x, is that right? Father of Rajiv, so phi f of l x; phi of f is already a function in the domain D, so that 'of an element' makes sense. This is what we are going to say.

It should satisfy, before that you have the function symbols, which are constants. phi of c is 0, that is already there. You should also have l of c as 0. This, we argued there on the constants, first to begin with. You should write l of c is equal to phi of c for constant c, or 0-ary functions c. Then for the variables we are telling, variables already l is defined; l of x is an object in the domain D. Now you have to go for the terms. Say, t 1 to t n are already terms. Now, you say l of f of t 1 to t n; now instead of one variable x, only I am taking many variables, that is the difference. This should be equal to phi f of, l should go inside, phi f of l of x; so it should be l of t 1, l of t n. If you have already extended l to t 1 to t n, we know l of t 1 to t n; so it is a recursive definition. It will start from the constant terms and for the variables, already you know l of x, so these are the basic cases. Then in the inductive case, if you know already defined for variables and constants, you can extend it by taking any function there. That defines how l is extended for taking care of the terms, is it clear?

This formula itself is very important. Slowly you will realize where it will come. Now, you see we have started with one interpretation having two components; one nonempty set and one association or a map, which takes predicates and functions symbol to relations and partial functions. Then we have brought up this l, a evaluation, right, we have three components now. Let us give it a name. We say that I l is equal to I, phi and l, is a state under the interpretation I. We will refer to it as a state under the interpretation I. It is called a state because of the language from the programs. Supposing in a program you have many variables. There is an open variable. It has free variables. You say, x is initialized to 0. Once you say x is initialized to 0, you get one state of the program. That is the language you use in the programs and let us continue with that. Now a variable has been assigned to a value; it is a state. Similarly, all the terms, because of the definition.

(Refer Slide Time: 28:45)



Now, we have to say, under these states how the formulas are satisfied or not satisfied, right. Not under the interpretation. We will talk about the states now. In this state, something is satisfied; in this state it is not satisfied. Finally, we will come to interpretation, later, slowly.

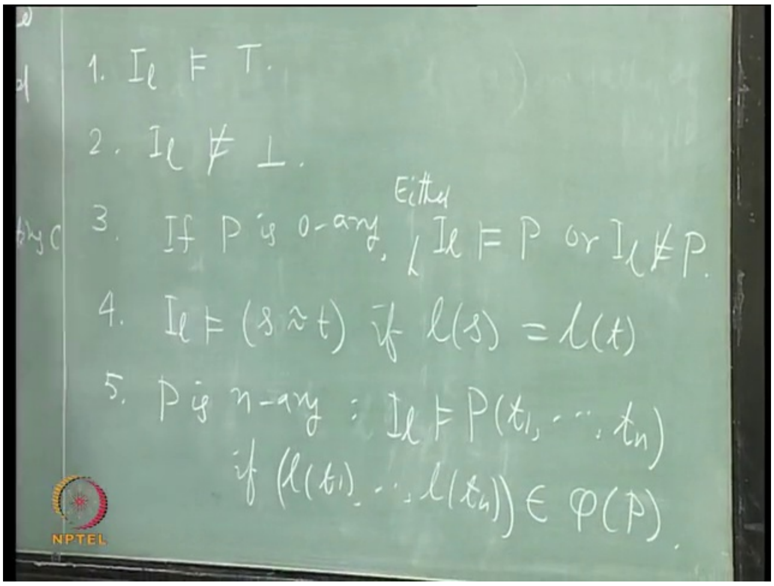
Now, you have to define how a state satisfies a formula. So there, we will say, if it is any state I l, it always satisfies top, you just declare this. Top has to be satisfied anywhere, of that should be equal to 1. You say I l satisfies top. Then we say, no state will satisfy bottom. So we say I l falsifies bottom, always; whatever state it may be. It is not unsatisfiable, then you really recursively finding it, so the first, these two states are easy to see, that is happening.

Then we have to go for, where, see, this is also a recursive definition. That means there should not be any connectives, no quantifiers, that will be the bases case. It should be something like P of something. How to interpret this P of something? This is what we want. But P means it can have many cases: like, a 0-ary or, it is equal to, equality relation or, it is any n-ary relation. So these three special cases we have to define. Let us write. If P is a 0-ary predicate, then there is no argument, there, it is a proposition; so we just define: either I l satisfies this, or I l does not satisfy this. It is a sentence; nothing else there.

Which otherwise you can write, I l of P equal to 0 or I l of P equal to 1. Well, that also you can write. Let us stick to one notation. I l satisfies P or it is either, or I l falsifies it. Next, we will be having identity relation P maybe equal to the identity relation, equal. There, we will write, if or we can directly write, I l satisfies s equal to t, if, what happens, this term is a definite description, this is also a definite description, then I say, this definite expression is equal to this definite expression, in my interpretation in my domain D.

Now, how do I say father of Rajiv is equal to father of Sanjay? Right. How do I say there are two terms now, father of Rajiv, father of Sanjay. It refers to the same person that is what I have to say. Now, in formal notation how do we say? s has already been mapped to somewhere by whom? l. So, l of s must be equal to l of t, that is all. This happens, if l of s is equal to l of t. That is the way we are going to interpret this equality relation, same as identity, no other way we are going to interpret it; so you are fixing it here.

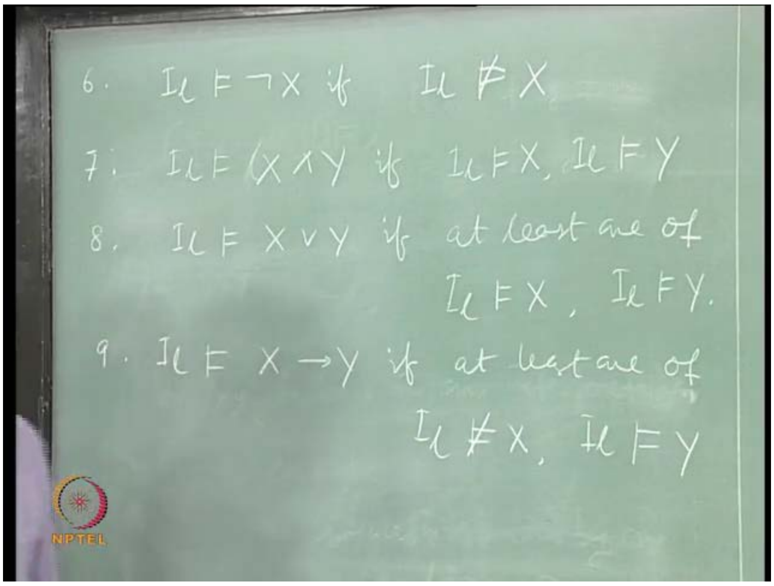
(Refer Slide Time: 34:40)



Next, if P is any n-ary relation, so p is n-ary; in that case what happens, you will have P of t 1 to t n that will be appearing; if P of t 1 to t n, now you will say that, t 1 has already been assigned to l of t 1, t 2 to l of t 2, t n to l of t n; they are elements in the domain D; now, P is a relation there, which is an n-ary relation. We will say that these two tuples of numbers to, or tuples of objects, l of t 1 to l of t n, if they are related as whatever P has been assigned to. I will say that I l satisfies P of t 1 to t n if l of t 1 to l of t n belongs to phi of P, because phi of P is the n-ary relation here. It is the n-ary relation; it is the subset of D to the power n. So, l of t 1 belongs to D, l of t n also belongs to D; that is an n tuple, which is an element of D to the power n; if that belongs to phi of P, then yes, otherwise no. These are some of the basic cases. What about the other steps?

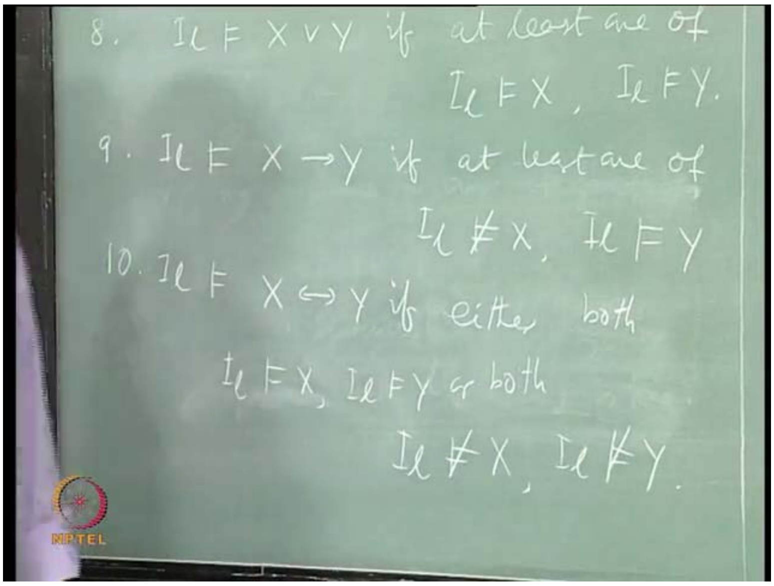
We should discuss for connectives and for the quantifiers. Next, we go. Let us say not. So, I l satisfies not x; it is propositional; if I l does not satisfy x; if I l falsifies x. This is a definition. All these, if and and only if. We do not write iff unnecessarily; this is a definition. There, what is the next one? All the connectives you have to decide now. Say, I l satisfies, and let not be for c, x and y there should be a bracket here. If, what happens, I l satisfies x and also I l satisfies y, if both of them are. Next, it satisfies x or y if at least one of I l satisfies x, I l satisfies y; at least one of these holds. Next, I l satisfies say x arrow y if at least one of I l falsifies x holds, or this one holds, satisfies y. Now you are experienced, no? x implies y is equivalent to not x or y; so we just write the definition.

(Refer Slide Time: 35:48)



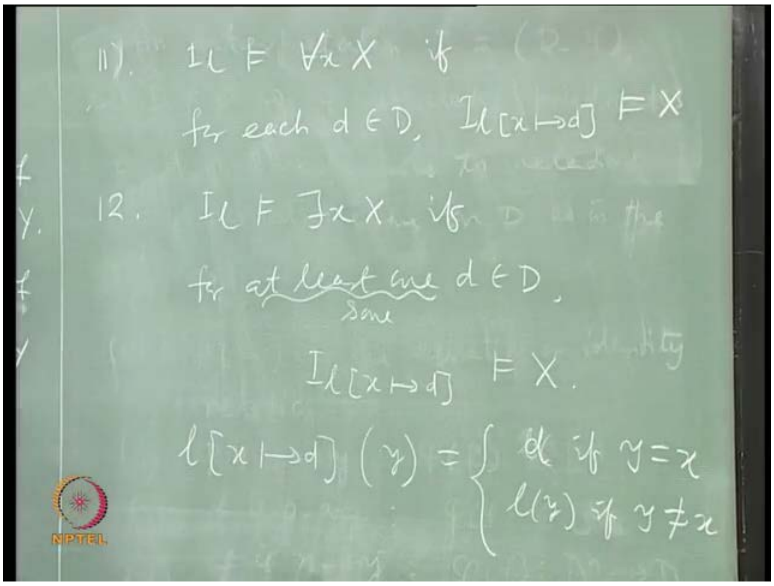
Next, I l satisfies x biconditional y if, or both we want, we want the other way. You can write two ways: one is taking from this clue, another is from the interpretation itself. Let us write the interpretation; if either both I l satisfies x, I l satisfies y 1, or both I l does not satisfy or falsifies x, and I l falsifies y. Once, both are of the same truth value that is what it says. Till now it looks like propositional. Now is the crucial thing; how do you take care of the quantifiers? You will be using those 'fixed to some element'.

(Refer Slide Time: 37:43)



So, we say that I l satisfies, of each x X if for each element d in the domain, I l x fixed to d satisfies X, just take care here. Suppose you have x equal to Px. Just try to see what we are doing. Then what we are doing, you replace x by d; because x is already fixed to d, whatever other variables we are not worried, they are same as what l assigns.

(Refer Slide Time: 39:01)

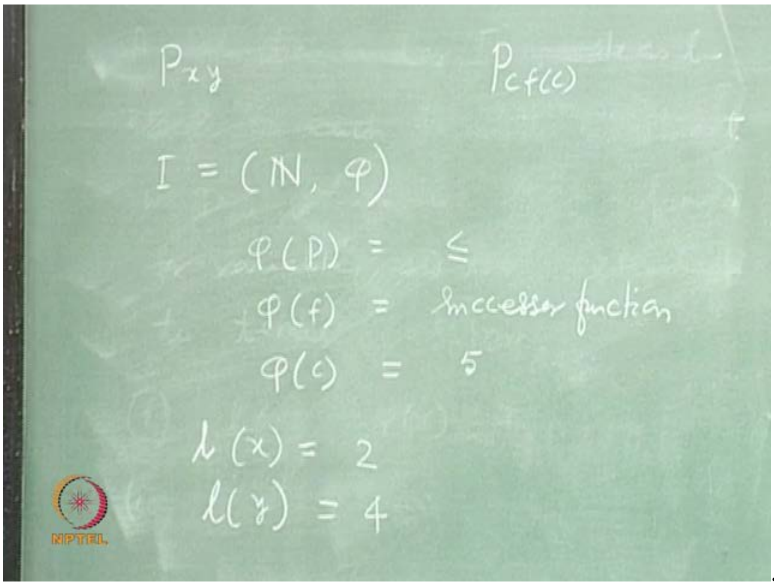


Now, once you fix x to d, it becomes P of d, in some sense, not exactly P of d; and it becomes d belongs to phi of P, fine, so Pd, let us say. Here we say, for each d in D, Pd holds. Therefore, for each x, Px holds; is it clear? Just to make it intuitively what we are doing. Then next one should be clear. Then you can formulate it yourself. I l satisfies there is x, X if for at least one d in D, we will write as some d; what happens, I l, where x fixed to d, satisfies X.

Here, we have used the notation that l x fixed to d of any y is equal to l of, is equal to d if y equal to x, and it is l of y if y is not equal to x. That means the state I l, in that, if you evaluate one universal quantifier and a formula, then you have to consider all states under the same interpretation I. All those states where x becomes fixed, all the others are as in l. So that now transfers the responsibility of each element in the domain to all valuations, all possible states that you can get from this by varying this x to d. Once this happens then we say that for each x X is satisfied, in that state.

Let us see an example. If you try to consciously remember this, it is difficult to remember. Unconsciously, remember it. That is why. Just like you have learnt your languages naturally, without knowing what words, or how many words I am learning every day. So, forget it. Then, unconsciously, you will be learning it; just try using it. Let us see how to proceed.

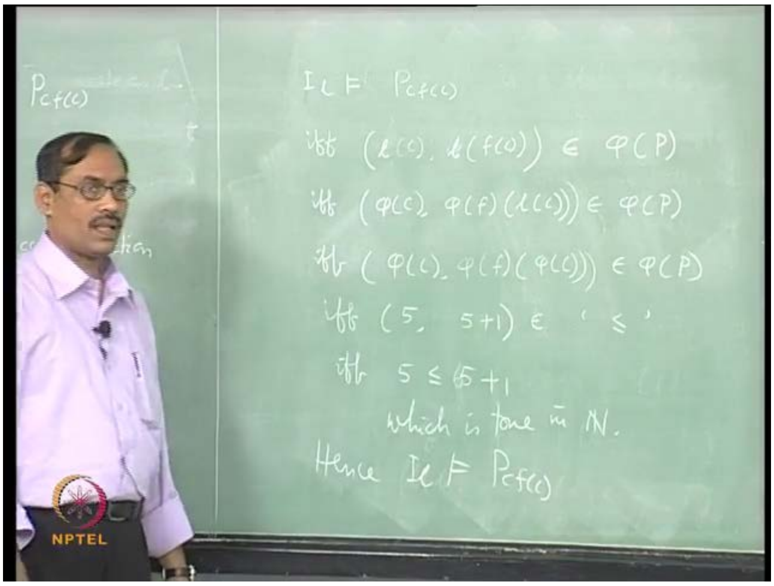
(Refer Slide Time: 42:43)



.

Suppose, I consider a formula of this form Pxy, and another, let us say, P c f of c. Let us take one interpretation I which is having its domain as natural numbers, and some association phi, some map. We will definitely tell what this map is. So, phi of P is less than or equal to, and we need phi of f also. So, phi of f is, say successor function. Then it will give you n plus 1. Now, we need also c. Let us say, phi of c is 5. Then this formula can be interpreted as, it is, nothing else is required, no l is required here; because l will agree with phi anyway by definition, but for Pxy you need something else; say, l, you have to define what is l of x, what is l of y. Let us write l of x is equal to 2, and l of y is equal to 4; something you are fixing arbitrarily. Then, with the formal semantics, how do we proceed? You have the state I l now. In that state I l, we are going to consider what is happening.

(Refer Slide Time: 44:29)



Let us say, I l satisfies Pcfc. Here, we can write, both the sides may not be same, definition, meaning is iff. It is not c comma f of c; let us go slowly. This will tell us that c is related to some element by phi, and then phi of f of c; it is not phi of f of c, it is l, because each definite description will go to some element by the valuation l; phi of f will come, not phi of f of c, is that ok? So the notation should be l here.

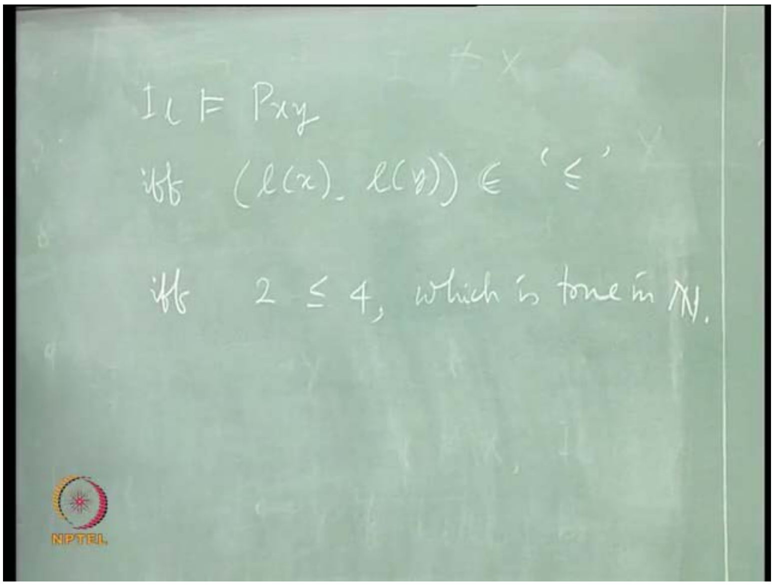
Similarly, here you should have taken l also. It is not phi of c, though they are same. Let us write it. l of c, this must belong to phi of P; phi of P is the relation. This happens if and only if, now l and phi will be same, agreeing on c, so you can write phi f c here also. So, phi of c, then l of phi of c is phi of f of l of c; this belongs to phi of P. You are just rewriting without thinking. This happens when phi of c is 5, and phi of f is successor function; so successor of phi of c belongs to the relation less than or equal to. It means we do not have six, if you do not use, it does not matter, that is all it says. Which is true.

Therefore, I l satisfies Pcf of c. This is a crucial thing, last step; hence. It is crucial because our assumption is that in every domain there is some inner mechanism of truth, which determines whether a given sentence there, is true or false. We do not know that, the formulas do not give that; it gives only how to translate from the formulas to sentences, in a given domain, that is all it gives. Then finally, the truth in the domain will decide it.

Suppose you get one formula which will give to one conjecture, after the translation, tween prime conjecture, or Goldbach conjecture, something, it will give. Now, you cannot decide there whether it is true or false; but all that you know, semantics only tells that it is either true or false, that is all; there it stops.

Such cases, you cannot really find out whether that is satisfiable or it is not satisfiable. Now, looking informally. These are the formal steps. This is how we have to proceed. Looking informally, c is interpreted as phi, f of c is interpreted as phi plus 1, P is less than or equal to. So, 5 is less than or equal to 5 plus 1. That is all. But then really, to go through it, we have to go through these steps; it is a recursive definition.

(Refer Slide Time: 49:03)



What about the other formula Pxy? Similarly, say I l of satisfies p x y if and only if x goes to l x by 2. So, we will write l of x comma l of y belongs to less than or equal to. This is how it will be translated; phi of P, which is less than or equal to. Then this will write l of x is 2 less than equal to 4, which is true in N. Therefore, it is correct. That is how you will be proceeding. We will takes some more examples later.