Mathematical Logic
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Module - 1
Lecture - 24
Hurdles in giving Meaning

In the scopes what we have seen is, defining the formulas first, and then sub-formulas of a formula. So we noted that if there is a formula you can think of a sub-formula. If you try it with an abbreviated formula then you might be wrongly concluding something to be a sub-formula which need not be a sub-formula. Because to start with, you should start with a formula itself not any string. Then you take sub-string of it which itself is a formula then you say it is a sub-formula of the original, right? You have to start with the formula, that is the question. If you start with an abbreviated formula there can be confusion which will conclude wrongly as a sub-formula; it may not be really a sub-formula because of the precedence rules.

Then we try to define the scope of a quantifier. It is simple. We just say that scope of an occurrence of a quantifier, it is not a quantifier, it is the occurrence, because in the same formula there can be several occurrences of for each x, right? So which one you are considering, you have to mention it. We say that scope of an occurrence of a quantifier, not of a quantifier. Suppose it is for each x. Then we say the scope of that occurrence of for each x is the sub-formula starting with that for each x. So starting with it, you are going to the right, reading to the right, not like urdu or arabic. This is the convention we are taking throughout.

So, you take a formula. Then look at an occurrence of a quantifier. Then find the sub-formula starting from that occurrence. Whatever is the sub-formula, that sub-formula will come from the original not from the abbreviated, if it comes from there then you find out in the abbreviated also, this is the scope, is that right? Then once you find the scope you say that every occurrence of that variable which is used by the quantifier there, that occurrence of the quantifier becomes bound. Suppose you write for each x if x is a man then x is mortal. Then both the occurrences of x in “if x is a man then x is a mortal”, both those occurrences become also bound, by the same occurrence of the quantifier for each x. We also say that the occurrence of x along with that quantifier is a bound occurrence, by default, because it is also inside the scope, is it clear?

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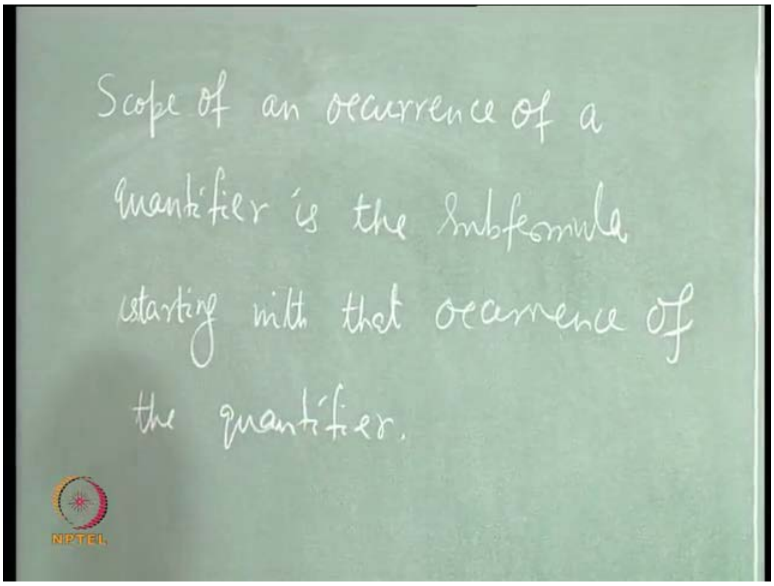

Let us write it. You say, scope of an occurrence of a quantifier is the sub-formula. So, always we have a formula in the context, that we are not mentioning here. Always, we have a given formula, in that context, you are only writing. Because scope of that occurrence means, occurring where? formula is required there, right?

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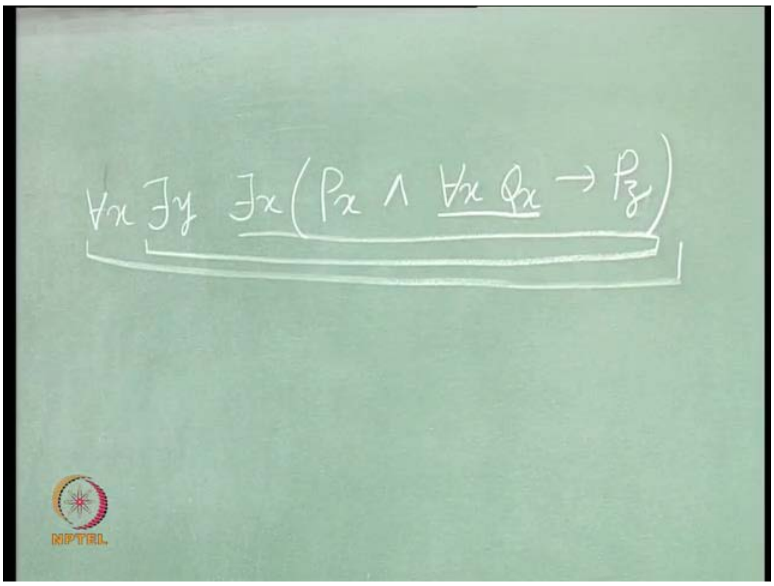

So, quantifier, is the sub-formula, so when you write sub-formula, again is sub-formula of the given formula, is sub-formula starting with that occurrence.

For example, you take for each x. Suppose you take this formula. Then the scope of this there is x should be equal to the whole of this, up to this, that is its scope. And the scope of this for each x, look at the precedence and there have been brackets; it will be the whole of this, since there is no bracket, precedence rule, for each x only goes along with this, right? So, for each x will be this much only. Similarly, scope of this there is y will be again whole of this.

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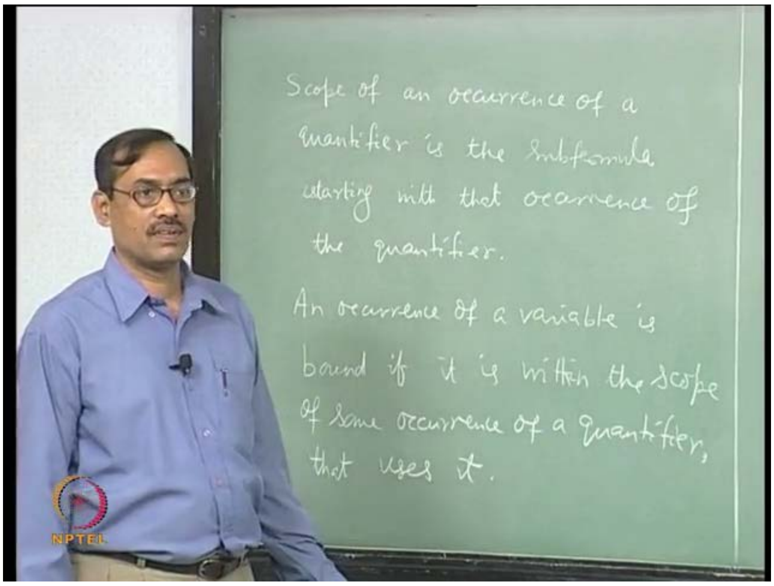

Then next we say that an occurrence of variable is a bound occurrence provided it is within the scope of some quantifier or some occurrence of a quantifier. An occurrence of a variable is bound if it is within the scope of some occurrence of a quantifier. So for example, here this occurrence of y is a bound occurrence this occurrence of x is a bound occurrence because it is within the scope.

Now, what about this x? This is also a bound occurrence, because it is within the scope of this. What about this z? Is it bound or not bound? We will say it is not bound, because our thing is, you have to say it is a bound occurrence if it is within occurrence of a quantifier, there is something to add, which uses the variable, right?

Because, here there is no quantifier which quantifies over z, we should not take it as a bound occurrence. No quantifier binds it; so you have to write occurrence of a quantifier that uses it. We say that an occurrence of a variable is free if it is not bound.

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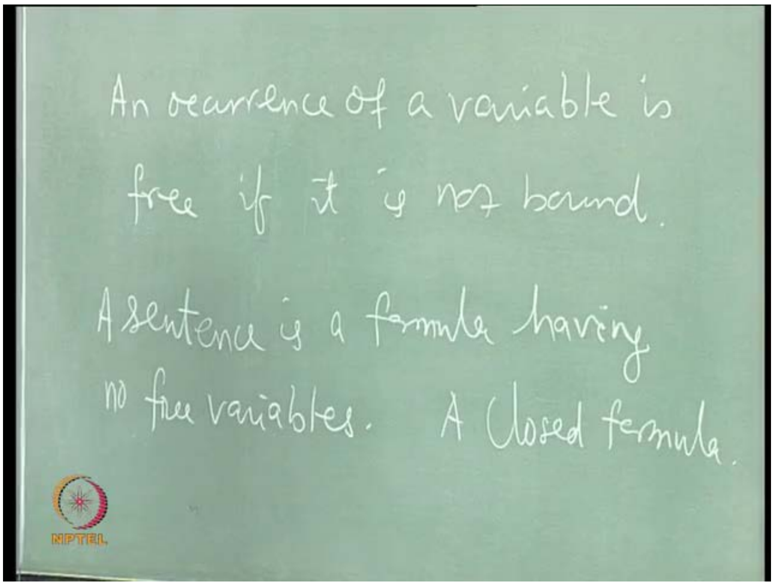

Let us say of a variable is a free occurrence if it is not bound. So any occurrence of any variable will be either bound or free because of this. Now, there is one more thing we should see. When this binding of the variables occur by the quantifiers. Here is one x, so this is a bound occurrence, because it is within the scope of this. I may also say that this is a bound occurrence, because it is within the scope of this. Also it is a bound occurrence, because it is within the scope of this for each x, right? Which occurrence binds it?

Student: Nearest.

The nearest right, so the innermost or if there are many occurrences of variables along with the quantifiers and there is an occurrence of a variable somewhere else, that occurrence will be called to be bound by the one which is the right most, among all these. If a variable occurs within the scopes of many occurrences of quantifiers, that use the same variable, then it is said to be bound by the one which is the right most among these, right? That is the right most among all these for each x. There is x, for each x within all the scope this occurrence of x lies, but it is bound by only one, this, which is the right most among these, is it clear?

We might need also not only occurrence but, the variable itself; whether it is free or bound. We say that a variable in a formula is bound if it has at least one bound occurrence. And again a variable is free in the formula if it has at least one free occurrence, right? So, that means a variable can be both free and bound though an occurrence cannot be both free and bound, it has to be either; but a variable can be.

Because that has many occurrences; depending on the occurrences, it will become. We may need this concept both the ways, so these. It is customary to tell that way instead of telling if it is not bounded it is free, or it is not free, it is bound. We say that a variable is bound if it has at least one bound occurrence and it is free if it has at least one free occurrence. Then there can be formulas having all the four possibilities, right? A variable is neither free nor bound, a variable is.

Student: Free.

Free not bound, variable is bound not free, a variable is both bound and free but, can the variable be not bound and not free? It is neither bound nor free, is it possible?

Student: Not possible.

In a formula you want, see, in a formula when you say a variable, it has to occur first, right? Suppose it has occurred. Then either it has occurred within the scope of some quantifier, it is bound by some quantifier, or, it is not. So take any occurrence. It is either free or bound. If it occurs, that occurrence, take any occurrence, it will be either bound or free. So, one of these should occur, right. One of these cases should be happening. That cases is not possible, is it clear?

Then we define a sentence to be one which is having no free variables. A sentence is a formula having no free variables. A sentence is also called a closed formula and all the other formulas are called open formulas. When you translate from English sentences, you will get really sentences, closed formulas only, not open formulas. But we have to really give meaning to both open and closed. We might have cases where open formulas will occur even if they do not come from natural languages. We will give meaning for both of them.

That means, in a closed formula you take any occurrence of any variable that has to be a bound occurrence. It cannot be a free occurrence. There is no free occurrence at all of any variable, then only you say it is a sentence, fine. Let us go back to our Aristotelian sentences, to see how they become closed formulas.

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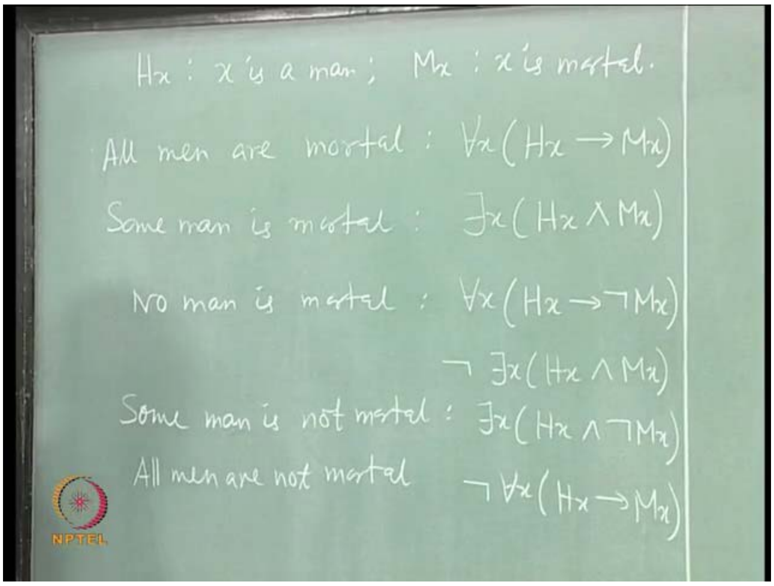

Say, all men are mortal. Suppose I write H of x as x is a man, and Mx means x is mortal. Then what is translation of this to FL?

Student: For each x.

For each x if x is a man then x is mortal. Now, some man is mortal there exists x.

Student: x is man.

x is a man and x is mortal. Now no man is mortal.

Student: For each x

For each x if x is a man then x is immortal, that we want to write. It is not the case that you will find a man who is mortal, right? That is what it is telling. It is not the case that you will find one man who is mortal. Some man is not mortal.

Student: Mortal.

Or there is one man who is immortal or you say it is not the case that

Student: Hx implies x.

Either of this. And you see that this is also the same thing as telling all men are not mortal.

Student: Not all men.

Not all men are mortal will be.

Student: Sir, is not that.

It is not that all men are mortal, so what will be that?

Student: For each x.

It is not that all men are mortal. Same thing as this also. They are same. Both these are same, we have seen.

Student: All men are not, is same thing is, continuously you are saying.

It is only yours, but, that is its meaning in natural languages, is it not? All children in this class are not brilliant. What do we say? It does not refer to you, right? You are not children, then what do you say?

Student: Not all children are brilliant.

Yes, are brilliant. It is not the case that all children here are brilliant. That means there exists at least one who not brilliant. So they give the same meaning. Sometimes any also confuses with not, any is also like on ambiguous quantifier. It sometimes means all, sometimes means some. Especially, when with not, it gives the meaning of some. See, you have to be observant about that, what is the meaning of the sentence. See, all these Aristotelian sentences are only having monadic predicates, unary predicates, right?

Aristotelian logic is simply a sub-logic of our first-order logic, where you get only monadic predicates, nothing more. And there is no equality predicate used. So it is called pure monadic logic. It was a long leap, coming from Aristotelian logic to first-order logic. It took almost 2000 years to reach this first-order logic, what we are discussing now. This was the end of all logic at that time. That is why we have to mention it specially. But then he had a very difficult way of tackling all these arguments, with this kinds of sentences. All the sentences were of these four kinds, that was his first observation.

Now, we know that it concerns only monadic logic. Binary predicates cannot be handled here. Those examples also, we will see how they will come nicely. But then with these he had to really struggle to give, which are valid arguments which are not. Because in his syllogism, there will be really three sentences, two will be premises, one will be a conclusion. Like your classic one: All men are mortal. Socrates is a man. Therefore, Socrates is mortal. In that syllogism, what happens is, the first sentence “Socrates is a man” can be written now in FL. See, Socrates is a constant. Let us write some a or s. Now, what happens, you say it is man; so s is a man, it will be simply Hs.

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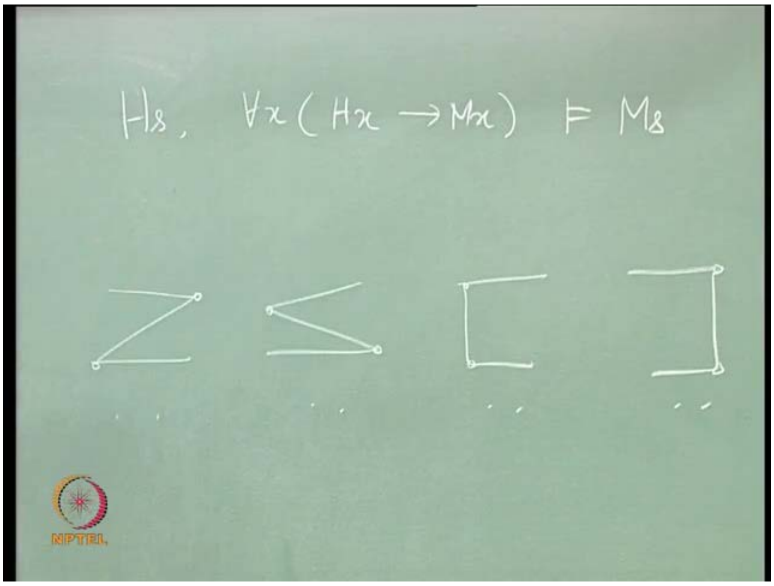

Now, we will have two premises there. Hs and then you have the other premise as for each x, each x implies Mx, and your conclusion is Ms. But in these two sentences, you will see that there is a common term. The common term is H, this predicate, men, right? Socrates is a man, all men are mortal, man is the common term there. And in the conclusion, you see that that man is gone. That is eliminated. That connection is exploited; it is eliminated; then you conclude something about the other two terms. That was is scheme of syllogisms. Always there will be a middle term. You exploit that middle term and then conclude connecting the other two minor and major terms. We have, how many possibilities are there? We found there are four possibilities for the sentence. “Socrates is a man” is of first category. That is also universal like sun raises. So we say that is a universal sentence.

Of these four types. You will see that the arguments will be brought up using these four types of sentences. Each argument will have again three sentences: two premises, one conclusion, and there is a middle term in both the premises, where the middle term rests. For example, here if I take “Socrates is a man”, “all men are mortal”. So, 'man' comes diagonally, right? So our first term, middle term, next middle term, next last term. That gives rise to a figure where the middles are joined. So, you will get four figures, in this sense. This is your middle term, again this is your middle term. There are four figures and each one will have a conclusion, so there is something else here.

In each figure there will be three places to be filled in. And all these can be filled in by four types of sentences. Each one of these is an argument, is just a syllogism. So how many syllogisms can be constructed? That was his first point, right? Let us see. There are four figures. Take one figure. In one figure, you have four possibilities. Here, four types of sentences, another four here, and one four here. So, 4 into 4 into 4 into 4; there are 256 types of syllogisms. So, he solves all these and says that there are only 19 out of this which are correct; all the others are wrong syllogisms. Once some argument is there, you just identify which syllogism it falls into. This one? You remembered all those 19, okey, this is valid. If it is not out of those 19, it is invalid. That was his way of dealing with it.

Now, we have a better way of course. We do not have to remember those things. But he also devised a nice way to remember that. For example, this one, this is one, a sentence, this is also another, a sentence, and the conclusion is also one, a sentence. He gave a name to this as BARBARA. There are 3 A’s, will be coming in that sequence, and some consonants are inside; so that you remember a name. Similarly, there is another, where first sentence is E another is A, next is E. E-sentence is the one: no man is mortal; they are called E-sentences. Then you say CELARENT.

Students: What are A and E?

A is for all men are mortal, E is for no man is mortal.

Student: And the other is?

I and O, right. I is for some man is mortal, O is for some man is not mortal. Then he devised. See, EAE, it gives CELARENT; so BARBARA, CELARENT, DARII, FERIO, CESARE, CAMESTRES, FESTINO, BAROKO, etc. You remember all those nineteen. And then find out this syllogism is this, it is valid, the other one is invalid. Forget it. That is how the procedure went. But now we are tackling something more, along with the monadic logic, pure monadic logic, we have equality symbol, we have some other things even, binary predicates, and so on. We cannot remember like this. We have to devise some better way of giving meaning and then working it out.

What we suggest is, first try to find out what is going on when you give meaning to these things. For example, we know what are the bound variables, what are the free variables, what are sentences. Once you know it, suppose I take one of the sentences, say for each x Hx implies Mx, one of the sentences. Now, how do I give meaning? Where will I find meaning to this? That is the first thing. Even to make it simpler, let us, Hs. So, H is a predicate, it is a unary predicate I know, s is some element, it is a particular constant. Constants can be given meaning as some element in a set. Let us say take, in human beings we give meaning to s as Socrates, H as `is a man'. So, this is translated as Socrates is a man, fine. But, it need not be "Socrates is a man", when you look at Hs because H can be anything, s can be anything. It will come something like, something is of some type, is of this type.

It has the capability of being interpreted that way. It is not necessarily so. I go back to "Socrates is a man", right. It might be something: this chalk is white, some sort of thing or this mineral water is infected; anything of that type can be taken there, is that right? Now, the thing is where to look for such meaning? What we see here is, you take first, considered on the set of the human beings, there, you find one particular human being, then associate that with s. A constant will be associated with an element of your domain. You have to start with a non-empty domain. If you take empty domain, almost everything will be vacuously true, there all, for all x.

It will have nothing for us. So we have to start with a non-empty domain. And the domain we do not know, how big is it, which domain it is. So we have to really transcend it. If you say something is valid it should be true in every domain. For example, you say

Px or not Px, Px implies Px, for each x Px implies Px; it should be valid in every domain. If x is P, then x is P. Some such things we have to really filter out later, the valid statements or valid sentences or even valid formulas without the constraint of every variable being bound, right, every occurrence being bound. To consider this, let us start with the simple one.

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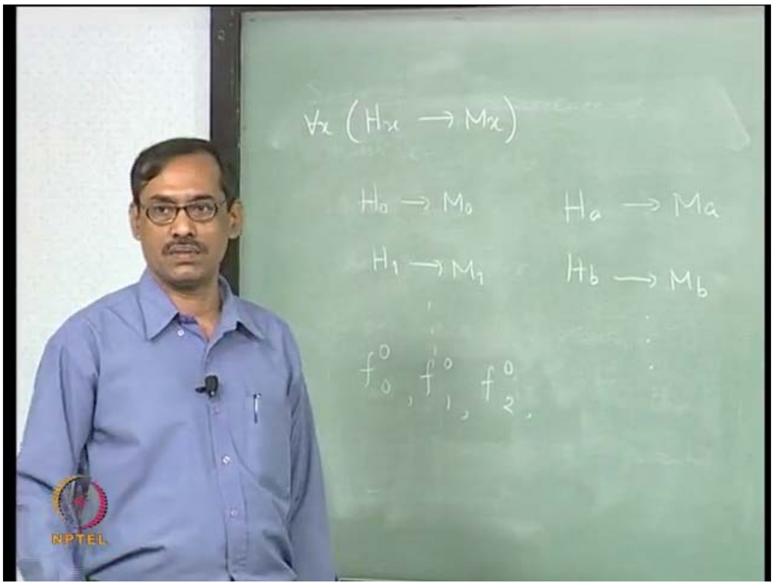

For each x Hx implies Mx. Suppose I want to interpret in the set of natural numbers, right? How do I interpret? How do I check whether this is true or not in natural numbers? My domain is natural numbers, not human beings. What I have to do is, I am thinking of H as something, some predicate, say Hx means h is a natural number, Mx means x is a real number. Now, you cannot say x is a real number when my domain is only natural numbers. The real number has no meaning there. So, we have to give something else. Say, we say Hx means x is a prime number, Mx means x is even. I have to verify now whether every prime number is even or not; that I have to verify. It may be true or false, does not matter, is that clear? That is how we will be going.

Abstractly, what I do, I will verify whether H0 implies M0 is true. My first question. Next, I will ask whether H1 implies M1 is true, right. I will continue there, I must verify for all the natural numbers; that is what the sentence demands. For each natural number H of that natural number implies M of that natural number, it should be true. Whatever this H and M may be, I will consider that later, let us say. But there are some hurdles like

0 for example, is not our constant, so H0 is not a formula, right? Our constants are not natural numbers. What is a constant in a first-order language or in first-order logic itself?

They are f 0's, so we have agreed to write them as a, b, c and so on. But not 0, 1, 2, 3, 4. They are very particular objects; they have some structure in that. But our constants are, have no structures. They are just syntactic entities: a, b, c, d, and so on. They can be substituted for x, can be some terms, here also, that is, in general, that is possible. But certainly I cannot say H0. H0 is not syntactically allowed in first-order logic; is the hurdle clear? This is the hurdle we are facing now.

See, 0 is not a constant in first-order logic. In first-order logic, we have the constants as f0, f1, and so on, with superscript 0. These are the constants: f 0 0, f 0 1, f 0 2; these are our constants. When I substitute here, you are thinking you have substituted a constant, but they are not constants. But we have to do something. That means either think of these constants as my 0, these constants as my 1, and so on. We have to interpret a certain way. I have to read these constants as those 0s in my domain. I have to associate the syntactic entities with my concrete things, natural numbers, which I am taking now, right? One association is required, agreed? We will associate that way later.

Now, even if we associate, now instead of 0, I am thinking of this as f 0 0. What about this H? Let us say a. a is allowed. Other f 0 0, we have put some convention; so it has come to Ha implies Ma. Similarly, Hb to Mb, and so on. I may think of this now, right? And I have to think what this H is. Again, I have to associate, like these constants were associated with the natural numbers, here. Similar way, I have to associate these predicates H and M with some concrete properties of natural numbers. Something is prime, something is even, something is odd, or some such thing, which talks about natural numbers. It is a relation over natural numbers, right, that I have to associate. We will associate.

But then what, here, we are doing is, we are forgetting this 'for each x' and considering the open formula, Hx implies Mx. Then in place of x, I am substituting the constants and trying to verify the truth of those new sentences, right. Some substitution has taken place. This x has been substituted by the constants now. In general you can substitute variables by terms, why only constants? Any term can come, like, you may say, it is H of successor of 0 implies M of successor of 0. So, 'successor of' is a function, say, it is f of

0. That is, instead of x, we can write f of 0, f of a; that is also allowed. Therefore, terms are also allowed, if constants are allowed, terms must also be allowed.

In general, any term can be there. But what about our terms? Our terms not only involve constants, they involve also variables. Say, f of a is allowed, f of x is also allowed. So, it should be true for x even, f of x even. I can write H of f of x implies M of f of x, whatever x you chose. There will be sub-cases of this; f is a definite function; it will be subset of this, right. So, it should allow for all the terms. Substitution should be done for all the terms, right, is that clear? That is what we are now addressing. How to substitute variables for terms? Whether everything is allowed, or some constant is there, that is our first concern.

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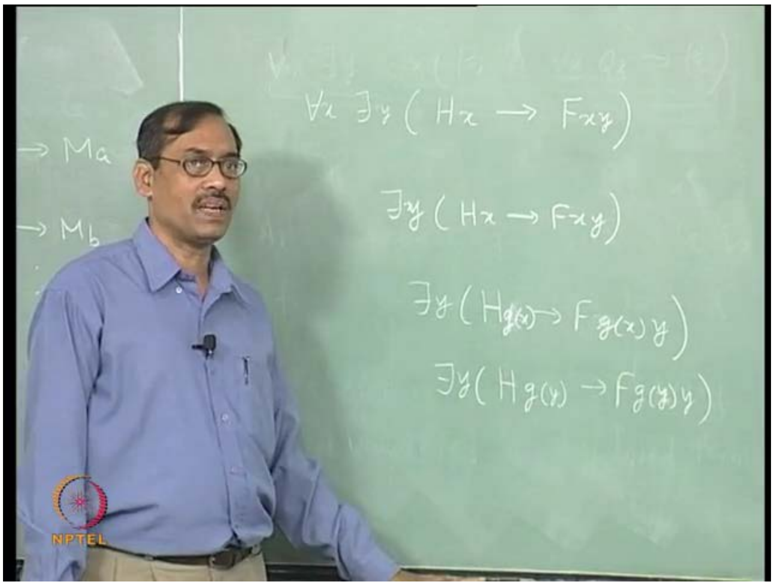

Let us take one more example. Say, for each x there is y, Hx implies Fxy. I am taking a binary predicate here, intentionally. I want there is y also. Here, we can interpret; or this might have come by translating the sentence if x is a human being then x has a father, right? Every human being has a father Fxy means y is father of x. Let us translate that way. Now, our first concern is if I go to all human beings, I have to verify without this for each x. I have a formula there is y Hx implies Fxy. Here, what I am going to verify? In place of x, I will substitute each a, one way right. Find whatever number may be, infinite number of sentences, and determine whether true, all those sentences are true or not. That is what it demands.

I see that this x can be replaced by any term also, not only constants, even any term. Also I can replace; that term may be something like elder brother of x right. That is also allowed. I see, there is y Hx implies, are h of f of x implies f, eliminate, g rather f g of x y. Now, how do I read the sentence? Say, x has been assigned to some particular person, Socrates, right? Now, what happens, this says: if Socrates' elder brother is a human being, g of x is: g of Socrates; so elder brother of Socrates is a human being, then that elder brother of Socrates has a father. That means elder brother of Socrates has a father. This is the sentence. So this is allowed, there is no problem.

Now, instead of this g x, suppose I write there is y H g of y implies f g of x g of y; y any time is allowed, so instead of g of x, I substitute g of y; in place of x, in the original, this sentence. How do you read this? There is a person, now y is quantified, right. So there is a person, it is really not a person, anything, there is an object whose elder brother is a person g of y. If that happens, if elder person of elder brother of somebody is a person, then what happens?

He is his own father.

Student: His father is father of.

That elder brother is his own father. That elder brother is his own father; this is absurd. See, the problem with substitution? Why this is happening? Let us see the reason. Why this is happening? This case, it did not give any problem. It says Socrates' elder brother has father, that is all. But this says there is one person whose elder brother is a person, then that elder brother is his own father.

Student: That substitution of for all x so.

Yes?

Student: Any.

Yes, x can be substituted for any term. So, I substitute g of x, now I substitute g of y.

Student: That was not much constraint on x as such so it did not affect it, but for y it is like some y, it is not like any y.

Well, what did it amount to is that, here, x and y are different variables. So, when you think of g of x, this x is a free variable; but now when you write g of y, this y is getting captured by the quantifier; it becomes a bound occurrence. We do not want such capturing to happen. If this happens, there will be problem. Earlier it was free, now it is not free, that is what we see. So a substitution should not make a free variable bound. It is becoming like that; but, there is now free variable here. So it is difficult to express this idea, right? We cannot say free variable becomes bound, because the free variable was x, x is not here. How can you say? But that occurrence, whatever you have written, that becomes bound.

After the substitution, a variable is becoming bound, that is happening. It is variable capturing. A variable is getting captured by the substitution itself. That should not happen. We have to introduce some more ideas, before even we go for this semantics, how to give meanings. See, you have to be very patient here, because we are tackling a very big logic. It is almost, it is capable of expressing almost everything. You have to very careful, and we are patiently going now.

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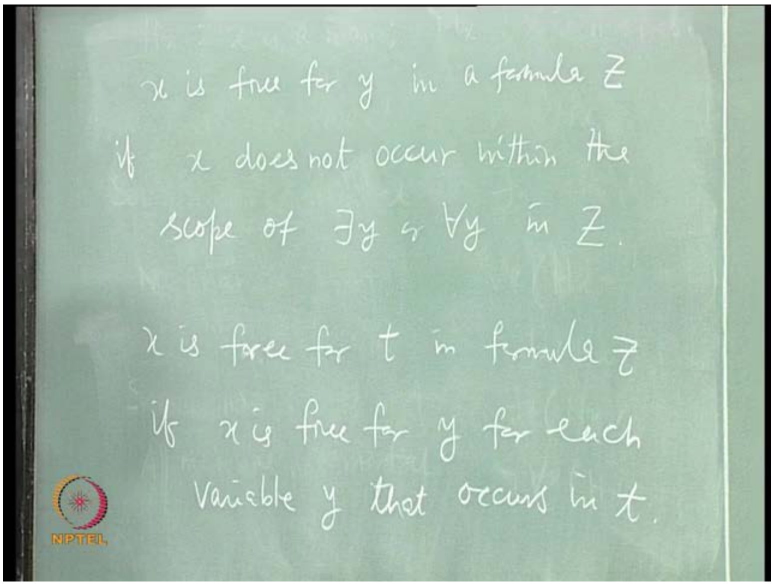

Suppose a formula is, again in a context, in that context only we are talking; a formula is given. Then in that, we have some variables. You say x and y are variables occurring there. You say, even it does not occur, we can see what is the vacuousness here. Let us take two variables x and y. We say x is free for y in a formula say Z. If, now we want to see that if it is free then substitutions will be allowed; if it is not free for y then x cannot be replaced by y, that is what we want to express. If x is free for y, then you can substitute x for y, replace x with that y, same y, right, so what happens? See, here it is getting problem because x is occurring within the scope of there is y; that should not happen. So, that we have to write. If x does not occur within the scope of there is y or for each y in Z.

So, x is free for y if x does not occur within the scope of a quantifier that uses y, right? Then it is free. so x becomes free for y; x can be replaced by y. Then we have to go for terms also, not only variables. We say that x is free for t, t is a term here, free for t in formula Z if x is free for y for each variable y that occurs in t. This is very essential. Now then, our aim is to go for the substitutions. We will give a notation for the substitutions also.

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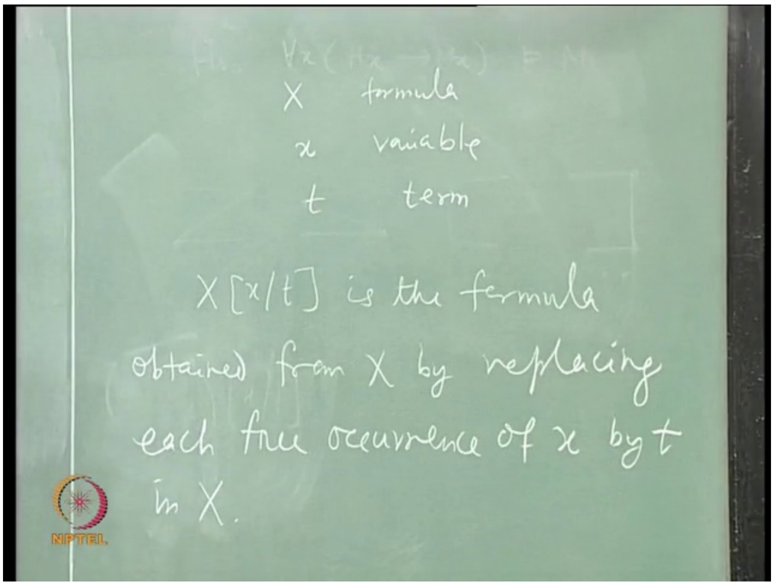

X is a formula, small x is a variable and t is a term. Then we will write X x by t for the formula which is obtained from capital X by replacing each occurrence of small x by t. That is what we want to do, fine. Do not write now. That is what we want to do. But there is again one more hurdle; we have to see that. Suppose you write Px for each x. Now here, if I write x by t; suppose in this formula, I substitute x by t, then I will get for each t Pt which is not a formula, because t is not allowed in for all. Only variables are allowed with for all, fine? But if I take Px there is no problem it is syntactically allowed.

Which means only free occurrences can be substituted, bound occurrences need not be substituted, we do not need them. They are already bound, there, not there, right, is it clear? Any bound occurrence, you see, as if it is not there. We will see later that it can be replaced by any other name; it is an named gap; so any gap can be introduced there, does not matter. What do we need here is that X x by t is the formula obtained from X by replacing each free occurrence of x by t in X. And whenever you use substitution we assume that x is free for t. We always assume that this x is free for t. That is our assumption. Sometimes, it is he told that a substitution is called admissible if x is free for t. In our terminology, all substitution will be admissible. We will accept them. Otherwise we will not use this substitution at all, fine. May be, we will stop it here. All that we have done is, we introduced the bound and free occurrences. And then when and which substitutions are admissible, fine. Then later we have to use this substitutions to give meaning to the sentences or formulas in first-order logic.